Distinguished non-Archimedean representations

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ABSTRACT. For a symmetric space (G, H), one is interested in understanding the vector space of H-invariant linear forms on a representation π of G. In particular an important question is whether or not the dimension of this space is bounded by one. We cover the known results for the pair $(G = \mathrm{R}_{E/F}\mathrm{GL}(n), H = \mathrm{GL}(n))$, and then discuss the corresponding $\mathrm{SL}(n)$ case. In this paper, we show that $(G = \mathrm{R}_{E/F}\mathrm{SL}(n), H = \mathrm{SL}(n))$ is a Gelfand pair when n is odd. When n is even, the space of H-invariant forms on π can have dimension more than one even when π is supercuspidal.

The latter work is joint with Dipendra Prasad.

1. Introduction

Let G be a group, and H the group of fixed points of an involution on G. A representation π of G is said to be distinguished with respect to H, if the space of H-invariant linear forms on π is nonzero. More generally, if the space $\mathrm{Hom}_H(\pi, \chi)$ is nonzero for a character χ of H, we say that π is χ-distinguished with respect to H. In this paper we are interested in the case when (G, H) is defined over a p-adic field.

The initial impetus for much of the research in this field came from the work of Harder, Langlands, and Rapoport [9] where they introduce the notion of distinguishedness (in terms of the non-vanishing of a certain period integral) when (G, H) is defined over the adeles of a number field. Specifically a careful analysis of distinguishedness for the pair $G = \mathrm{R}_{F/\mathbb{Q}}\mathrm{GL}(2), H = \mathrm{GL}(2)$, where F is a real quadratic extension of $\mathbb{Q}$, was required to settle the Tate conjecture for the Hilbert modular surface associated to G.

The philosophy (due to Jacquet) is that distinguishedness for the pair (G, H) often characterise the image of a lift from a suitable group H'. For instance if $G = \mathrm{GL}_n(E)$ and $H = \mathrm{GL}_n(F)$, where E/F a quadratic extension of p-adic (or number) fields, the H-distinguished representations are the ones in the image of the base change map from a suitable unitary group [5]. Conversely if H is a unitary group (with respect to E/F), then H' is $\mathrm{GL}_n(F)$ (see the series of papers starting with [13] and [14]).

From now on let E/F be a quadratic extension of p-adic fields. The groups that we consider will be defined over F. For a representation π of G, one is interested in a better understanding of the space $\mathrm{Hom}_H(\pi, 1)$. A symmetric pair (G, H) is said to have the multiplicity one property (equivalently (G, H) is called a Gelfand pair) if the above space has dimension less than or equal to one for all irreducible admissible representations π of G. The pair $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ is known to have the multiplicity one property ([5],[10]). Here it is not very hard to deduce the multiplicity one property, as it is relatively easy to show that all the double cosets of H in G are fixed by an involution. Nevertheless proving that a pair (G, H) is a Gelfand pair can turn out to be quite hard. For instance this is the case when $G = \mathrm{GL}_{2n}(F)$ and $H = \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ [15].

An example of a non-Gelfand pair is obtained by taking $G = \mathrm{GL}_n(E)$ and $H = \mathrm{U}(n, E/F)$, the quasi-split unitary group with respect to E/F. But in this case, $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, 1)$ is conjectured to be bounded by one, if π is a supercuspidal representation (see [11] for details). Such a pair is often called a supercuspidal Gelfand pair. It is known that if "almost all" double cosets HgH are fixed by an involution of G, then (G, H) is a supercuspidal Gelfand pair [12].

It is natural to ask whether there is a symmetric space for which the multiplicity one property fails even for supercuspidal representations. There are such spaces, and an example is $(G = \mathrm{SL}_n(E), H = \mathrm{SL}_n(F))$, n being an even integer [2]. This can be deduced from the formula for the dimension of $\mathrm{Hom}_H(\pi, 1)$ proved in [2]. This is stated as Theorem 4.3 in this paper, and we also sketch a proof of it. Also worth pointing out is the fact that the multiplicity formula in this context resembles closely to the Labesse-Langlands multiplicity formula for the multiplicity of a representation in the cuspidal spectrum of $\mathrm{SL}_2(\mathbb{A}_F)$ (for a number field F) [17, 21].

Interestingly, when n is odd, $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ is a Gelfand pair. We record this in the following theorem.

THEOREM 1.1. *Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$. Then if n is odd, $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}(\pi, 1) \leq 1$.*

The plan of the paper is as follows. We recall some of the main results in the case $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ in Section 2. The case $(\mathrm{GL}_n(E), \mathrm{U}(n))$ is dealt with in the next section. Final two sections deal with the symmetric pair $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$, and the proof of Theorem 1.1. This is joint work with Dipendra Prasad.

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2. The symmetric space $(\mathrm{R}_{E/F}\mathrm{GL}(n), \mathrm{GL}(n))$

Let F be a p-adic field and let E be a degree two extension of F. Let σ denote the nontrivial element of $\mathrm{Gal}(E/F)$. We recall some of the main theorems and conjectures about representations of $G = \mathrm{GL}_n(E)$ distinguished with respect to $H = \mathrm{GL}_n(F)$.

As already pointed out in the introduction (G, H) is a Gelfand pair. The key here is that there is an involution of G that fixes all the double cosets HgH. Specifically one knows the following lemma (Proposition 10, [5]).

PROPOSITION 2.1. *The involution $g \mapsto g^{-\sigma}$ fixes the double cosets of H in G.*

PROOF (SKETCH). Consider the map from G/H to the set $S = \{g \in G \mid gg^\sigma = 1\}$ given by

$$gH \mapsto gg^{-\sigma}.$$

This map is a bijection (it is clearly well defined and injective, and surjectivity follows by Hilbert 90). Next step is to show that two elements of S which are conjugate in G are conjugate in H. Now consider the elements $gg^{-\sigma}$ and $g^{-\sigma}g$. Since these two elements are conjugate in G, they should be conjugate in H. The proof follows from this. □

Thus any distribution on G that is H-bi-invariant is invariant under the above involution. Also if an irreducible admissible representation π of G admits an H-invariant linear form, then so does the contragredient $\pi^\vee$ of π. Therefore the multiplicity one property follows from the well-known lemma due to Gelfand.

Proposition 2.1 also proves the following result (see Proposition 12, [**5**]).

THEOREM 2.2. *If π is an irreducible admissible representation of G which is distinguished by H, then $\pi^\vee \cong \pi^\sigma$.*

Also note that an obvious necessary condition for π being distinguished with respect to H is that the central character ω_π of π restricts trivially to F^*. Are these conditions sufficient too? A precise conjectural answer is given below (here and elsewhere $\omega_{E/F}$ signifies the quadratic character associated to the extension E/F).

CONJECTURE 1 (Jacquet). *Let π be an irreducible admissible representation of $\mathrm{GL}_n(E)$ such that $\omega_\pi|_{F^*} = 1$ and $\pi^\vee \cong \pi^\sigma$. Then π is H-distinguished if n is odd. If n is even, π is either distinguished or $\omega_{E/F}$-distinguished with respect to $H = \mathrm{GL}_n(F)$.*

When E/F is unramified, and π a supercuspidal, this is proved by D. Prasad [**19**] where the proof eventually boils down to a similar result about stable representations of finite groups of Lie type. More recently the conjecture is settled by Anthony Kable [**16**] whenever π is a square integrable representation. The proof uses the theory of the twisted tensor L-function (aka the Asai L-function) [**6**], and has two main parts. First he proves the following identity which relates the twisted L-function with the Rankin-Selberg L-function.

$$(1) \qquad L(s, \pi \times \pi^\sigma) = L(s, As(\pi))L(s, As(\pi) \otimes \omega_{E/F})$$

Then it is proved that if $L(s, As(\pi))$ has a pole at $s = 0$, then π is H-distinguished. Thus the Jacquet's conjecture follows as it is well known that $L(s, \pi \times \pi^\vee)$ has a pole at $s = 0$ (necessarily simple when π is in the discrete series).

In fact the Asai L-function $L(s, As(\pi))$ having a pole at $s = 0$ characterises π being distinguished for a discrete series π. The other direction is proved in [**3**]. Thus we have the following proposition.

PROPOSITION 2.3. *Let π be a square integrable representation of $\mathrm{GL}_n(E)$. Then π is distinguished with respect to $\mathrm{GL}_n(F)$ if and only if $L(s, As(\pi))$ has a pole at $s = 0$.*

Combining with the identity (1), this proves the following.

COROLLARY 2.4. *Let π be a square integrable representation of $\mathrm{GL}_n(E)$. Then π cannot be both distinguished and $\omega_{E/F}$-distinguished with respect to $\mathrm{GL}_n(F)$.*

Note that the central character considerations make this corollary obvious when n is odd, and hence the nontrivial part is the case of an even n.

As already mentioned in the introduction, conjecturally, H-distinguished representations lie in the image of the base change map from $\mathrm{U}(n, E/F)$ to $\mathrm{GL}_n(E)$. There are two base change maps from $\mathrm{U}(n)$ to $\mathrm{GL}_n(E)$, stable and unstable. The precise conjecture is as follows (see [5]).

CONJECTURE 2 (Flicker-Rallis). *Let π be an irreducible admissible representation of $\mathrm{GL}_n(E)$. Then π is H-distinguished if and only if it is an unstable (resp. stable) base change lift from $\mathrm{U}(n)$ if n is even (resp. odd).*

Note that Conjecture 1 follows from Conjecture 2.

When π is in the discrete series, the images of the two base change maps are believed to form two disjoint sets. Thus corollary 2.4 is in agreement with this conjecture.

Dually we have (see [13, 14]):

CONJECTURE 3 (Jacquet-Ye). *The representation π is distinguished with respect to $\mathrm{U}(n)$ if and only if it is a base change lift from $\mathrm{GL}_n(F)$.*

REMARK 1. *A finite field analogue of the above two conjectures can be found in [8]. Moreover it is proved there that both the pairs have the multiplicity one property (over a finite field).*

Conjecture 2 is known when $n = 2$. In [5], Flicker deduces it from a similar global result. A local proof is provided in [1]. There, closely following a method due to Hiroshi Saito [20], images of the base change lifts are characterised in terms of the local factors of the representation, which characterise distinguishedness by a result of Jeff Hakim [10]. Recently, in a joint work with C.S. Rajan, we have been able to prove Conjecture 2 for $n = 3$, when π is in the discrete series. This is achieved by closely analysing properties of the Asai L-function [4]. Conjecture 3 (globally) is known for $n = 2, 3$ from the work of Jacquet and Ye.

3. The symmetric space $(\mathrm{R}_{E/F}\mathrm{U}(n), \mathrm{U}(n))$

In this case $G = \mathrm{GL}_n(E)$, and $H = \mathrm{U}(n)$. The pair (G, H) is conjectured to be a supercuspidal Gelfand pair, and the conjecture has been proved in several cases (see [11]). In this section we only recall how this can be seen when $n = 2$.

Consider the group $\mathrm{GL}_2^+(F) = \{g \in \mathrm{GL}_2(F) \mid \det g \in N_{E/F}(E^*)\}$. Then we know that $E^*\mathrm{GL}_2^+(F) = E^*\mathrm{U}(2)$. Using this identity one proves the following proposition.

PROPOSITION 3.1. *Let π be an irreducible admissible representation of $\mathrm{GL}_2(E)$ with central character $\omega_\pi = \mu \circ N_{E/F}$ for a character μ of F^*. Then π is $\mathrm{U}(2)$-distinguished if and only if it is either μ-distinguished or $\mu\omega_{E/F}$-distinguished with respect to $\mathrm{GL}_2(F)$.*

Moreover a μ-distinguished (or $\mu\omega_{E/F}$-distinguished) functional is a $\mathrm{U}(2)$-distinguished functional. Conversely a $\mathrm{U}(2)$-distinguished functional uniquely determines a nonzero functional on which $\mathrm{GL}_2(F)$ acts by μ (or $\mu\omega_{E/F}$). Thus for a supercuspidal representation, the space of H-invariant linear forms has dimension less than or equal to one by Corollary 2.4.

On the other hand there are principal series representations of $\mathrm{GL}_2(E)$ which may admit a two dimensional space of H-invariant linear forms. For instance, take $\pi = Ps(\chi, \chi^{-1})$ where χ is a character of E^* such that $\chi = \chi^\sigma$. Then π is both distinguished and $\omega_{E/F}$-distinguished with respect to $\mathrm{GL}_2(F)$ [**10**], and the corresponding functionals are $\mathrm{U}(2)$-distinguished.

4. The symmetric space $(\mathrm{R}_{E/F}\mathrm{SL}(n), \mathrm{SL}(n))$

Representation theory of $\mathrm{SL}_n(E)$ can be understood in terms of the representation theory of $\mathrm{GL}_n(E)$ [**7**]. An irreducible admissible representation π of $\mathrm{SL}_n(E)$ comes in the restriction of an irreducible admissible representation π' of $\mathrm{GL}_n(E)$. It is a theorem of M. Tadic that the restriction to $\mathrm{SL}_n(E)$ of an irreducible admissible representation of $\mathrm{GL}_n(E)$ is multiplicity free [**22**]. If two representations of $\mathrm{GL}_n(E)$ restrict to the same representation of $\mathrm{SL}_n(E)$, then they are equivalent up to a twist by a character of E^*. Constituents in the direct sum decomposition of a representation of $\mathrm{GL}_n(E)$ form an L-packet of $\mathrm{SL}_n(E)$.

Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$ which is distinguished with respect to $\mathrm{SL}_n(F)$. Let π' be a representation of $\mathrm{GL}_n(E)$ such that π occurs in the restriction of π'. Thus π' is distinguished with respect to $\mathrm{SL}_n(F)$. Consequently its central character $\omega_{\pi'}$ is trivial on the nth roots of unity in F^*. It follows that $\omega_{\pi'}|_{F^*} = \eta^n$ for a character η of F^*. Thus we can assume, after twisting by a character of E^* if necessary, that $\omega_{\pi'}$ restricts trivially to F^*. Now the space $\mathrm{Hom}_{\mathrm{SL}_n(F)}(\pi', 1)$ has the structure of a $\mathrm{GL}_n(F)$-module (with the obvious action) on which $F^*\mathrm{SL}_n(F)$ acts trivially. Thus it is a direct sum of characters of F^* as a $\mathrm{GL}_n(F)$ module. Equivalently π' is χ-distinguished with respect to $\mathrm{GL}_n(F)$ for a character χ of F^*. Hence there is no loss of generality in assuming that π' is distinguished with respect to $\mathrm{GL}_n(F)$. Also $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}(\pi', 1)$ equals the number of characters χ of F^* for

which π' is χ-distinguished with respect to $\mathrm{SL}_n(F)$. This is so since $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi', \chi) \leq 1$.

Consider the subgroup of $\mathrm{GL}_n(E)$ defined by

$$\mathrm{GL}_n(E)^+ = \{g \in \mathrm{GL}_n(E) \mid \det g \in F^* E^{*n}\}.$$

Also let us fix a nontrivial additive character ψ of E which has trivial restriction to F. Assume that π' is tempered (this is because we need to use Proposition 4.1).

In the restriction of π' to $\mathrm{GL}_n(E)^+$, exactly one representation is ψ-generic, say π^+. We contend that the $\mathrm{SL}_n(F)$-invariant linear forms on π' are nontrivial only on the space of π^+ (among all the irreducible constituents of $\pi'|_{\mathrm{GL}_n(E)^+}$). For the above claim we require the following proposition (Corollary 1.2, [**3**]).

PROPOSITION 4.1. *Let π' be a tempered representation of $\mathrm{GL}_n(E)$ that is $\mathrm{GL}_n(F)$-distinguished. Let ψ be a nontrivial additive character of E that has trivial restriction to F. Then the (nontrivial) $\mathrm{GL}_n(F)$-invariant linear form on π' can be realized on the the ψ-Whittaker model of π' by $\ell(W) = \int_{N_n(F) \backslash P_n(F)} W(p) dp$. Here $P_n(F)$ is the mirabolic subgroup of $\mathrm{GL}_n(F)$, and $N_n(F)$ is the unipotent radical of the Borel subgroup of $\mathrm{GL}_n(F)$.*

Further we have,

PROPOSITION 4.2. *All the constituents of the restriction of a representation of $\mathrm{GL}_n(E)^+$ to $\mathrm{SL}_n(E)$ admit the same number of linearly independent $\mathrm{SL}_n(F)$-invariant functionals.*

PROOF. Indeed the constituents are conjugates of one another under the inner conjugation action of $\mathrm{GL}_n(F)$ on $\mathrm{SL}_n(F)$ (as $\mathrm{GL}_n(F)\mathrm{SL}_n(E)E^* = \mathrm{GL}_n(E)^+$). $\square$

Thus $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}(\pi, 1)$ is zero if π does not occur in the restriction of π^+ to $\mathrm{SL}_n(E)$. Moreover this dimension is the same nonzero number for all π that appear in the restriction of π^+ to $\mathrm{SL}_n(E)$. From the preceding discussion it is clear that this dimension is

$$q(\pi) = q(\pi') = \frac{|X_{\pi'}|}{|Z_{\pi'}|/|Y_{\pi'}|}$$

where

$$X_{\pi'} = \{\chi \in \hat{F}^* \mid \pi' \text{ is } \chi - \text{distinguished}\},$$

$$Y_{\pi'} = \{\mu \in \hat{E}^* \mid \pi' \otimes \mu \cong \pi', \mu|_{F^*} = 1\},$$

$$Z_{\pi'} = \{\mu \in \hat{E}^* \mid \pi' \otimes \mu \cong \pi'\},$$

since $|Z_{\pi'}|$ is the cardinality of the L-packet of π, and $|Y_{\pi'}|$ is the number of constituents in the restriction of π' to $\mathrm{GL}_n(E)^+$ (by the result of Tadic quoted in the beginning of this section). Note that the cardinalities of the sets above do not depend on the choice of π'.

Notice that $q(\pi)$ is a non-negative integer. This is so since the group $Z_{\pi'}/Y_{\pi'}$ acts freely on $X_{\pi'}$ (assume $X_{\pi'}$ is non-empty, $q(\pi) = 0$ otherwise), and hence $q(\pi)$ is the number of orbits.

All this information can be clubbed together as follows. Define a pairing between $Z_{\pi'}$ and the L-packet of π by

$$< \mu, \pi >= \mu(a)$$

where π is ψ_a-generic. (ψ_a is the additive character of E given by $\psi_a(x) = \psi(ax)$). Then

$$\frac{1}{|Y_{\pi'}|} \sum_{\mu \in Y_{\pi'}} < \mu, \pi >$$

is one if π comes in the restriction of π^+, and zero otherwise. Thus we have (see Theorem 1.4, [**2**]),

THEOREM 4.3. *Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$ that comes in the restriction of a tempered representation of $\mathrm{GL}_n(E)$. Then,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}(\pi, 1) = \frac{q(\pi)}{|Y_\pi|} \sum_{\mu \in Y_\pi} < \mu, \pi > .$$

REMARK 2. *There is another way of looking at the number $q(\pi)$. Define two equivalence relations on the set of all twists of π' that are distinguished with respect to $\mathrm{GL}_n(F)$ as follows:*

$$\pi_1' \sim_w \pi_2' \Longleftrightarrow \pi_1' \cong \pi_2' \otimes \mu, \mu \in \hat{E}^*$$

$$\pi_1' \sim_s \pi_2' \Longleftrightarrow \pi_1' \cong \pi_2' \otimes \mu, \mu \in \hat{E}^*, \mu|_{F^*} = 1.$$

Then $q(\pi)$ is the number of strong equivalence classes in a weak equivalence class of π'.

5. Proof of Theorem 1.1

Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$ which is distinguished with respect to $\mathrm{SL}_n(F)$. Let π' be a representation of $\mathrm{GL}_n(E)$ such that π comes in the restriction of π' to $\mathrm{SL}_n(F)$.

We need to introduce the group

$$Y_{\pi'}' = \{\mu \in \hat{E}^* \mid \pi' \otimes \mu \cong \pi', \mu|_{E^1} = 1\}$$

where E^1 denotes the norm one elements of E^*. We claim that the map

$$\chi \mapsto \chi \circ N_{E/F}$$

establishes an injection from $X_{\pi'}$ to $Y'_{\pi'}$.

This is of course vacuously true if $X_{\pi'}$ is empty. Otherwise, as before, we can assume that π' is distinguished with respect to $\mathrm{GL}_n(F)$. Then this is a well defined map follows from an application of Theorem 2.2. Injectivity is a consequence of the fact that π' cannot be both distinguished and $\omega_{E/F}$-distinguished when n is an odd integer. (This can be seen by considering central characters. This fact is not true in general when n is even. See also Corollary 2.4). If we assume the truth of Jacquet's conjecture, this map is in fact a bijection, but we do not need that.

Thus

$$q(\pi) \leq \frac{|Y_{\pi'}||Y'_{\pi'}|}{|Z_{\pi'}|}.$$

If n is an odd integer, note that $Y_{\pi'} \bigcap Y'_{\pi'} = \{1\}$. Therefore the quantity on the right side of the above inequality is $\frac{|Y_{\pi'}Y'_{\pi'}|}{|Z_{\pi'}|}$. This forces $q(\pi)$ to be zero or one, since we know that it is a non-negative integer.

Now let

$$\pi'\big|_{\mathrm{GL}_n(E)^+} = \oplus_i \pi_i^+$$

be the direct sum decomposition of π' restricted to $\mathrm{GL}_n(E)^+$, where π_i^+ are inequivalent. Let a_i denote the common dimension (by Proposition 4.2) of the space of $\mathrm{SL}_n(F)$-invariant forms on constituents of $\pi_i^+\big|_{\mathrm{SL}_n(E)}$. We conclude that

$$\sum_i a_i = q(\pi).$$

Thus a_i cannot be more than one, proving the theorem.

When n is an even integer, $Y_{\pi'} \bigcap Y'_{\pi'} \neq \{1\}$ in general, and this results in higher multiplicities. We restrict ourselves to just one example.

To this end, consider a quadratic extension K of F different from E, and let η be a character of K^* with trivial restriction to F^*. Let L denote the compositum of E and K. Also assume that K/F is such that we can choose η so that $\eta^8 \neq 1$. Let π_0 be the representation of $\mathrm{GL}_2(F)$ obtained by automorphically inducing η, and let π' be the base change lift of π_0 to $\mathrm{GL}_2(E)$.

Our assumption on η guarantees that π' is a supercuspidal representation, and that $|Z_{\pi'}| = 2$ (see [17, 21]). This forces $|Y'_{\pi'}| = 2$ (as either μ or $\mu \circ N_{E/F}$ has to be in $Y'_{\pi'}$ if $\mu \in Z_{\pi'}$). Since $n = 2$, $Y_{\pi'} = Y'_{\pi'}$. Thus in this case $q(\pi) = 2$.

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