INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

Anant R. Shastri

August 29, 2012

Essential Singularity

- The singular part of a function around a pole consisted finitely many negative powers of $z - a$.

Essential Singularity

- The singular part of a function around a pole consisted finitely many negative powers of $z - a$.
- A natural question is what happens when we allow infinitely many terms.

Essential Singularity

- The singular part of a function around a pole consisted finitely many negative powers of $z - a$.
- A natural question is what happens when we allow infinitely many terms.
- Of course, we need to assume that such an 'infinite sum' is convergent

Essential Singularity

- Let $\sum_{m\geq 1} b_m t^m$ be a power series of infinite radius of convergence. Then for any $a \in \mathbb{C}$ the sum $f(z) = \sum_m b_m(z-a)^{-m}$ makes sense for all $z \neq a$ and defines a holomorphic function f in $\mathbb{C} \setminus \{a\}$.

Essential Singularity

- Let $\sum_{m\geq 1} b_m t^m$ be a power series of infinite radius of convergence. Then for any $a \in \mathbb{C}$ the sum $f(z) = \sum_m b_m (z-a)^{-m}$ makes sense for all $z \neq a$ and defines a holomorphic function f in $\mathbb{C} \setminus \{a\}$.
- What kind of singularity f has at a?

Essential Singularity

- Let $\sum_{m\geq 1} b_m t^m$ be a power series of infinite radius of convergence. Then for any $a \in \mathbb{C}$ the sum $f(z) = \sum_m b_m(z-a)^{-m}$ makes sense for all $z \neq a$ and defines a holomorphic function f in $\mathbb{C} \setminus \{a\}$.
- What kind of singularity f has at a?
- Answer depends on how many terms b_m are non zero.

Essential Singularity

- The point a is a removable singularity iff all $b_m = 0$.

Essential Singularity

- The point a is a removable singularity iff all $b_m = 0$.
- It is a pole iff only finitely many and at least one of b_m are non zero.

Essential Singularity

- The point a is a removable singularity iff all $b_m = 0$.
- It is a pole iff only finitely many and at least one of b_m are non zero.
- It is the third type that we are interested in, now.

Essential Singularity

Let $z = a$ be an isolated singularity of $f \not\equiv 0$. It may turn out that $z = a$ is neither a removable singularity i.e., $\lim_{z\to a}(z-a)f(z) \neq 0$, nor a pole, i.e., $\lim_{z-a} f(z) \neq \infty$. Such a singularity is called an **essential singularity.**

Essential Singularity

It can be proved that if f has an essential singularity at a, then there is a holomorphic function g in a nbd of a and a power series $\sum_{m\geq 1} b_m t^m$ of infinite radius convergence with infinitely many $b_m \neq 0$ such that

$$f(z) = \sum_{m\geq 1} b_m (z-a)^{-m} + g(z)$$

where g is holomorphic in a nbd of a.

Essential Singularity: Example

- $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots \quad , z \neq 0.$

Essential Singularity: Example

- $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots \quad , z \neq 0.$
- We have

$$\lim_{x \to 0+} f(x) = \infty; \quad \lim_{x \to 0-} f(x) = 0$$

and $|f(\imath y)| = 1$ for all $y \in \mathbb{R}$. Thus $\lim z \to 0 f(z)$ does not exist nor we have $\lim_{z\to 0} |f(z)| = \infty$.

Essential Singularity: Example

- $f(z) = e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \cdots \quad , z \neq 0.$
- We have

$$\lim_{x \to 0+} f(x) = \infty; \quad \lim_{x \to 0-} f(x) = 0$$

and $|f(\imath y)| = 1$ for all $y \in \mathbb{R}$. Thus $\lim z \to 0 f(z)$ does not exist nor we have $\lim_{z \to 0} |f(z)| = \infty$.
- Therefore $z = 0$ is an essential singularity of $e^{1/z}$.

Singularity at Infinity

- The discussion of isolated singularity can be carried out for the point $z = \infty$ as well.

Singularity at Infinity

- The discussion of isolated singularity can be carried out for the point $z = \infty$ as well.
- To begin with we need that the function is defined and holomorphic in a neighborhood of infinity, i.e., in $|z| > M$ for some sufficiently large M.

Singularity at Infinity

- We say that ∞ is a removable singularity or a pole of f iff

$$\lim_{z\to\infty}\left|\frac{f(z)}{z^n}\right| = 0 \tag{1}$$

for some integer n. (If this integer can be chosen to be ≤ 1, then ∞ is a removable singularity, otherwise, it is a pole.)

Singularity at Infinity

- We say that ∞ is a removable singularity or a pole of f iff

$$\lim_{z\to\infty}\left|\frac{f(z)}{z^n}\right| = 0 \tag{1}$$

for some integer n. (If this integer can be chosen to be ≤ 1, then ∞ is a removable singularity, otherwise, it is a pole.)

- (1) is the same as saying

$$\lim_{w\to 0}|w^n f(1/w)| = 0. \tag{2}$$

Singularity at Infinity

- Therefore, ∞ is removable singularity or a pole is the same as saying that 0 is a removable singularity or a pole for $g(z) = f(1/z)$.

Singularity at Infinity

- Therefore, ∞ is removable singularity or a pole is the same as saying that 0 is a removable singularity or a pole for $g(z) = f(1/z)$.
- The simplest examples are polynomial functions of degree $d \geq 1$. They have a pole only at ∞ and the order of the pole is d.

Singularity at Infinity

- More generally, any rational function of positive degree d has a pole of order d at ∞; if the degree d is ≤ 0, then it is a removable singularity.

Singularity at Infinity

- More generally, any rational function of positive degree d has a pole of order d at ∞; if the degree d is ≤ 0, then it is a removable singularity.
- The following theorem, which can be proved in different ways is a converse to this.

Singularity at Infinity

- Theorem
  Let f be a meromorphic function on $\mathbb{C}$. Suppose ∞ is a removable singularity or a pole of f. Then f is a rational function.

Singularity at Infinity

- Theorem

  Let f be a meromorphic function on $\mathbb{C}$. Suppose ∞ is a removable singularity or a pole of f. Then f is a rational function.

  - **Proof:** By definition $\lim_{z\to\infty} \dfrac{f(z)}{z^n} = 0.$
  - Observe that the condition implies that f has finitely many poles.

Singularity at Infinity

- Theorem

  Let f be a meromorphic function on $\mathbb{C}$. Suppose ∞ is a removable singularity or a pole of f. Then f is a rational function.

  - **Proof:** By definition $\lim\limits_{z\to\infty} \dfrac{f(z)}{z^n} = 0.$
  - Observe that the condition implies that f has finitely many poles.
  - Therefore there exists a polynomial $Q(Z)$ such that $g(z) = Q(z)f(z)$ is an entire function with ∞ as a removable singularity.

Singularity at Infinity

Now we apply Cauchy's estimate. The given condition implies that for $|z| >> 0$ we have, $|\frac{g(z)}{z^n}| < \epsilon$.

Singularity at Infinity

Now we apply Cauchy's estimate. The given condition implies that for $|z| >> 0$ we have, $|\frac{g(z)}{z^n}| < \epsilon$. Fix $z \in \mathbb{C}$ and take C to be the circle $|z - \xi| = r$ for sufficiently large r and use the Cauchy's integral formula

$$|g^{(n)}(z)| = \left|\frac{n!}{2\pi\imath}\int_C \frac{g(\xi)d\xi}{(\xi - z)^{n+1}}\right| \leq \frac{n!\epsilon}{2\pi r}\int_C |d\xi| = n!\epsilon.$$

Singularity at Infinity

It follows that the n^{-th} derivative of g vanishes identically. Therefore is g a polynomial of degree at most $n-1$. ♠Thereofore f is a rational function.

Residues

The height of glory of Cauchy's integration theory is fully attained in the theory of residues. This comes as a logical conclusion of the results that we have seen earlier, viz., relation between the value of line integrals, nature of isolated singularities, and existence of primitives.

Residues

- Definition

  Let f be a function on a domain U with the set of isolated singularities denoted by S. To each $a \in S$, let C_a be a circle with center a, and contained in U. Put $P_a = \int_{C_a} f(z)dz$. The number P_a is called the **fundamental period of** f **at** a.

Residues

- (i) Clearly $P_a(f)$ is independant of the radius of the circle C_a.

Residues

- (i) Clearly $P_a(f)$ is independant of the radius of the circle C_a.
- (ii) We know that the function $1/(z-a)$ has period $2\pi\imath$, i.e., $\int_C (z-a)^{-1} dz = 2\pi\imath$.

Residues

- (i) Clearly $P_a(f)$ is independant of the radius of the circle C_a.
- (ii) We know that the function $1/(z-a)$ has period $2\pi\imath$, i.e., $\int_C (z-a)^{-1} dz = 2\pi\imath$.
- Treating this as a normalizing factor, we define,

$$\boxed{R_a := \frac{P_a}{2\pi\imath} = \frac{1}{2\pi i}\int_{C_a} f(z)dz =: \mathrm{Res}_a(f).} \quad (3)$$

- We call R_a the *residue* of f at $z = a$.

Residues

Theorem

The residue of f at an isolated singularity $z = a$ is the unique number w such that the function

$$g(z) := f(z) - \frac{w}{z-a} \tag{4}$$

has a primitive in the whole of the annulus $0 < |z - a| < \delta$.

Residues

Proof: First to prove the uniqueness of w.

Residues

Proof: First to prove the uniqueness of w. Recall the fact that $(z-a)^{-1}$ does not have a primitive in the annulus. If there were two such numbers w_1, w_1 such that $f(z) - \frac{w_j}{z-a}$ have primitives in the annulus, then their difference will have a primitive as well.

Residues

Proof: First to prove the uniqueness of w. Recall the fact that $(z-a)^{-1}$ does not have a primitive in the annulus. If there were two such numbers w_1, w_1 such that $f(z) - \frac{w_j}{z-a}$ have primitives in the annulus, then their difference will have a primitive as well. Dividing out by the constant $w_1 - w_2$ this will imply that $(z-a)^{-1}$ has a primitive, which is a contradiction.

Residues

Proof: First to prove the uniqueness of w. Recall the fact that $(z-a)^{-1}$ does not have a primitive in the annulus. If there were two such numbers w_1, w_1 such that $f(z) - \frac{w_j}{z-a}$ have primitives in the annulus, then their difference will have a primitive as well. Dividing out by the constant $w_1 - w_2$ this will imply that $(z-a)^{-1}$ has a primitive, which is a contradiction. So, the number w is unique.

Residues

We shall now prove that for $w = R_a(f)$,
$g(z) = f(z) - \frac{w}{z-a}$ has a primitive in the annulus.

Residues

We shall now prove that for $w = R_a(f)$, $g(z) = f(z) - \frac{w}{z-a}$ has a primitive in the annulus. From Primitve Existence Theorem, it is enough to prove that for all oriented simple closed curves γ in the annulus, we have

$$\int_\gamma g(z)dz = 0.$$

Residues

There are two cases to be considered.

Residues

There are two cases to be considered.

Case 1:γ encircles the point a

OR

Residues

There are two cases to be considered.

Case 1: γ encircles the point a

OR

Case 2: it does not.

Residues

- CASE 1. We can choose the circle C_a around a so small that the disc D_a bounded by C_a is contained in the inside of γ.

Residues

- CASE 1. We can choose the circle C_a around a so small that the disc D_a bounded by C_a is contained in the inside of γ.
- Now apply Cauchy's theorem version-II, to the region $R \setminus D_a$ to conclude that

$$\int_\gamma f(z)dz = \int_{C_a} f(z)dz$$

Residues

For the same reason we also have

$$\int_{\gamma} \frac{dz}{z-a} = \int_{C_a} \frac{dz}{z-a} = 2\pi i.$$

Residues

For the same reason we also have

$$\int_\gamma \frac{dz}{z-a} = \int_{C_a} \frac{dz}{z-a} = 2\pi i.$$

Therefore,

$$\int_\gamma g(z)dz = \int_C f(z)dz - R_a(f)\int_\gamma \frac{dz}{z-a} = 0.$$

Residues

CASE 2: By Cauchy's theorem (I-version) both $\int_\gamma f(z)dz$ and $\int_\gamma \frac{dz}{z-a}$ vanish.

$$\int_\gamma g(z)dz = 0.$$

This completes the proof of the theorem. ♠

Residues

- The following theorem is of practical importance in computing the residues at poles.

Residues

- The following theorem is of practical importance in computing the residues at poles.

- Theorem

  Let a be a pole of order n of f and let $g(z) = (z-a)^n f(z)$. Then g is complex differentiable at a also, and the residue of f at a is given by

$$R_a(f) = \frac{g^{(n-1)}(a)}{(n-1)!}$$

Residues

- **Proof:** We have,

$$f(z) = \frac{b_n}{(z-a)^n} + \cdots + \frac{b_1}{z-a} + g_n(z). \qquad (5)$$

Residues

- **Proof:** We have,

$$f(z) = \frac{b_n}{(z-a)^n} + \cdots + \frac{b_1}{z-a} + g_n(z). \qquad (5)$$

- Upon integration all terms in this sum vanish except the term involving b_1 which gives you $R_a(f) = b_1$. Multiplying both sides of (5) by $(z-a)^n$, we get

  $$g(z) = b_n + \cdots + b_1(z-a)^{n-1} + g_n(z)(z-a)^n. \, (6)$$

  where g_n is holomorphic at a.

Residues

- **Proof:** We have,

$$f(z) = \frac{b_n}{(z-a)^n} + \cdots + \frac{b_1}{z-a} + g_n(z). \qquad (5)$$

- Upon integration all terms in this sum vanish except the term involving b_1 which gives you $R_a(f) = b_1$. Multiplying both sides of (5) by $(z-a)^n$, we get

$$g(z) = b_n + \cdots + b_1(z-a)^{n-1} + g_n(z)(z-a)^n. \,(6)$$

where g_n is holomorphic at a.
- Differentiate $(n-1)-$times and put $z = a$. ♠

Residues: A Remark

At a simple pole $z = a$ of f, note that the residue $R_a(f)$ is never zero.
For, in this case, we have $f(z) = \frac{b_1}{z-a} + g(z)$ where g is holomorphic and $b_1 = R_a(f)$.

Residues: Examples

- (i) Let $f(z) = e^z/(z^2 - 1), z \neq \pm 1$. Then $z = \pm 1$ are simple poles of f.

Residues: Examples

- (i) Let $f(z) = e^z/(z^2 - 1), z \neq \pm 1$. Then $z = \pm 1$ are simple poles of f.
- To compute the residue at $z = 1$, we write $g(z) = (z-1)f(z) = e^z/(z+1)$ and find $g(1) = e/2$. Therefore $Res_1 = e/2$. Similarly $R_{-1} = -e^{-1}/2$.

Residues: Examples

- (ii) Let $f(z) = (\sinh z)/z^3 := (e^z - e^{-z})/2z^3$. Clearly $z = 0$ is a pole. What is the order of this pole?

Residues: Examples

- (ii) Let $f(z) = (\sinh z)/z^3 := (e^z - e^{-z})/2z^3$. Clearly $z = 0$ is a pole. What is the order of this pole?
- Caution is needed in this type of examples. For, sinh has a zero of order 1 at 0. Hence it follows that the order of the pole of f at 0 is 2. Therefore the residue is given by the value of $((\sinh z)/z)'$ at $z = 0$. This can be computed as follows:

Examples

$$R_0 = \lim_{z\longrightarrow 0}((\sinh z)/z)'$$
$$= \lim_{z\longrightarrow 0}\frac{z\cosh z - \sinh z}{z^2}$$
$$= \lim_{z\longrightarrow 0}\frac{\cosh z - z(\sinh z) - \cosh z}{2z}$$
$$= \frac{1}{2}\lim_{z\longrightarrow 0}(-z\cosh z - \sinh z) = 0.$$

Examples

- Alternatively, the Taylor' expansion can be employed, whenever the method above becomes cumbersome. For instance, when the order of the pole is very high.

Examples

- Alternatively, the Taylor' expansion can be employed, whenever the method above becomes cumbersome. For instance, when the order of the pole is very high.
- In this example, we know that
$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots .$$

Examples

- Alternatively, the Taylor' expansion can be employed, whenever the method above becomes cumbersome. For instance, when the order of the pole is very high.
- In this example, we know that
  $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$.
- Therefore, it follows immediately that the $(1/z)$-term is missing from the expression for $\frac{\sinh z}{z^3}$. Hence, $R_0 = 0$.

More Examples

- (i) Consider the case when $f(z) = g(z)p(z)$ where g is given by a Laurent series and p is a polynomial:

$$g(z) = \sum_{-\infty}^{\infty} a_n z^n; \qquad p(z) = \sum_{0}^{m} \alpha_k z^k.$$

More Examples

- (i) Consider the case when $f(z) = g(z)p(z)$ where g is given by a Laurent series and p is a polynomial:

$$g(z) = \sum_{-\infty}^{\infty} a_n z^n; \qquad p(z) = \sum_{0}^{m} \alpha_k z^k.$$

- Then the residue of f at 0 is given by

$$R_0(f) = a_{-1}\alpha_0 + \cdots + a_{-k-1}\alpha_k + \cdots + a_{-m-1}\alpha_m.$$

Residues: Examples

For example, if $g(z) = e^{1/z}$ then, the residue of f at 0 is :

$$\frac{\alpha_m}{(m+1)!} + \cdots + \frac{\alpha_1}{2!} + \alpha_0.$$

More Examples

- (ii) Let f have a simple pole at z_0 and g be holomorphic. Then $R_{z_0}(fg) = g(z_0)R_{z_0}(f)$.

More Examples

- (ii) Let f have a simple pole at z_0 and g be holomorphic. Then $R_{z_0}(fg) = g(z_0)R_{z_0}(f)$.
- To see this write

$$f(z) = \frac{b_{-1}}{z - z_0} + \sum_0^\infty b_j(z - z_0)^j; \quad g(z) = \sum_0^\infty c_j(z - z_0)^j,$$

valid in a neighborhood of z_0.

- Clearly, the Laurent series for fg which is the Cauchy product of these two, has the coefficient of $(z - z_0)^{-1}$ equal to $c_0 b_{-1}$.

Real Integrals

- How to use complex integration theory in computing definite real integrals?

Real Integrals

- How to use complex integration theory in computing definite real integrals?
- Given a real definite integral, associate a complex integration, evaluate it, and then take real or(the imaginary) part.

Real Integrals

- How to use complex integration theory in computing definite real integrals?
- Given a real definite integral, associate a complex integration, evaluate it, and then take real or(the imaginary) part.
- We perceive several obstacles in this approach.

Real Integrals

- How to use complex integration theory in computing definite real integrals?
- Given a real definite integral, associate a complex integration, evaluate it, and then take real or(the imaginary) part.
- We perceive several obstacles in this approach.
- For instance, the complex integration theory is always about integration over closed curves.

Real Integrals

- How to use complex integration theory in computing definite real integrals?
- Given a real definite integral, associate a complex integration, evaluate it, and then take real or(the imaginary) part.
- We perceive several obstacles in this approach.
- For instance, the complex integration theory is always about integration over closed curves.
- When it works it works like magic. Not always the best.

Trigonometric Integrals

- Let us show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad -1 < a < 1.$$

Trigonometric Integrals

- Let us show that
$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \ -1 < a < 1.$$
- Observe that for $a = 0$, there is nothing to prove. So let us assume that $a \neq 0$. We want to convert the integrand into a function of a complex variable and then set $z = e^{\imath\theta}$, $0 \leq \theta \leq 2\pi$, so that the integral is over the unit circle C.

Trigonometric Integrals

- Since, $z = e^{\imath\theta} = \cos\theta + \imath \sin\theta$, we have, $\sin\theta = (z - z^{-1})/2\imath$, and $dz = \imath e^{\imath\theta} d\theta$, i.e., $d\theta = dz/\imath z$.

Trigonometric Integrals

- Since, $z = e^{\imath\theta} = \cos\theta + \imath \sin\theta$, we have, $\sin\theta = (z - z^{-1})/2\imath$, and $dz = \imath e^{\imath\theta} d\theta$, i.e., $d\theta = dz/\imath z$.
- Therefore,

$$\begin{aligned} I &= \int_C \frac{dz}{\imath z(1 + a(z - z^{-1})/2\imath)} \\ &= \int_C \frac{2dz}{az^2 + 2\imath z - a} = \frac{2}{a}\int_C \frac{dz}{(z - z_1)(z - z_2)}, \end{aligned}$$

- where, z_1, z_2 are the two roots of the polynomial $z^2 + \frac{2\imath}{a}z - 1$.

Trigonometric Integrals

- Note that

$$z_1 = \frac{(-1 + \sqrt{1 - a^2})\imath}{a}, \; z_2 = \frac{(-1 - \sqrt{1 - a^2})\imath}{a}.$$

Trigonometric Integrals

- Note that
$$z_1 = \frac{(-1+\sqrt{1-a^2})\imath}{a}, \;\; z_2 = \frac{(-1-\sqrt{1-a^2})\imath}{a}.$$
- It is easily seen that $|z_2| > 1$. Since $z_1 z_2 = -1$, it follows that $|z_1| < 1$.

Trigonometric Integrals

- Therefore on the unit circle C, the integrand has no singularities and the only singularity inside the circle is a simple pole at $z = z_1$.

Trigonometric Integrals

- Therefore on the unit circle C, the integrand has no singularities and the only singularity inside the circle is a simple pole at $z = z_1$.
- The residue at this point is given by

$$R_{z_1} = 2/a(z_1 - z_2) = 1/\imath\sqrt{1 - a^2}.$$

- Hence by the Residue Theorem, we have:

$$I = 2\pi\imath R_{z_1} = 2\pi/\sqrt{1 - a^2}.$$

In summary, we have a theorem:

Trigonometric Integrals

Theorem

Trigonometric integrals : *Let $\phi(x,y) = p(x,y)/q(x,y)$ be a rational function in two variables such that $q(x,y) \neq 0$ on the unit circle. Then*

$$I_\phi := \int_0^{2\pi} \phi(\cos\theta, \sin\theta) d\theta = 2\pi \left(\sum_{|z|<1} R_z(\tilde{\phi}) \right),$$

where, $\tilde{\phi}(z) = \dfrac{1}{z}\phi\left(\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2\imath}\right)$.

Improper Integrals

- We shall begin with a brief introduction to the theory of improper integrals.

Improper Integrals

- We shall begin with a brief introduction to the theory of improper integrals.
- Chiefly there are two types of them. One type arises due to the infiniteness of the interval on which the integration is being taken.

Improper Integrals

- We shall begin with a brief introduction to the theory of improper integrals.
- Chiefly there are two types of them. One type arises due to the infiniteness of the interval on which the integration is being taken.
- The other type arises due to the fact that the integrand is not defined (shoots to infinity) at one or both end point of the interval.

Improper Integrals

Definition

When $\int_a^b f(x)dx$ is defined for all $b > a$ we define

$$\int_a^{\infty} f(x)dx := \lim_{b \longrightarrow \infty} \int_a^b f(x)dx, \qquad (7)$$

if this limit exists.

Improper Integrals

Definition

When $\int_a^b f(x)dx$ is defined for all $b > a$ we define

$$\int_a^\infty f(x)dx := \lim_{b \longrightarrow \infty} \int_a^b f(x)dx, \qquad (7)$$

if this limit exists. Likewise we define

$$\int_{-\infty}^b f(x)dx := \lim_{a \longrightarrow -\infty} \int_a^b f(x)dx, \qquad (8)$$

if this limit exists.

Improper Integrals

Definition

continued: Also, we define

$$\int_{-\infty}^{\infty} f(x)dx := \int_{0}^{\infty} f(x)dx + \int_{-\infty}^{0} f(x)dx, \quad (9)$$

provided both the integrals on the right exist.

Improper Integrals

Recall the Cauchy's criterion for the limit. It follows that the limit (7) exists iff *given* $\epsilon > 0$ *there exists* $R > 0$ *such that for all* $b > a > R$ *we have,*

$$\left|\int_a^b f(x)dx\right| < \epsilon. \tag{10}$$

Improper Integrals

Recall the Cauchy's criterion for the limit. It follows that the limit (7) exists iff *given* $\epsilon > 0$ *there exists* $R > 0$ *such that for all* $b > a > R$ *we have,*

$$\left|\int_a^b f(x)dx\right| < \epsilon. \tag{10}$$

In many practical situations the following theorem and statements which can be easily derived out of it come handy in ensuring the existence of the improper integral of this type.

Improper Integrals

Theorem

Existence of Improper Integrals : *Suppose f is a continuous function defined on $[0, \infty)$ and there exists $\alpha > 1$ such that $x^\alpha f(x)$ is bounded. Then $\int_0^\infty f(x)\,dx$ exists.*

Improper Integrals

However, the condition in the above theorem is not always necessary. For instance, the function $f(x) = \dfrac{\sin x}{x}$ does not satisfy this condition. Nevertheless $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ exists as will be seen soon.

Cauchy's Principal Value

- Observe that there are several legitimate ways of taking limits in (9). One such is to take the limit of $\int_{-a}^{a} f(x)dx$, as $a \longrightarrow \infty$.
- This is called the *Cauchy's Principal Value* of the improper integral and is denoted by,

$$PV\left(\int_{-\infty}^{\infty} f(x)dx\right) := \lim_{a\longrightarrow\infty} \int_{-a}^{a} f(x)dx. \quad (11)$$

Cauchy's Principal Value: An Example

- As an example consider $f(x) = x$.

Cauchy's Principal Value: An Example

- However, this limit, even if it exists, is, in general, not equal to the improper integral defined in (9), above.
- As an example consider $f(x) = x$.
- Then the Cauchy's PV exists but the improper integral does not. However, if the improper integral exists, then it is also equal to its principle value. This observation is going to play a very important role in the following application.

An Example

- **Example** Let us consider the problem of evaluating

$$I = \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx$$

An Example

- **Example** Let us consider the problem of evaluating

$$I = \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx$$

- Denoting the integrand by f, we first observe that f is an even function and hence

$$I = \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx$$

which in turn is equal to its PV.

An Example

- Thus we can hope to compute this by first evaluating
$$I_R := \int_{-R}^{R} f(x)dx$$
and then taking the limit as $R \longrightarrow \infty$.

An Example

- Thus we can hope to compute this by first evaluating
$$I_R := \int_{-R}^{R} f(x)dx$$
and then taking the limit as $R \longrightarrow \infty$.
- Step I: By merely replacing the real variable by a complex variable, we get a rational function of a complex variable whose restriction to the real axis is the given function.

An Example

- Step II: We join the two end points R and $-R$ by an arc in the upper-half space, say, the semi-circle! So let C_R denote the semi-circle running from R to $-R$ in the upper-half space.

An Example

- Step II: We join the two end points R and $-R$ by an arc in the upper-half space, say, the semi-circle! So let C_R denote the semi-circle running from R to $-R$ in the upper-half space.
- Draw a picture by yourself.

An Example

- Step III: Let γ_R denote the closed contour obtained by tracing the line segment from $-R$ to R and then tracing C_R. We shall compute

$$J_R := \int_{\gamma_R} f(z)dz$$

using residue computation.

An Example

- Step III: Let γ_R denote the closed contour obtained by tracing the line segment from $-R$ to R and then tracing C_R. We shall compute

$$J_R := \int_{\gamma_R} f(z)dz$$

using residue computation.

- Step IV: When the number of singular points of the integrand is finite, J_R is a constant for all large R. This is the crux of the matter.

A Lucky Example

- We then hope that in the limit, the integral on the unwanted portions tends to zero, so that $\lim_{R\longrightarrow\infty} J_R$ itself is equal to I. Are we lucky enough?

A Lucky Example

- We then hope that in the limit, the integral on the unwanted portions tends to zero, so that $\lim_{R\longrightarrow\infty} J_R$ itself is equal to I. Are we lucky enough?
- Step III, is precisely where we use the residue theorem.
- The zeros of the denominator $q(z) = z^4 + 5z^2 + 4$ are $z = \pm\imath, \ \pm 2\imath$ and *luckily* they do not lie on the real axis.(This is important.)

A Lucky Example

- They are also different from the roots of the numerator. Also, for $R > 2$, two of them lie inside γ_R. (We do not care about those in the lower half-space.)

A Lucky Example

- They are also different from the roots of the numerator. Also, for $R > 2$, two of them lie inside γ_R. (We do not care about those in the lower half-space.)
- Therefore by the Residue theorem, we have, $J_R = 2\pi\imath(R_\imath + R_{2\imath})$. The residue computation easily shows that $J_R = \pi/2$.

A Lucky Example

- Observe that $f(z) = p(z)/q(z)$, where $|p(z)| = |z^2 - 1| \le R^2 + 1$, and similarly $|q(z)| = |(z^2 + 1)(z^2 + 4)| \ge (R^2 - 1)(R^2 - 4)$.

A Lucky Example

- Observe that $f(z) = p(z)/q(z)$, where $|p(z)| = |z^2 - 1| \leq R^2 + 1$, and similarly $|q(z)| = |(z^2+1)(z^2+4)| \geq (R^2-1)(R^2-4)$.
- Therefore

$$|f(z)| \leq \frac{R^2+1}{(R^2-1)(R^2-4)} =: M_R.$$

- This is another lucky break that we have got.

A Lucky Example

- Note that M_R is a rational function of R of degree -2. For, now we see that

$$\left|\int_{C_R} f(z)dz\right| \leq M_R \left|\int_{C_R} dz\right| = M_R R\pi.$$

A Lucky Example

- Note that M_R is a rational function of R of degree -2. For, now we see that

$$\left|\int_{C_R} f(z)dz\right| \leq M_R \left|\int_{C_R} dz\right| = M_R R\pi.$$

- Since M_R is of degree -2, it follows that $M_R R\pi \longrightarrow 0$ as $R \longrightarrow \infty$.

A Lucky Example

- Thus, we have successfully shown that the limit of $\int_{C_R} f(z)dz$ vanishes at infinity.

A Lucky Example

- Thus, we have successfully shown that the limit of $\int_{C_R} f(z)dz$ vanishes at infinity.
- To sum up, we have,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2}\lim_{R\longrightarrow\infty}\int_{-R}^{R} f(x)dx$$
$$= \frac{1}{2}\lim_{R\longrightarrow\infty} J_R = \frac{\pi}{4}.$$

A Lucky Example

We can summerise what we have done in this example as a theorem:

Theorem

Let f be a rational function without any poles on the real axis and of degree ≤ -2. Then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi\imath \sum_{w\in \boldsymbol{H}} R_w(f).$$

Equally-lucky-but-with-a-difference Example

- For $f(x) = (\cos 3x)(x^2 + 1)^{-2}$, evaluate

$$\int_{-\infty}^{\infty} f(x)dx$$

Equally-lucky-but-with-a-difference Example

- For $f(x) = (\cos 3x)(x^2 + 1)^{-2}$, evaluate

$$\int_{-\infty}^{\infty} f(x)dx$$

- Except that now the integrand is a rational function multiplied by a trigonometric quantity; this does not seem to cause any trouble as compared to the example above, because the multiplier is a bounded function.

An Example with a difference

- For we can consider

$$F(z) = e^{3\imath z}(z^2 + 1)^{-2}$$

to go with and later take only the real part of whatever we get. The denominator has poles at $z = \pm\imath$ which are double poles but that need not cause any concern.

Equally-lucky-but-with-a-difference Example

- When $R > 1$, the contour γ_R encloses $z = \imath$ and we find the residue at this point of the integrand, and see that $J_R = 2\pi/e^3$.

Equally-lucky-but-with-a-difference Example

- When $R > 1$, the contour γ_R encloses $z = \imath$ and we find the residue at this point of the integrand, and see that $J_R = 2\pi/e^3$.
- Yes, the bound that we can find for the integrand now has different nature!

An Example with a difference

- Putting $z = x + \imath y$ we know that $|e^{3\imath z}| = |e^{-3y}|$. Therefore,

$$|f(z)| = \left|\frac{e^{3\imath z}}{(z^2+1)^2}\right| \le \left|\frac{e^{-3y}}{(R^2-1)^2}\right|.$$

Since, e^{-3y} remains bounded by 1 for all $y > 0$ we are done. Thus, it follows that the given integral is equal to $2\pi/e^3$.