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# On the conductor of certain local $L$-functions

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**Abstract.** The conductor formula of Bushnell, Henniart and Kutzko [BHK98] computes the conductor of a pair of supercuspidal representations of general linear groups over a $p$-adic field. This is the conductor of the tensor product lift. In this paper, we give an explicit formula for the conductors of the symmetric and exterior square lifts, under the assumption that $p \neq 2$.

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## 1. Introduction

Let $F$ be a $p$-adic field. Let $\mathfrak{o}_F$ be the ring of integers of $F$ and let $\mathfrak{p}_F$ denote the unique maximal ideal of $\mathfrak{o}_F$. Let $\psi$ be a continuous nontrivial additive character of $F$. Let $n(\psi)$ be the largest integer $n$ such that $\psi$ is trivial on $\mathfrak{p}^{-n}$. If $\pi_i$'s are irreducible admissible representations of $GL_{n_i}(F)$ for $i = 1, 2$ respectively, then Jacquet, Piatetski-Shapiro and Shalika [JPSS83] attach an $L$-function $L(s, \pi_1 \times \pi_2)$ and an epsilon factor $\epsilon(s, \pi_1 \times \pi_2, \psi)$ to the given data, where $s \in \mathbb{C}$. The epsilon factor $\epsilon(s, \pi_1 \times \pi_2, \psi)$ satisfies the following relation

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = q^{-f(\pi_1 \times \pi_2, \psi)s} \epsilon(0, \pi_1 \times \pi_2, \psi).$$

The integer $f(\pi_1 \times \pi_2, \psi)$ for an unramified character $\psi$ of $F$, i.e., when $n(\psi) = 0$, is called the conductor for the pair $(\pi_1, \pi_2)$ and is denoted by $f(\pi_1 \times \pi_2)$. The conductor formula of Bushnell, Henniart and Kutzko [BHK98] gives an explicit formula for $f(\pi_1 \times \pi_2)$, when $\pi_1$ and $\pi_2$ are supercuspidal representations of general linear groups. The ingredients in this explicit conductor formula are in terms of the data attached to $\pi_1$ and $\pi_2$ via the theory of compact induction of Bushnell and Kutzko [BK93]. It follows from the explicit conductor formula that $f(\pi_1 \times \pi_2)$ is a non-negative integer. We refer to Section 5 for more details regarding the formula of Bushnell, Henniart and Kutzko.

Let $W'_F$ be the Weil-Deligne group of $F$ and let $\rho_{\pi_i} : W'_F \to GL_{n_i}(\mathbb{C})$ be the Langlands parameters of $\pi_i$, where $i = 1, 2$. Then, under the local Langlands correspondence, the conductor for pairs $f(\pi_1 \times \pi_2)$ equals the Artin conductor of $\rho_{\pi_1} \otimes \rho_{\pi_2}$. Thus, $f(\pi_1 \times \pi_2)$ is nothing but the conductor of the tensor product lift $\pi_1 \boxtimes \pi_2$, which is an irreducible admissible representation of $GL_{n_1 n_2}(F)$.

A natural question is to ask for explicit conductor formulas in the spirit of the formula of Bushnell, Henniart and Kutzko [BHK98] for other functorial lifts. In particular, we ask this for the following three functorial lifts: (i) the symmetric square lift from $GL_n(F)$ to $GL_{\frac{n(n+1)}{2}}(F)$, (ii) the exterior square lift from $GL_n(F)$ to $GL_{\frac{n(n-1)}{2}}(F)$, and (iii) the Asai lift from $GL_n(F)$ to $GL_{n^2}(F_0)$ when $F/F_0$ is a quadratic extension.

It is conceivable that the conductor formulas in these three cases should follow from the formula of Bushnell, Henniart and Kutzko. This is because these functorial lifts are intimately connected to the tensor product lift. Indeed, we have:

$$\rho_\pi \otimes \rho_\pi \cong \mathrm{Sym}^2 \rho_\pi \oplus \wedge^2 \rho_\pi$$

and

$$\rho_\pi \otimes \rho_\pi^\sigma \cong r(\rho_\pi)|_{W'_F},$$

where $\sigma$ is the nontrivial automorphism of $F/F_0$ and $r$ denotes tensor induction from $W'_F$ to $W'_{F_0}$. Since the Artin conductor is additive, the explicit conductor formula for the tensor product gives rise to conductor formulas for the sum of conductors of the symmetric and exterior square lifts and also the sum of conductors of the Asai lift and the twisted Asai lift. Here, the twisted Asai lift is the Asai lift tensored with the quadratic character associated to $F/F_0$.

In other words, we will have explicit conductor formulas for these three functorial lifts the moment we can calculate the difference of the relevant conductors. It turns out that this is not very hard if we assume that the residue characteristic is odd. Under our assumption, the necessary calculations can be carried out on the Galois side of the local Langlands correspondence.

The Asai case has been done in [AM15]. The following is [AM15, Theorem 1.4].

**Theorem 1.1.** *Let $F/F_0$ be a quadratic extension of $p$-adic fields. If it is ramified, assume also that $p \neq 2$. Let $\kappa$ be a character of $F^\times$ which restricts to the quadratic character $\omega_{F/F_0}$ of $F_0^\times$ associated to $F/F_0$. Let $\pi$ be an irreducible supercuspidal representation of $GL_n(F)$ and let $r(\pi)$ be its Asai lift to $GL_{n^2}(F_0)$. Then,*

$$f(r(\pi)) + \deg L(s, \pi, r) = f(r(\pi) \otimes \omega_{F/F_0}) + \deg L(s, \pi \otimes \kappa, r).$$

*Here, $L(s, \pi, r)$ is the Asai $L$-function of $\pi$.*

For an irreducible admissible representation $\pi$ of $GL_n(F)$, let $\mathrm{Sym}^2(\pi)$ (resp. $\wedge^2(\pi)$) be its symmetric square (resp. exterior square) lift to $GL_{\frac{n(n+1)}{2}}(F)$ (resp. $GL_{\frac{n(n-1)}{2}}(F)$). Let $L(s, \pi, \mathrm{Sym}^2)$ (resp. $L(s, \pi, \wedge^2)$) denote the symmetric square (resp. exterior square) $L$-function. In this paper, we prove:

**Theorem 1.2.** *Let $\pi$ be an irreducible supercuspidal representation of $GL_n(F)$, where $F$ is a $p$-adic field with $p \neq 2$. Then,*

$$\begin{aligned} &f(\mathrm{Sym}^2(\pi)) + \deg L(s, \pi, \mathrm{Sym}^2) \\ &\quad = f(\wedge^2(\pi)) + \deg L(s, \pi, \wedge^2) + f(\pi) + \deg L(s, \pi). \end{aligned}$$

Theorem 1.2, together with the conductor formula of Bushnell, Henniart and Kutzko (cf. Theorem 5.1), gives explicit conductor formulas for these functorial lifts, thanks to the identity

$$f(\pi \times \pi) = f(\mathrm{Sym}^2(\pi)) + f(\wedge^2(\pi)). \tag{1}$$

and

$$f(\pi \times \pi^\sigma) = \begin{cases} f(r(\pi)) & \text{if } F/F_0 \text{ is unramified} \\ f(r(\pi)) + f(r(\pi) \otimes \omega_{F/F_0}) - n^2 & \text{if } F/F_0 \text{ is ramified,} \end{cases} \tag{2}$$

where in the second case, we have assumed that $p \neq 2$.

We leave the statement of these explicit conductor formulas to Section 6 (cf. Theorem 6.1). We hope that these explicit conductor formulas will have applications in number theory. In this regard, we refer to the recent work of White [Whi15] where an easy upper bound for the conductor of the symmetric square $L$-function plays a crucial role in explicating an error term that comes in a beyond endoscopic problem [Whi15, Proposition 4.4].

**Remark 1.3.** The degree of the $L$-functions appearing in Theorem 1.1 and Theorem 1.2 can also be made explicit in terms of the inducing data of the representation $\pi$. This is either $n/e$ or $n/2e$, where $e$ is the $\mathfrak{o}_F$-period of the principal $\mathfrak{o}_F$-order in $M_n(F)$ in the simple stratum defining a maximal simple type occurring in $\pi$ [AM15, Theorems 1.1,1.2 and 1.3].

**Remark 1.4.** Theorem 1.2, interpreted on the Galois side of the local Langlands correspondence, is a particular case of a slightly more general statement about Adams operators, and the statement is neater when presented in terms of the Swan conductor instead of the Artin conductor. We do this in Section 4 (cf. Theorem 4.1) on a suggestion by Guy Henniart.

## 2. Preliminaries

### 2.1 Artin conductor

We first recall the definition of the Artin conductor [Ser79, Chapter VI]. Let $F$ be a $p$-adic field with ring of integers $\mathfrak{o}_F$ with its unique maximal ideal $\mathfrak{p}_F$ and residue field $k_F = \mathfrak{o}_F/\mathfrak{p}_F$. Let $E$ be a finite Galois extension of $F$ with Galois group $G = \mathrm{Gal}(E/F)$. Let $\mathfrak{o}_E$ and $k_E$ denote the ring of integers and the residue field of $E$ respectively. The integer $e = e(E/F)$ (resp. $f = f(E/F)$) denotes the ramification index of $E/F$ (resp. the degree of $k_E/k_F$). Now $\mathfrak{o}_E$ is generated by some $x \in \mathfrak{o}_E$ as an $\mathfrak{o}_F$-algebra; $\mathfrak{o}_E = \mathfrak{o}_F[x]$. Define the order function on $G$ by

$$i_G(s) = \upsilon_E(s(x) - x),$$

where $\upsilon_E$ is some normalized valuation defined on $E$. It is easy to see that

$$i_G(tst^{-1}) = i_G(s)$$

for all $s, t \in G$. Now define an integer valued function on $G$ by

$$a_G(s) = \begin{cases} -f \cdot i_G(s), & s \neq 1 \\ f \sum_{s \neq 1} i_G(s), & s = 1. \end{cases} \tag{3}$$

Clearly, $a_G$ is a class function on $G$. We have [Ser79, Chapter VI, Theorem 1]:

**Theorem 2.1.** *The function $a_G$ is the character of a representation of $G$.*

If $\rho$ is a (virtual) representation of $G$, its Artin conductor is defined as the multiplicity of $\rho$ in the Artin representation;

$$f(\rho) = (\chi_\rho, a_G)_G = \frac{1}{|G|} \sum_{s \in G} \chi_\rho(s) \overline{a_G(s)},$$

where $\chi_\rho$ is the (virtual) character of $\rho$. Note that the Artin conductor is non-negative if $\rho$ is a representation.

The ramification groups $G_i \lhd G$ are defined by

$$G_i = \{g \in G \mid i_G(g) \geq i + 1\}.$$

We have [Ser79, Chapter VI, Propositions 1 and 2]:

**Proposition 2.2.** *Let $F_{\mathrm{nr}} \subset E$ be the maximal unramified extension of $F$ in $E$.*

$$a_G(g) = \begin{cases} f \cdot a_{G_0}(g) & \text{if } g \in G_0 \\ 0 & \text{if } g \notin G_0. \end{cases}$$

*Moreover,*

$$a_{G_0} = \sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \mathrm{Ind}_{G_i}^{G_0} u_i,$$

*where $u_i$ is the augmentation representation of $G_i$.*

The next proposition relates $a_G$ restricted to a subgroup $H$ to $a_H$ [Ser79, Chapter VI, Proposition 4].

**Proposition 2.3.** *Let $H < G$ and let $F'$ be its fixed field. Let $\delta_{F'/F}$ denote the discriminant of $F'/F$. Then,*

$$a_G|_H = v_F(\delta_{F'/F}) \cdot r_H + f(F'/F) \cdot a_H,$$

*where $r_H$ is the regular representation of $H$.*

We will also need the following result due to Henniart and Tunnell [Ser79, Chapter VI, Exercise 2]. For this, let $\phi_{E/F}$ be the function defined by

$$1 + \phi_{E/F}(m) = \frac{1}{|G_0|}[|G_0| + |G_1| + \ldots + |G_m|].$$

**Proposition 2.4.** *For an irreducible representation $\rho$ of $G$, let $c_\rho$ be the largest integer $c$ such that $\rho$ is not trivial on $G_c$. Then,*

$$(\rho(g), a_G(g))_G = \dim \rho \cdot (1 + \phi_{E/F}(c_\rho)).$$

**Corollary 2.5.** *Let $\tau$ be a (virtual) representation of $G_0$ whose constituents are not trivial on $G_1$. Then,*

$$(\tau(g), a_{G_0}(g))_{G_0} = \left(1 - \frac{|G_1|}{|G_0|}\right) \cdot \dim \tau + \frac{|G_1|}{|G_0|} \cdot (\tau(g), a_{G_1}(g))_{G_1}.$$

*Proof of Corollary 2.5.* Suffices to do this when $\tau$ is an irreducible representation of $G_0$. Now the assertion is immediate if further $\tau$ restricts irreducibly to $G_1$. Indeed, by Proposition 2.4,

$$\begin{aligned}\frac{(\tau(g), a_{G_0}(g))_{G_0}}{\dim\tau} &= \frac{1}{|G_0|}[|G_0| + |G_1| + \ldots + |G_{c_\tau}|] \\ &= \frac{|G_1|}{|G_0|}\cdot\frac{1}{|G_1|}[|G_1| + |G_1| + |G_2| + \ldots + |G_{c_\tau}|] \\ &\quad + \left(1 - \frac{|G_1|}{|G_0|}\right) \\ &= \left(1 - \frac{|G_1|}{|G_0|}\right) + \frac{|G_1|}{|G_0|}\cdot\frac{(\tau(g), a_{G_1}(g))_{G_1}}{\dim\tau}.\end{aligned}$$

In the general case, note that the restriction of $\tau$ to $G_1$ is a direct sum of conjugates of one irreducible representation, since $G_1$ is normal in $G_0$. Thus all the constituents have the same dimension. Also, one of them is trivial on $G_i$ if and only if each one is, and hence if and only if $\tau$ is trivial on $G_i$.

**Remark 2.6.** Note that if all the constituents of $\tau$ are trivial on $G_1$ and nontrivial on $G_0$, then $(\tau(g), a_{G_0}(g))_{G_0} = \dim\tau$.

Combining Corollary 2.5 with Proposition 2.3, we get:

**Proposition 2.7.** *Let $\tau$ be a (virtual) representation of $G$ whose constituents are not trivial on $G_1$. Then,*

$$(\tau(g), a_G(g))_{G_0} = \frac{|G_1|}{|G_0|}\cdot(\tau(g), a_G(g))_{G_1}.$$

*Proof of Proposition 2.7.* Since

$$a_G|_{G_0} = f\cdot a_{G_0},$$

we have

$$(\tau(g), a_G(g))_{G_0} = f\cdot(\tau(g), a_{G_0}(g))_{G_0}.$$

Similarly,

$$a_G|_{G_1} = f\left(\frac{|G_0|}{|G_1|} - 1\right)\cdot r_{G_1} + f\cdot a_{G_1},$$

and therefore

$$(\tau(g), a_G(g))_{G_1} = f\left(\frac{|G_0|}{|G_1|} - 1\right)\cdot\dim\tau + f\cdot(\tau(g), a_{G_1}(g))_{G_1},$$

and Proposition 2.7 follows.

## 2.2 Artin $L$-function

As before, let $W'_F$ denote the Weil-Deligne group of $F$. For a reductive algebraic group $G$ defined over $F$, let ${}^LG$ be its Langlands dual. Given a Langlands parameter $\rho : W'_F \to {}^LG$ and a finite dimensional representation $r : {}^LG \to GL(V)$, we have its Artin $L$-function $L(s, \rho, r)$ defined as follows. If $N$ is the nilpotent endomorphism of $V$ associated to $r \circ \rho$, then

$$L(s, \rho, r) = \frac{1}{\det\left(1 - (r \circ \rho)(\mathrm{Frob})|_{(\mathrm{Ker}\ N)^I}\, q^{-s}\right)}$$

where Frob is the geometric Frobenius and $I$ is the inertia subgroup of the Weil group of $F$. Thus, $L(s, \rho, r) = P(q^{-s})^{-1}$ for some polynomial $P(X)$ with $P(0) = 1$, and by the degree of $L(s, \rho, r)$ we mean the degree of $P(X)$. If $\pi = \pi(\rho)$ denotes the $L$-packet of irreducible admissible representations of $G(F)$ corresponding to $\rho$ under the conjectural Langlands correspondence, then its Langlands $L$-function, denoted by $L(s, \pi, r)$, is expected to coincide with $L(s, \rho, r)$.

For an irreducible supercuspidal representation $\pi$ of $GL_n(F)$, let $[\mathfrak{A}, m, 0, \beta]$ be the simple stratum defining a maximal simple type occurring in the irreducible supercuspidal representation $\pi$ [BK93]. Here, $\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in $M_n(F)$, $m \geq 0$ is an integer called the level of $\pi$, and $\beta \in M_n(F)$ is such that $F[\beta]$ is a field with $F[\beta]^\times$ normalizing $\mathfrak{A}$. Let $e = e(\mathfrak{A}|\mathfrak{o}_F)$ be the $\mathfrak{o}_F$-period of $\mathfrak{A}$; this quantity in fact equals the ramification index $e(F[\beta]/F)$ of $F[\beta]/F$. Then, $e$ divides $n$, and by [BK93, Lemma 6.2.5],

$$\deg L(s, \pi_1^\vee \times \pi_2) = \begin{cases} \frac{n}{e} & \text{if } \pi_2 \cong \pi_1 \otimes \mu \text{ for an unramified character } \mu \text{ of } F^\times \\ 0 & \text{otherwise.} \end{cases}$$

Analogous to the above result, the degree of $L(s, \pi, r)$, when $r = \mathrm{Sym}^2, \wedge^2$ or the Asai representation (tensor induction from $W'_F$ to $W'_{F_0}$), is determined in [AM15]. This degree is always either 0 or $n/e$ or $n/2e$. Note that when $r : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is the identity map, $L(s, \pi, r)$ is the standard $L$-function, denoted by $L(s, \pi)$, which has degree 0.

**Remark 2.8.** In many cases, including $r = \otimes, \mathrm{Sym}^2, \wedge^2$ or the Asai representation, an associated $L$-function attached to $\pi$ can also be defined via the Rankin-Selberg method or via the Langland-Shahidi method. All the three definitions are expected to coincide and they are known to coincide in several cases of our interest [Sha84,Sha90,AR05,Hen10,Mat11,KR12]. However, we do not need to know this fact in the sequel.

## 3. Proof of Theorem 1.2

Let $\pi$ be an irreducible supercuspidal representation of $GL_n(F)$. Let

$$\rho = \rho_\pi : W_F' \to GL_n(\mathbb{C})$$

be its Langlands parameter. Let $E$ be the Galois extension of $F$ cut out by the kernel of $\rho_\pi$. Let $G = \mathrm{Gal}(E/F)$ be its Galois group. Let $a_G$ denote the Artin character of $G$.

For a (virtual) representation $\rho$ of $G$, let $f(\rho)$ denote its Artin conductor. Let $L(s,\rho)$, $L(s,\rho,\mathrm{Sym}^2)$ and $L(s,\rho,\wedge^2)$ denote the standard, symmetric square, and the exterior square $L$-functions of $\rho$. Recall that the degree of $L(s,\rho,r)$ is nothing but the number of unramified characters occurring in the direct sum decomposition of $r \circ \rho$, for a representation $r$ of $GL_n(\mathbb{C})$.

In order to prove Theorem 1.2, we prove:

**Proposition 3.1.** *Let $E/F$ be a finite Galois extension of $p$-adic fields with $p \neq 2$. Let $\rho$ be an irreducible representation of $G = \mathrm{Gal}(E/F)$. Then,*

$$\begin{aligned} &f(\mathrm{Sym}^2(\rho)) + \deg L(s,\rho,\mathrm{Sym}^2) \\ &\quad = f(\wedge^2(\rho)) + \deg L(s,\rho,\wedge^2) + f(\rho) + \deg L(s,\rho). \end{aligned}$$

We assume that $\rho|_{G_1} \neq 1$. The case $\rho|_{G_1} = 1$ is easier and will be handled at the end.

In the following, we continue to denote the character of the representation $\rho$ by $\rho$ or $\rho(g)$, where $g \in G$. Note that in the latter notation,

$$(\mathrm{Sym}^2\rho)(g) - (\wedge^2\rho)(g) = \rho(g^2).$$

Thus, for the proof of Proposition 3.1, we claim:

$$(\rho(g^2), a_G(g))_G + \nu = (\rho(g), a_G(g))_G, \tag{4}$$

where $\nu$ is the number (sum of coefficients) of unramified characters in the (virtual) representation $\rho(g^2)$.

First we observe an elementary lemma.

**Lemma 3.2.** *If $H$ is a subgroup of $G$ with odd cardinality, then*

$$a_G(h^2) = a_G(h)$$

*for all $h \in H$.*

*Proof of Lemma 3.2.* By the definition of $a_G$, it follows that $a_G(g^l) \leq a_G(g)$ for any $g \in G$ and for any $l \in \mathbb{N}$, and in particular $a_G(g^2) \leq a_G(g)$. On the other hand, since $H$ is an odd group, $h = (h^2)^l$ for some positive integer $l$. Thus, $a_G(h) = a_G((h^2)^l) \leq a_G(h^2)$ as well.

We will apply Lemma 3.2 in the following when $H$ is the wild inertia group $G_1$ which is an odd group by our assumption that $p \neq 2$. We have:

$$\begin{aligned}(\rho(g^2), a_G(g))_{G_1} &= \frac{1}{|G_1|} \sum_{g \in G_1} \rho(g^2)\overline{a_G(g)} \\ &= \frac{1}{|G_1|} \sum_{g \in G_1} \rho(g^2)\overline{a_G(g^2)} \\ &= \frac{1}{|G_1|} \sum_{g \in G_1} \rho(g)\overline{a_G(g)} \\ &= (\rho(g), a_G(g))_{G_1}\end{aligned}$$

where the second equality is by Lemma 3.2 and the third equality is because $g \mapsto g^2$ is a bijection on any odd group.

Now we apply Proposition 2.7 to the (virtual) representations $\rho(g^2)$ and $\rho(g)$. Note that we are in the case $\rho|_{G_1} \neq 1$, and therefore Proposition 2.7 gives

$$(\rho(g), a_G(g))_{G_0} = \frac{|G_1|}{|G_0|} \cdot (\rho(g), a_G(g))_{G_1}.$$

In order to apply Proposition 2.7 to $\rho(g^2)$, we first write

$$\rho(g^2) \cong \rho'(g) \oplus \rho''(g)$$

as a sum of two virtual representations where $\rho'(g)$ has no irreducible constituent trivial on $G_1$ and $\rho''(g)$ has all its irreducible constituents trivial on $G_1$. Applying Proposition 2.7 to $\rho'(g)$, we get:

$$(\rho'(g), a_G(g))_{G_0} = \frac{|G_1|}{|G_0|} \cdot (\rho'(g), a_G(g))_{G_1}.$$

On the other hand, we have:

$$(\rho''(g), a_G(g))_{G_0} = f \cdot (\rho''(g), a_{G_0}(g))_{G_0} = f \cdot \nu_1,$$

where $\nu_1$ is the number of tamely ramified characters appearing in $\rho(g^2)$.

Now we compute $(\rho(g^2), a_G(g))_{G_0}$. We have:

$$\begin{aligned}(\rho(g^2), a_G(g))_{G_0} &= (\rho'(g), a_G(g))_{G_0} + (\rho''(g), a_G(g))_{G_0} \\ &= \frac{|G_1|}{|G_0|} \cdot (\rho'(g), a_G(g))_{G_1} + f \cdot \nu_1 \\ &= \frac{|G_1|}{|G_0|} \cdot [(\rho(g^2), a_G(g))_{G_1} - (\rho''(g), a_G(g))_{G_1}] + f \cdot \nu_1 \\ &= \frac{|G_1|}{|G_0|} \cdot [(\rho(g), a_G(g))_{G_1} - (\rho''(g), a_G(g))_{G_1}] + f \cdot \nu_1,\end{aligned}$$

where we have made use of our earlier observation that $(\rho(g^2), a_G(g))_{G_1} = (\rho(g), a_G(g))_{G_1}$. We also have:

$$(\rho''(g), a_G(g))_{G_1} = f\left(\frac{|G_0|}{|G_1|} - 1\right)\dim \rho''.$$

Thus, we get:

$$(\rho(g^2), a_G(g))_{G_0} = (\rho(g), a_G(g))_{G_0} + f\left(\frac{|G_1|}{|G_0|} - 1\right)\dim \rho'' + f \cdot \nu_1.$$

We now claim that

$$\dim \rho'' = \nu_0 + \nu_1 = 0,$$

where $\nu_0$ is the number of times the trivial character of $G_0$ appears in $\rho(g^2)|_{G_0}$.

Granting the claim, we see that

$$(\rho(g^2), a_G(g))_{G_0} + f \cdot \nu_0 = (\rho(g), a_G(g))_{G_0},$$

or in other words

$$(\rho(g^2), a_{G_0}(g))_{G_0} + \nu_0 = (\rho(g), a_{G_0}(g))_{G_0}. \tag{5}$$

Observe that the identity (4) is evident from the identity (5) (cf. Proposition 2.4). This finishes the proof of Proposition 3.1 and hence Theorem 1.2, modulo the above claim. The claim follows from the following lemma by taking $G = G_0$, $H = G_1$ and the representation $\tau$ to be $\rho|_{G_0}$. Since $G/H$ is cyclic, all the irreducible characters of $G$ which are trivial on $H$ are one dimensional.

**Lemma 3.3.** *Let $G$ be a finite group and $H$ a normal subgroup of $G$ of odd order. Let $\tau$ be a representation of $G$ whose irreducible components do not contain the trivial representation of $H$. Write*

$$\tau(g^2) = \oplus_\chi m(\chi)\chi,$$

*where $m(\chi) \in \mathbb{Z}$ and the $\chi$'s are irreducible characters of $G$. Then,*

$$\sum_{\chi|_H=1} m(\chi) \dim \chi = 0.$$

*Proof of Lemma 3.3.* Without loss of generality, we may assume that the representation $\tau$ itself is irreducible. We need to show that the virtual representation $\tau(g^2)$ restricted to $H$ does not contain 1. Now,

$$(\tau(h^2), 1)_H = \frac{1}{|H|} \cdot \sum_{h\in H} \tau(h^2) = \frac{1}{|H|} \cdot \sum_{h\in H} \tau(h) = (\tau(h), 1)_H$$

since $h \mapsto h^2$ is a bijection on $H$. The latter term is 0 by the assumption on $\tau$.

Finally, suppose $\rho|_{G_1} = 1$. If $\rho$ is in fact trivial on $G_0$, Proposition 3.1 is obvious as then $\rho$ is a one dimensional character with $f(\rho) = 0$. If $\rho$ is nontrivial on $G_0$ but trivial on $G_1$, then $\rho$ restricted to $G_0$ is a sum of nontrivial one dimensional characters of $G_0$ each of which is trivial on $G_1$. The identity (5) is obvious in this case too.

**Remark 3.4.** That $G_1$ is an odd group has played a crucial role in the proof of Proposition 3.1. It is not clear what possible modification one has to make to Theorem 1.2 for it to accommodate the even residue characteristic case. We illustrate some of the difficulties involved in the even case in Section 7. Our proof here making use of the oddness of $G_1$ is in the spirit of [Pra99, Proposition 4].

## 4. A more general situation

This section is suggested to us by Guy Henniart.

Recall that in Section 3, the key point in proving Proposition 3.1 was the identity (4), which in turn followed from the identity (5):

$$(\rho(g^2), a_{G_0}(g))_{G_0} + \nu_0 = (\rho(g), a_{G_0}(g))_{G_0},$$

provided $\rho$ restricted to $G_0$ is nontrivial. We were interested in the Adams operator

$$\Psi^2(\rho)(g) = \rho(g^2) = \mathrm{Sym}^2\rho \ominus \wedge^2\rho,$$

because we were finally interested in explicit conductor formulas for the symmetric and exterior square lifts. The proof crucially made use of the fact that $p$ is odd, one of the key points repeatedly used in the proof being the fact $g \mapsto g^2$ is a bijection on any odd group.

We could similarly ask the question for the Adams operator

$$\Psi^k(\rho)(g) = \rho(g^k),$$

where $\rho$ is an irreducible representation of $G$ which is not trivial on $G_0$. It follows that our proof goes through mutatis mutandis under the assumption $(k, p) = 1$, establishing the identity

$$(\rho(g^k), a_G(g))_G + \nu = (\rho(g), a_G(g))_G, \tag{6}$$

where $\nu$ is the number of unramified characters in $\rho(g^k)$.

Perhaps a more appealing way of writing down (6) is in terms of the Swan conductor. Recall that the Swan conductor of a (virtual) representation $\tau$, denoted by $\mathrm{Sw}(\tau)$, is by definition

$$f(\tau) + \dim \tau^{G_0} = \mathrm{Sw}(\tau) + \dim \tau,$$

where $\tau^{G_0}$ is the space of invariants of $\tau$ under the inertia subgroup $G_0$. Note that,

$$\dim \tau^{G_0} = \deg L(s, \tau)$$

is the number of unramified characters in $\tau$. Thus, (6) implies:

**Theorem 4.1.** *Let $E/F$ be a finite Galois extension of $p$-adic fields. Let $\rho$ be an irreducible representation of $G = \mathrm{Gal}(E/F)$. Then,*

$$\mathrm{Sw}(\Psi^k(\rho)) = \mathrm{Sw}(\rho),$$

*if $(k, p) = 1$.*

**Remark 4.2.** It appears that Theorem 4.1 is known to the experts. We have not seen it stated exactly like this though it does follow from [Sna94, Theorem 6.1.43], where it is proved by first developing the theory of explicit Brauer induction. Our proof is more straightforward and uses only the elementary properties of the Artin conductor.

## 5. Conductor for pairs

Let $\pi_i$ be irreducible supercuspidal representations of $GL_{n_i}(F)$ for $i = 1, 2$. The aim of this section is to give a brief description of the conductor formula for the pair $\pi_1 \times \pi_2$ as given in [BHK98], which will put our formulas for other functorial lifts in perspective. To state the explicit formula, we need to know the classification of irreducible supercuspidal representations of general linear groups in terms of compact induction [BK93].

5.1

Let $\pi$ be an irreducible supercuspidal representation of $G = GL_n(F)$ for some positive integer $n$. By [BK93], $\pi$ contains a (unique up to conjugation) maximal simple type $(K, \tau)$, where $K$ is a compact open subgroup of $GL_n(\mathfrak{o}_F)$ and $\tau$ is a smooth irreducible representation of $K$. Moreover, every supercuspidal representation of $GL_n(F)$ arises out of a maximal simple type.

If $\pi$ has a fixed vector for $1 + \mathfrak{p}_F M_n(\mathfrak{o}_F)$, then the stratum attached to $\pi$ can be taken to be of the form $[M_n(\mathfrak{o}_F), 0, 0, 0]$. Otherwise, the maximal simple type $(K, \tau)$ occurring in $\pi$ is given by a simple stratum, denoted by $[\mathfrak{A}, m, 0, \beta]$. The ring $\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in $A = M_n(F)$. The non-negative integer $m$ is called the level of $\pi$. The element $\beta \in GL_n(F)$ is such that the algebra $F[\beta]$ is a field with the property that $F[\beta]^\times$ normalizes $\mathfrak{A}$. The ramification index $e(F[\beta]/F)$ of $F[\beta]/F$ is denoted by $e$, which is also the $\mathfrak{o}_F$-period of $\mathfrak{A}$. The conductor of $\pi$ is given by the following formula:

$$f(\pi) = n\left(1 + \tfrac{m}{e}\right).$$

Given such a $\pi$, there is a constant $\mathfrak{c}(\beta)$ attached to $\pi$ which has both algebraic and analytic interpretations. We do not give the details here but refer to [BHK98, §6.4] for the description of the constant $\mathfrak{c}(\beta)$.

5.2

Let $[\mathfrak{A}_i, m_i, 0, \beta_i]$ be the simple strata attached to the maximal simple types contained in irreducible supercuspidal representations $\pi_i$ of $G_i = GL_{n_i}(F)$ for $i = 1, 2$. The integers $e_i$'s are the ramification indices of $F[\beta_i]/F$. Define the integers $e$ and $m$ by

$$e = \mathrm{lcm}\{e_1, e_2\},$$
$$m/e = \max\{m_1/e_1, m_2/e_2\}.$$

The conductor formula for pairs depends on the interactions between the representations in terms of the inducing data attached to them. There are three possibilities: (i) $\pi_1$ and $\pi_2$ are unramified twists of each other, (ii) $\pi_1$ and $\pi_2$ are *completely distinct*, and (iii) $\pi_1$ and $\pi_2$ admit a *common approximation*. We do not get into defining (iii) but refer to [BHK98, §6] instead. Suffices to say that when $\pi_1$ and $\pi_2$ admit a common approximation, there is a best common approximation and this is an object of the form $([\Lambda, m, 0, \gamma], l, \vartheta)$, where the stratum $[\Lambda, m, 0, \gamma]$ is determined by $\pi_1$ and $\pi_2$, $0 \leq l < m$ is an integer, and $\vartheta$ is a character of a compact group attached to the data coming from $\pi_1$ and $\pi_2$. The representations $\pi_1$ and $\pi_2$ are said to be completely distinct, if either both $\pi_1$ and $\pi_2$ are of level zero and the representations $\pi_i$ are not unramified twists of each other or if at least one of $\pi_1$ and $\pi_2$ is of positive level then either $m_1/e_1 \neq m_2/e_2$ or $m_1/e_1 = m_2/e_2$ but $\phi_{\pi_1} \neq \phi_{\pi_2}$, where $\phi_{\pi_i}$ is the characteristic polynomial (see [BK93, §2.3.2]) of $\pi_i$.

We now state the conductor formula for pairs due to Bushnell, Henniart and Kutzko [BHK98].

**Theorem 5.1 (Bushnell, Henniart and Kutzko).** *For $i = 1, 2$, let $\pi_i$ be an irreducible supercuspidal representation of $GL_{n_i}(F)$. Define the quantities $m_i, e_i, \beta_i$, $e$ and $m$ as above.*

(1) *Suppose $n_1 = n_2 = n$ and $\pi_1$ and $\pi_2$ are unramified twists of each other. Let $\beta = \beta_1$ and $d = [F[\beta] : F]$. Then,*

$$f(\pi_1^\vee \times \pi_2) = n^2 \left(1 + \frac{\mathfrak{c}(\beta)}{d^2}\right) - \deg L(s, \pi_1^\vee \times \pi_2).$$

(2) *Suppose $\pi_1$ and $\pi_2$ are completely distinct. Then,*

$$f(\pi_1^\vee \times \pi_2) = n_1 n_2 \left(1 + \frac{m}{e}\right).$$

(3) *Suppose $\pi_2$ is not equivalent to an unramified twist of $\pi_1$ and that the representations $\pi_i$ are not completely distinct. Let $([\Lambda, m, 0, \gamma], l, \vartheta)$ be a best common approximation to $\pi_i$, and assume that the stratum $[\Lambda, m, l, \gamma]$ is simple. Put $d = [F[\gamma] : F]$. Then,*

$$f(\pi_1^\vee \times \pi_2) = n_1 n_2 \left(1 + \frac{\mathfrak{c}(\gamma)}{d^2} + \frac{l}{de}\right).$$

**Remark 5.2.** Observe that in (2) and (3), $\deg L(s, \pi_1^\vee \times \pi_2) = 0$.

## 6. Conductors of functorial lifts

In this section we state the explicit conductor formulas for $\mathrm{Sym}^2(\pi)$, $\wedge^2(\pi)$ and $\mathrm{Asai}(\pi)$. These formulas are immediate from Theorem 1.1, Theorem 1.2, Theorem 5.1 and the identities (1) and (2). Note that the Asai case is already part of [AM15].

Let $\pi$ be an irreducible supercuspidal representation of $GL_n(F)$. There is a maximal simple type occurring in $\pi$ which is given by a simple stratum of the form $[\mathfrak{A}, m, 0, \beta]$. Let $e$ denote the ramification index of $F[\beta]/F$. Let $r(\pi)$ denote the representation $\mathrm{Sym}^2(\pi)$ or $\wedge^2(\pi)$ or $\mathrm{Asai}(\pi)$. In the Asai case, we assume that $F$ is given to be a quadratic extension of a $p$-adic field $F_0$.

If $\rho_\pi$ is the Langlands parameter of $\pi$, note that

$$\dim r \circ \rho_\pi = \begin{cases} \frac{n(n+1)}{2} & r = \mathrm{Sym}^2 \\ \frac{n(n-1)}{2} & r = \wedge^2 \\ n^2 & r = \mathrm{Asai}. \end{cases}$$

Let $\pi'$ denote the contragredient $\pi^\vee$ of $\pi$ if $r = \mathrm{Sym}^2$ or $r = \wedge^2$ and the conjugated contragredient $\pi^{\sigma\vee}$ if $r$ is the Asai representation, where $\sigma$ is the nontrivial element of $\mathrm{Gal}(F/F_0)$.

We define the integer $\alpha \in \{1, 2\}$ by

$$\alpha = \begin{cases} 1 & \text{if } r \text{ is the Asai representation and } F/F_0 \text{ is unramified} \\ 2 & \text{in all the other cases.} \end{cases}$$

Let $\varepsilon \in \{0, \pm 1\}$ be defined as

$$\varepsilon = \begin{cases} 0 & r = \mathrm{Asai} \\ 1 & r = \mathrm{Sym}^2 \\ -1 & r = \wedge^2. \end{cases}$$

Now we are in a position to state the explicit conductor formula for these functorial lifts.

**Theorem 6.1.** *Let $F$ be a $p$-adic field with the assumption that $p \neq 2$ unless $r$ is the Asai representation and $F/F_0$ is unramified. Let $\pi$ be an irreducible supercuspidal representation of $GL_n(F)$. Define the quantities $m, e, \mathfrak{c}(\beta), \alpha$ and $\varepsilon$ as above. Let $\rho_\pi$ denote the Langlands parameter of $\pi$.*

(1) *Suppose $\pi'$ and $\pi$ are unramified twists of each other. Let $d = [F[\beta] : F]$. Then,*

$$f(r(\pi)) = \dim(r \circ \rho_\pi) + \frac{n^2}{\alpha} \cdot \frac{\mathfrak{c}(\beta)}{d^2} + \varepsilon \cdot \frac{n}{2} \cdot \frac{m}{e} - \deg L(s, \pi, r).$$

(2) *Suppose $\pi'$ and $\pi$ are completely distinct. Then,*

$$f(r(\pi)) = \dim(r \circ \rho_\pi) + \frac{n^2}{\alpha} \cdot \frac{m}{e} + \varepsilon \cdot \frac{n}{2} \cdot \frac{m}{e}.$$

(3) *Suppose $\pi'$ and $\pi$ are not unramified twists of each other and that they are not completely distinct. Let $([\Lambda, m, 0, \gamma], l, \vartheta)$ be a best common approximation to $\pi'$ and $\pi$, and assume that the stratum $[\Lambda, m, l, \gamma]$ is simple. Put $d = [F[\gamma] : F]$. Then,*

$$f(r(\pi)) = \dim(r \circ \rho_\pi) + \frac{n^2}{\alpha} \left( \frac{\mathfrak{c}(\gamma)}{d^2} + \frac{l}{de} \right) + \varepsilon \cdot \frac{n}{2} \cdot \frac{m}{e}.$$

**Remark 6.2.** As in the case of Theorem 5.1, $\deg L(s,\pi,r) = 0$ in cases (2) and (3).

Theorem 6.1 follows from Theorem 1.1 (resp. Theorem 1.2) and Theorem 5.1 when $r =$ Asai (resp. $r = \mathrm{Sym}^2$ or $\wedge^2$).

To this end, when $r =$ Asai, note that

$$\deg L(s,\pi,r) + \deg L(s,\pi \otimes \kappa, r) = \deg L(s,\pi \times \pi^\sigma)$$

and Theorem 5.1 yields $f(\pi \times \pi^\sigma)$ which in turn is given by

$$f(\pi \otimes \pi^\sigma) = \begin{cases} f(r(\pi)) & \text{if } F/F_0 \text{ is unramified,} \\ f(r(\pi)) + f(r(\pi) \otimes \omega_{F/F_0}) - n^2 & \text{if } F/F_0 \text{ is ramified.} \end{cases}$$

On the other hand, Theorem 1.1 gives an expression for

$$f(r(\pi)) - f(r(\pi) \otimes \omega_{F/F_0}).$$

Combining all this information, we get the formula for $f(r(\pi))$.

For $r = \mathrm{Sym}^2$ or $\wedge^2$, we make use of the identity

$$\deg L(s,\pi,\mathrm{Sym}^2) + \deg L(s,\pi,\wedge^2) = \deg L(s,\pi \times \pi)$$

together with

$$f(\pi \times \pi) = f(\mathrm{Sym}^2\pi) + f(\wedge^2\pi)$$

and appeal once again to Theorem 5.1 to evaluate this explicitly. On the other hand, Theorem 1.2 gives us an expression for

$$f(\mathrm{Sym}^2\pi) - f(\wedge^2\pi).$$

Combining all this information, we get the formulas for both $f(\mathrm{Sym}^2\pi)$ and $f(\wedge^2\pi)$.

**Remark 6.3.** The Asai case of Theorem 6.1 is [AM15, Theorem 6.1].

**Remark 6.4.** It is interesting to note that the conductor formulas for $r =$ Asai, $\mathrm{Sym}^2$ and $\wedge^2$ can all be expressed in a uniform way by defining the numbers $\alpha \in \{1,2\}$ and $\varepsilon \in \{0,\pm 1\}$ as is done above. In the Asai case, the integer $\alpha$ is nothing but the ramification index of the quadratic extension involved. However, it is not very clear as to why the symmetric square and exterior square cases should be thought of as belonging to a "ramified" situation. We do not have any further explanation for the invariant $\varepsilon$ too other than that defining these invariants this way provides a uniform setting to state all the conductor formulas together.

## 7. Examples in the even residue characteristic case

In proving the conductor formulas of functorial lifts, we have crucially used the assumption that the residue characteristic is odd. It is not clear what form the conductor formulas should take in the even residue characteristic case. Note that the conductor formula of Bushnell, Henniart and Kutzko is valid in any residue characteristic.

In this section, we give some examples to show that the formulas that we have stated do not give the conductor formulas when $p = 2$. A few of the examples listed below also hint at the difficulty involved in modifying the conductor formulas to accommodate the even residue characteristic case.

### 7.1 Example

Let $\chi$ be a ramified character of $F^\times$ such that $\chi^2$ is not unramified. We assume that $f(\chi) = m + 1 \geq v_F(2)$, where $m$ is the level of $\chi$.

**Lemma 7.1.** *$\chi^{-1}$ and $\chi$ are not completely distinct.*

*Proof of Lemma 7.1.* To see this, note that if $\chi$ corresponds to an element $\beta = u\varpi_F^{-m}$, then $\chi^{-1}$ corresponds to $-\beta$, where $\varpi_F$ is a uniformizer of $F$ and $u \in \mathfrak{o}_F^\times$. Indeed, if $\chi(1+x) = \psi_F(\beta x)$ for $x \in \mathfrak{p}_F^{[m/2]+1}$, where $\psi_F$ is a nonzero additive character of $F$ with $n(\psi_F) = 1$ (see [BH06, §15.9]), then for $x \in \mathfrak{p}_F^{[m/2]+1}$, we have

$$\chi^{-1}(1+x) = (\psi_F(\beta x))^{-1} = \psi_F((-\beta)x).$$

The characteristic polynomial of $\chi$ is given by $\phi_\chi(X) = X - u \pmod{\mathfrak{p}_F}$ and the characteristic polynomial of $\chi^{-1}$ is given by $\phi_{\chi^{-1}}(X) = X + u \pmod{\mathfrak{p}_F}$. Clearly, $\phi_\chi(X) = \phi_{\chi^{-1}}(X)$, since $p = 2$. Hence $\chi$ and $\chi^{-1}$ are not completely distinct.

Since $\chi^2$ is not an unramified character, it is case (3) of Theorem 5.1 that is applicable. Thus, we get:

$$f(\chi^2) = 1 + l,$$

where $l < m$. On the other hand, case (3) of Theorem 6.1, if it were applicable, would give:

$$f(\chi^2) = 1 + \frac{l+m}{2}.$$

Since $l < m$ from the definition of common approximation, this is not consistent with the correct answer given by Theorem 5.1 (3).

## 7.2 Example

Let $F/F_0$ be a quadratic extension of $p$-adic fields. Let $r$ denote the Asai lift from $GL_n(F)$ to $GL_{n^2}(F_0)$. For a character $\chi$ of $F^\times$, its Asai lift $r(\chi)$ is nothing but the restriction of $\chi$ to $F_0^\times$.

Assume $F/F_0$ is ramified. Let $\chi$ be a character of $F^\times$ with trivial restriction to $F_0^\times$ so that $r(\chi) = 1$ and $f(r(\chi)) = 0$. In this case,

$$f(r(\chi) \otimes \omega_{F/F_0}) = f(\omega_{F/F_0}) \geq 2,$$

since $p = 2$. Thus, Theorem 1.1 does not hold in this case.

The above example may be suggestive of modifying the degree terms in Theorem 1.1 by multiplying by $f(\omega_{F/F_0})$. However, just this modification would not do in general as can be seen by choosing $F/F_0$ and a character $\chi$ of $F^\times$ such that

$$0 \neq f(\chi|_{F_0^\times}) < f(\omega_{F/F_0}).$$

## 7.3 Example

Finally, we analyse the example constructed in [Ser60, §4]. Let $L$ be the extension of degree 16 of $\mathbb{Q}_2$ as in [Ser60, §4]

$$\begin{tikzcd}[column sep=small]
 & E \arrow[dl, dash, "K \cong Q_8"'] \arrow[dr, dash, "H \cong Q_8"] & \\
F' = \mathbb{Q}_2(\imath) \arrow[dr, dash] & & F = \mathbb{Q}_2(\sqrt{5}) \arrow[dl, dash] \\
 & F_0 = \mathbb{Q}_2 &
\end{tikzcd}$$

with both $H = \mathrm{Gal}(E/F)$ and $K = \mathrm{Gal}(E/F')$ isomorphic to the quaternions $Q_8$ and $\mathrm{Gal}(E/F_0)$ isomorphic to the generalized quaternions $G$ of order 16. In this example, the ramification filtrations are given by

$$G \rhd G_0 = G_1 \cong Q_8 \rhd G_2 = G_3 \cong C_2 \rhd G_4 = \{1\},$$
$$H = H_0 = H_1 \rhd H_2 = H_3 \cong C_2 \rhd H_4 = \{1\},$$

and

$$K \rhd K_0 = K_1 \cong C_4 \rhd K_2 = K_3 \cong C_2 \rhd K_4 = \{1\}.$$

Thus, the three nontrivial characters of $H$ are of conductor 2 and the two dimensional irreducible representation of $H$ has conductor 5. The group

$K$ has one nontrivial unramified character, two characters of conductor 2 and the two dimensional irreducible representation has conductor 6. Among the three nontrivial characters of $G$, two are of conductor 2 and one is unramified. Among the three irreducible representations of dimension 2, two have conductor 5 and one has conductor 4.

Let $1, \alpha, \beta, \alpha\beta$ be the four characters and $\rho$ be the two dimensional irreducible representation of $Q_8$. Then, $\mathrm{Sym}^2\rho \cong \alpha \oplus \beta \oplus \alpha\beta$ and $\wedge^2\rho = 1$. Thus, for $H$,

$$f(\mathrm{Sym}^2\rho) + \deg L(s, \rho, \mathrm{Sym}^2) = f(\wedge^2\rho) + \deg L(s, \rho, \wedge^2) + f(\rho) = 6.$$

Whereas for $K$,

$$f(\mathrm{Sym}^2\rho) = 4,\ f(\wedge^2\rho) = 0,\ f(\rho) = 6$$

and

$$\deg L(s, \rho, \mathrm{Sym}^2) = \deg L(s, \rho, \wedge^2) = 1.$$

In particular, not only that Theorem 1.2 does not hold in this case, a common scaling of the degree terms would not do either.

Let $\omega_H$ (resp. $\omega_K$) be the nontrivial character of $G/H$ (resp. $G/K$). Note that $\omega_H$ is unramified and $\omega_K$ has conductor 2. Let $\rho_1, \rho_2$ be the two dimensional irreducible representations of $G$ of conductor 5 and let $\tau$ denote the two dimensional irreducible representation of $G$ with conductor 4. Note that $\rho_2 \cong \rho_1 \otimes \omega_H \cong \rho_1 \otimes \omega_K$ and that all the four characters are self-twists for $\tau$. We have:

$$\mathrm{Sym}^2\rho_i \cong \omega_H\omega_K \oplus \tau,\ \wedge^2\rho_i = 1,\ \mathrm{Sym}^2\tau \cong 1 \oplus \omega_H \oplus \omega_K,\ \wedge^2\tau = \omega_H\omega_K.$$

Thus, Theorem 1.2 holds true for $\rho_i$ but not for $\tau$.

The Asai lift of the two dimensional representation $\rho$ of $H$ (or $K$) to $G$ is isomorphic to $1 \oplus \omega_H\omega_K \oplus \tau$. Thus,

$$f(r(\rho) \otimes \omega) = f(r(\rho))\ \&\ \deg L(s, r(\rho) \otimes \omega) = \deg L(s, r(\rho))$$

for both $\omega = \omega_H$ and $\omega = \omega_K$.

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## References


[AM15] U. K. Anandavardhanan and Amiya Mondal, On the degree of certain local $L$-functions, *Pacific J. Math.*, **276** (2015) no. 1, 1–17.

[AR05] U. K. Anandavardhanan and C. S. Rajan, Distinguished representations, base change, and reducibility for unitary groups, *Int. Math. Res. Not.* (2005), no. 14, 841–854.

[BH06] Colin J. Bushnell and Guy Henniart, The local Langlands conjecture for GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, **335** (2006).

[BHK98] Colin J. Bushnell, Guy M. Henniart and Philip C. Kutzko, Local Rankin-Selberg convolutions for $\mathrm{GL}_n$: explicit conductor formula, *J. Amer. Math. Soc.*, **11** (1998) no. 3, 703–730.

[BK93] Colin J. Bushnell and Philip C. Kutzko, The admissible dual of $\mathrm{GL}(N)$ via compact open subgroups, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, **129** (1993).

[Hen10] Guy Henniart, Correspondance de Langlands et fonctions $L$ des carrés extérieur et symétrique, *Int. Math. Res. Not. IMRN* (2010) no. 4, 633–673.

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro and J. A. Shalika, Rankin-Selberg convolutions, *Amer. J. Math.*, **105** (1983) no. 2, 367–464.

[KR12] Pramod Kumar Kewat and Ravi Raghunathan, On the local and global exterior square $L$-functions of $\mathrm{GL}_n$, *Math. Res. Lett.*, **19** (2012) no. 4, 785–804.

[Mat11] Nadir Matringe, Distinguished generic representations of $\mathrm{GL}(n)$ over $p$-adic fields, *Int. Math. Res. Not. IMRN*, (2011) no. 1, 74–95.

[Pra99] Dipendra Prasad, Some remarks on representations of a division algebra and of the Galois group of a local field, *J. Number Theory*, **74** (1999) no. 1, 73–97.

[Ser60] Jean-Pierre Serre, Sur la rationalité des représentations d'Artin, *Ann. of Math. (2)*, **72** (1960) 405–420.

[Ser79] Jean-Pierre Serre, Local fields, Graduate texts in mathematics, Springer-Verlag, New York, Translated from the French by Marvin Jay Greenberg, **67** (1979).

[Sha84] Freydoon Shahidi, Fourier transforms of intertwining operators and Plancherel measures for $\mathrm{GL}(n)$, *Amer. J. Math.*, **106** (1984) no. 1, 67–111.

[Sha90] Freydoon Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for $p$-adic groups, *Ann. of Math. (2)*, **132** (1990) no. 2, 273–330.

[Sna94] Victor P. Snaith, Explicit Brauer induction, Cambridge studies in advanced mathematics, Cambridge University Press, Cambridge, with Applications to Algebra and Number Theory, **40** (1994).

[Whi15] Paul-James White, Beyond endoscopy for the symmetric square $L$-function in the level aspect, preprint (2015).