INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

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Equally-lucky-but-with-a-difference Example

- For $f(x) = (\cos 3x)(x^2 + 1)^{-2}$, evaluate

$$\int_{-\infty}^{\infty} f(x)dx$$

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- Except that now the integrand is a rational function multiplied by a trigonometric quantity; this does not seem to cause any trouble as compared to the example above, because the multiplier is a bounded function.

An Example with a difference

- For we can consider

$$F(z) = e^{3\imath z}(z^2 + 1)^{-2}$$

to go with and later take only the real part of whatever we get. The denominator has poles at $z = \pm\imath$ which are double poles but that need not cause any concern.

Equally-lucky-but-with-a-difference Example

- When $R > 1$, the contour γ_R encloses $z = \imath$ and we find the residue at this point of the integrand, and see that $J_R = 2\pi/e^3$.

Equally-lucky-but-with-a-difference Example

- When $R > 1$, the contour γ_R encloses $z = \imath$ and we find the residue at this point of the integrand, and see that $J_R = 2\pi/e^3$.
- Yes, the bound that we can find for the integrand now has different nature!

An Example with a difference

- Putting $z = x + \imath y$ we know that $|e^{3\imath z}| = |e^{-3y}|$. Therefore,

$$|f(z)| = \left|\frac{e^{3\imath z}}{(z^2+1)^2}\right| \leq \left|\frac{e^{-3y}}{(R^2-1)^2}\right|.$$

Since, e^{-3y} remains bounded by 1 for all $y > 0$ we are done. Thus, it follows that the given integral is equal to $2\pi/e^3$.

Jordan's Inequality

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In the previous lecture, we had several lucky breaks. The next step is going to get us into some real trouble. Consider the problem of evaluating the Cauchy's Principal Value of

$$I = \int_{-\infty}^{\infty} f(x)dx,$$

where

$$f(x) = (x \sin\ x)/(x^2 + 2x + 2).$$

Jordan's Inequality

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Taking $F(z) = g(z)e^{\imath z}$, we see that, for $z = x$, we see that $f(x) = \Im(F(x))$.

Jordan's Inequality

Also, write,
$g(z) = z/(z^2 + 2z + 2) = z/(z - z_1)(z - z_2)$ where,
$z_1 = \imath - 1$ and $z_2 = -\imath - 1$, to see that
$|g(z)| \leq R/(R - \sqrt{2})^2 =: M_R, \; R > 2$, say. And of course, this implies that $\int_{C_R} F(z)dz$ is bounded by $\pi R M_R$, which does not tend to zero as $R \longrightarrow \infty$.
Hence, this is of no use!

Jordan's Inequality

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- Lemma

  Jordan's Inequality

$$J := \int_0^{\pi} e^{-R\sin\theta} d\theta < \pi/R, \quad R > 0.$$

Jordan's Inequality

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- Conclude that $\sin\theta > 2\theta/\pi$, for $0 < \theta < \pi/2$. Hence obtain the inequality,

$$e^{-R\sin\theta} < e^{-2R\theta/\pi}$$

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- Conclude that $\sin\theta > 2\theta/\pi$, for $0 < \theta < \pi/2$. Hence obtain the inequality,

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- Use this to obtain,

$$\begin{aligned} J &:= 2\int_0^{\pi/2} e^{-2R\sin\theta} d\theta < 2\int_0^{\pi/2} e^{-R\theta/\pi} d\theta \\ &= 2\pi(1 - e^{-R})/2R < \pi/R, \quad R > 0. \end{aligned}$$

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$$\left|\int_{C_R} F(z)dz\right| = \left|\int_0^{\pi} g(Re^{\imath\theta})e^{\imath Re^{\imath\theta}}\imath Re^{\imath\theta}d\theta\right|$$

$$< \; M_R R\int_0^{\pi} e^{-R\sin\theta}d\theta \; < \; M_R\pi.$$

Jordan's Inequality

- Let us now use this in the computation of the integral I above. We have,
- $$\left|\int_{C_R} F(z)dz\right| = \left|\int_0^\pi g(Re^{\imath\theta})e^{\imath Re^{\imath\theta}}\imath Re^{\imath\theta}d\theta\right|$$
$$< \; M_R R\int_0^\pi e^{-R\sin\theta}d\theta \; < \; M_R\pi.$$
- Since $M_R\pi \longrightarrow 0$ as $R \longrightarrow \infty$, we get
$$\Im(\lim_{R\longrightarrow\infty} J_R) = I.$$

Jordan's Inequality

We leave the calculation of the residue to the reader. [Answer: $\dfrac{\pi}{e}(\cos 1 + \sin 1)$.]

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We now have enough ideas to prove the following theorem,
the conditions of which are met if f is a rational function of degree ≤ -1 having no real poles.

Jordan's Inequality

Theorem

Let f be a holomorphic function in $\mathbb{C}$ except possibly at finitely many singularities none of which is on the real line. Suppose that $\lim_{|z|\to\infty} f(z) = 0$. Then for any real $a > 0$,

$$PV\left(\int_{-\infty}^{\infty} f(x)e^{\imath ax}\,dx\right) = 2\pi\imath \sum_{w\in \boldsymbol{H}} R_w[f(z)e^{\imath az}],$$

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- First of all observe that $\frac{\sin x}{x}$ is an even function and hence,

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$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2} PV \left(\int_{-\infty}^\infty \frac{\sin x}{x}\,dx \right).$$
- The associated complex function $F(z) = e^{\imath z}/z$ has a singularity on the x-axis and that is going to cause trouble if we try to proceed the way we did so far.

Bypassing a Pole

- Common sense tells us that, since 0 is the point at which we are facing trouble, we should simply avoid this point by going around it via a small semi-circle around 0 in the upper half-plane. (This would not have been possible if we remained within the real axis.)

Bypassing a Pole

- Common sense tells us that, since 0 is the point at which we are facing trouble, we should simply avoid this point by going around it via a small semi-circle around 0 in the upper half-plane. (This would not have been possible if we remained within the real axis.)
- Thus consider the closed contour $\gamma_{r,R}$ as shown in the figure.

Bypassing a Pole

Given any meromorphic function $F(z)$, with a simple pole at 0, and finitely many poles $\{z_j\} \subset H$, the upper half space, in order to compute $\displaystyle\int_{-\infty}^{\infty} F(z)dz$, here is the recipe and the justification for the same:

Bypassing a Pole

(i) Choose the closed curve $\gamma_{r,R}$ as in the figure and put $I(r,R) := \int_{\gamma_{r,R}} F(z)dz$.

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(ii) Choose R large enough and $r > 0$ small enough so that all the singularities of $F(z)$ are inside $\gamma_{r,R}$. Put $I_{r,R} = 2\pi\imath \sum_{z_j \in H}(Res_{z_j}(F))$.

Bypassing a Pole

- Use Jordan's inequaltiy, to see that $\lim_{R\to\infty}\int_{C_R} F(z)dz = 0$. This will work only if F is of a particular form. In the present case, we have $F(z) = e^{iz}/z$ and so it is OK.

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- Next Compute $\lim_{r\to 0}\int_{C_r} F(z)dz$. This is the crucial new step.

Bypassing a Pole

- Since 0 is a simple pole of $F(z)$ we can write
$$F(z) = \frac{b_1}{z} + g(z)$$
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- Therefore,
$$\int_{C_r} F(z)dz = b_1 \int_0^{\pi} \imath d\theta + \int_{C_r} g(z)dz$$
$= -b_1\pi\imath + (G(r) - G(-r))$, where,
$G'(z) = g(z)$.

Bypassing a Pole

Therefore upon taking the limit as $r \to 0$ the second term vanishes. Hence

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Finally, set

$$\int_{-\infty}^{\infty} F(z)dz = \pi\imath R_0(F) + I(r, R) = \pi\imath + \sum_{z_j} Res_{z_j}(F).$$

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Take the real or imaginary part as the case may be.

Bypassing a Pole

In this particular case, Since $F(z) = e^{iz}/z$ is holomorphic in the upper half plane and $R_0(F) = 1$ where $F(z) = e^{\imath z}/z$. Therefore, upon taking the imaginary part, we get,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Branch Cuts

Example

Consider the problem of evaluating the integral

$$I = \int_0^\infty \frac{x^{-\alpha}}{x+1} dx, \quad 0 < \alpha < 1.$$

This integral is important in the theory of Gamma function $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$

Branch Cuts

Observe that the integral converges because in $[0, 1]$, we can compare it with $\displaystyle\int_0^1 x^{-\alpha} dx$, whereas, in $[1, \infty)$, we can compare it with $\displaystyle\int_1^\infty x^{-\alpha-1} dx$.

Branch Cuts

Branch Cuts

The problem that we face here is that the corresponding complex function $f(z) = z^{-\alpha}$ does not have any single valued branch in any neighborhood of 0. So, an idea is to cut the plane along the positive real axis, take a well defined branch of $z^{-\alpha}$, perform the integration along a contour as shown in the figure below and then let the cuts in the circles tend to zero. The crux of the matter lies in the following observation:

Branch Cuts

Let $f(z)$ *be a branch of* z^α *in* $\mathbb{C} \setminus \{x \ : \ x \geq 0\}$. *Suppose for any* $x_0 > 0$, *the limit of* $f(z)$ *as* $z \longrightarrow x_0$ *through upper-half plane is equal to* $x_0^{-\alpha}$. *Then the limit of* $f(z)$ *as* $z \longrightarrow x_0$ *through lower-half plane is equal to* $x_0^{-\alpha} e^{-2\pi\imath\alpha}$.

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Let $f(z)$ be a branch of z^α in $\mathbb{C} \setminus \{x \ : \ x \geq 0\}$. Suppose for any $x_0 > 0$, the limit of $f(z)$ as $z \longrightarrow x_0$ through upper-half plane is equal to $x_0^{-\alpha}$. Then the limit of $f(z)$ as $z \longrightarrow x_0$ through lower-half plane is equal to $x_0^{-\alpha} e^{-2\pi\imath\alpha}$.
This easily follows from the periodic property of the exponential. Now, let us choose such a branch $f(z)$ of $z^{-\alpha}$ and integrate $g(z) = \dfrac{f(z)}{z+1}$ along the closed contour as shown in the figure. (Draw the figure yourself)

Branch Cuts

When the radius r of the inner circle is smaller than 1 and radius R of the outer one is bigger that 1, this contour goes around the only singularity of $g(z)$ exactly once, in the counter clockwise sense.

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When the radius r of the inner circle is smaller than 1 and radius R of the outer one is bigger that 1, this contour goes around the only singularity of $g(z)$ exactly once, in the counter clockwise sense. Hence,

$$\int_{\gamma} \frac{f(z)}{z+1}\, dz = 2\pi\imath e^{-\pi\imath\alpha} \tag{1}$$

Branch Cuts

Put

$$J_r = \int_{C_r} \frac{f(z)}{z+1} dz; \quad J_R = \int_{C_R} \frac{f(z)}{z+1} dz$$

where c_r, C_R are the two circular part of the countour γ.

We now let the two segments L_1, L_2 approach the interval $[r, R]$. This is valid, since in a neighborhood of $[r, R]$, there exist continuous extensions f_1 and f_2 of g_1 and g_2 where g_1 and g_2 are restrictions of g to upper half plane and lower half plane respectively.

Branch Cuts

In the limiting case, equation (1) becomes

$$\int_r^R \frac{x^{-\alpha}}{x+1}\,dx + J_R - \int_r^R \frac{x^{-\alpha}e^{-2\pi\imath\alpha}}{x+1}\,dx - J_r = 2\pi\imath e^{-\pi\imath\alpha}.$$

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Now we let $r \longrightarrow 0$ and $R \longrightarrow \infty$. It is easily checked that $|J_R| \leq 2\pi R^{1-\alpha}/(R+1)$ and $|J_r| \leq 2\pi r^{1-\alpha}/(r+1)$. Hence the limits of these integrals are both 0.

Branch Cuts

Therefore,

$$(1 - e^{-2\pi\imath\alpha}) \int_0^\infty \frac{x^{-\alpha}}{x+1}\, dx = 2\pi\imath e^{-\pi\imath\alpha}.$$

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Hence,

$$\int_0^\infty \frac{x^{-\alpha}}{x+1}\, dx = \frac{\pi}{\sin \pi\alpha}, \quad 0 < \alpha < 1.$$

There are different ways of carrying out the branch cut. See for example the book by Churchill and Brown, for one such. We shall cut out all this and describe yet another method here.

Cut-out the branch-cuts

Theorem

Let ϕ be a meromorphic function on $\mathbb{C}$ having finitely many poles none of which belongs to $[0, \infty)$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ be such that $\lim_{z\to 0} z^a\phi(z) = 0 = \lim_{z\to\infty} z^a\phi(z)$. *Then the following integral exists and*

$$I_a := \int_0^\infty x^{a-1}\phi(x)\,dx = \frac{2\pi\imath}{1 - e^{2\pi\imath a}} \sum_{w\in\mathbb{C}} R_w(z^{a-1}\phi(z)). \quad (2)$$

Cut-out the branch-cuts

Proof: First substitute $x = t^2$ and see that

$$I_a = \int_0^\infty x^{a-1}\phi(x)dx = 2\int_0^\infty t^{2a-1}\phi(t^2)dt. \quad (3)$$

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Next choose a branch $g(z)$ of z^{2a-1} in $-\pi/2 < argz < 3\pi/2$. Observe that $g(-x) = (-1)^{2a-1}g(x) = e^{2\pi\imath a}$, for $x > 0$.

Cut-out the branch-cuts

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} & z^{2a-1}\phi(z^2)dz \\
&= \int_0^{\infty} g(x)\phi(x^2)dx + \int_{-\infty}^{0} g(x)\phi(x^2)dx \\
&= \int_0^{\infty} g(x)\phi(x^2)dx - \int_0^{\infty} e^{2\pi\imath a} g(x)\phi(x^2)dx \\
&= (1 - e^{2\pi\imath a}) \int_0^{\infty} z^{2a-1}\phi(z^2)dz
\end{aligned}$$

Cut-out the branch-cuts

Therefore, the integral I_a is given by

$$\begin{aligned} &\frac{2}{1 - e^{2\pi\imath a}} \int_{-\infty}^{\infty} z^{2a-1}\phi(z^2)dz \\ = \;& \tfrac{4\pi\imath}{1-e^{2\pi\imath a}} \textstyle\sum_{z\in \boldsymbol{H}} R_z(z^{2a-1}\phi(z^2)). \end{aligned} \tag{4}$$

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Therefore, the integral I_a is given by

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If we set $f(z) = z^{a-1}\phi(z)$ then $zf(z^2) = z^{2a-1}\phi(z^2)$. Observe that $zf(z^2)$ has no poles on the real axis.

Cut-out the branch-cuts

Therefore, the sum of the residues of $zf(z^2)$ in $\boldsymbol{H}$ is equal to half the sum of the residues in the entire plane. Finally, we have seen, in exercise 12 of Tut 6 that the sum of the residues of $zf(z^2)$ and that of $f(z)$ are the same. The formula (2) follows. ♠

Cut-out the branch-cuts

It may be noted that the assignment $a \mapsto I_a$ is called *Mellin's transform* corresponding to ϕ. Coming back to the special case when $\phi(z) = \dfrac{1}{z+1}$, we have $R_{-1}\dfrac{z^{a-1}}{z+1} = (-1)^{a-1} = -e^{\pi \imath a}$. Hence,

$$\int_0^\infty \frac{x^{(a-1)}dx}{x+1} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1. \tag{5}$$

Cut-out the branch-cuts

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Observe that the condition that a is not an integer is crucial for the non existence of the branch of z^{α} throughout a neighborhood of 0. On the other hand, that is what guarantees the existence of the integral.

Winding Number

We have seen the importance of the formula:

$$\boxed{\int_\gamma \frac{f(z)}{z-a}dz. = f(a)\int_\gamma \frac{dz}{z-a}} \tag{6}$$

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In order to bring out the true strength of (6), we need to understand the integral on the right side of (6) in a more general set-up than what we have done so far, i.e., when γ is a circle centered at a. Let us take up this task now.

Winding Number

Lemma

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed contour not passing through a given point z_0. Then the integral $w = \displaystyle\int_\gamma \frac{dz}{z - z_0}$ is an integer multiple of $2\pi\imath$.

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Proof: Enough to prove that $e^w = 1$. For $a \leq t \leq b$, define

$$\alpha(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\,ds; \quad g(t) = e^{-\alpha(t)}(\gamma(t) - z_0).$$

Winding Number

Since γ is continuous and differentiable except at finitely many points, so is g.

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Since γ is continuous and differentiable except at finitely many points, so is g.
Moreover, wherever g is differentiable, we have

$$\begin{aligned} g'(t) &= -e^{-\alpha(t)}\alpha'(t)(\gamma(t) - z_0) + e^{-\alpha(t)}\gamma'(t) \\ &= e^{-\alpha(t)}(-\gamma'(t) + \gamma'(t)) = 0. \end{aligned}$$

Winding Number

Therefore,
$g(t) = g(a) = \gamma(a) - z_0,$ for all $t \in [a, b]$ and hence,

$$e^{\alpha(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0},$$

for all $t \in [a, b]$. Since $\gamma(a) = \gamma(b)$, it now follows that $e^w = e^{\alpha(b)} = e^{\alpha(a)} = 1$. ♠

Winding Number

Definition

Let γ be a closed contour not passing through a point z_0. Put

$$\int_\gamma \frac{dz}{z - z_0} = 2\pi\imath m.$$

Then the number m is called the *winding number of the closed contour* γ *around the point* z_0 and is denoted by $\eta(\gamma, z_0)$.

Winding Number

Thus

$$\boxed{\boxed{\eta(\gamma, z_0) := \frac{1}{2\pi\imath}\int_\gamma \frac{dz}{z - z_0}.}} \tag{7}$$

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In order to understand the concept of winding number let us examine it a little closely.

Winding Number

Take $z_0 = 0$ and γ to be any circle around 0. Then we have seen that

$$\int_\gamma \frac{dz}{z} = 2\pi\imath.$$

In other words, $\eta(\gamma, 0) = 1$. So we can say that γ *winds* around 0 exactly once and this coincides with our geometric intution.

Winding Number

Now let γ be any simple closed contour contained in the interior of an open disc in the upper half plane. Since $1/z$ is holomorphic in that disc, by Cauchy's Theorem on convex domains or otherwise (it has a primitive), it follows that $\displaystyle\int_\gamma \frac{dz}{z} = 0$. That means $\eta(\gamma, 0) = 0$. Hence in this case, we see that the winding number is zero which again conforms with our geometric understanding.

Winding Number

More generally, if γ is contained in a disc, then for all points a outside this disc, we have $\eta(\gamma, a) = 0$. This is a simple consequence of Cauchy's theorem for discs or by simply observing that $1/(z-a)$ has a primitive on the disc. Once again this conforms with our general understanding that such a contour does not go around a.

Winding Number

Let us now consider the curve $\gamma(t) = e^{2\pi \imath n t}$, defined on the interval $[0, 1]$ for some integer n. This curve traces the unit circle $n-$times in the counter clockwise direction. This tallies with the computation of

$$\int_\gamma \frac{dz}{z} = 2\pi \imath n.$$

Winding Number

By Proposition on the continuity of integrated function, it follows that $z \mapsto \eta(\gamma, z)$ is a continuous function on $\mathbb{C} \setminus Im(\gamma)$. Being an integer valued continuous function, it must be a constant on any path. Hence it will be a constant on each path connected subset of $\mathbb{C} \setminus Im(\gamma)$.

Winding Number: Examples

- Let us find the value of
$$\int_{|z|=1} \frac{e^{az}}{z-a}\, dz.$$

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- Let us find the value of
$$\int_{|z|=1} \frac{e^{az}}{z-a}\, dz.$$
- Observe that e^{az} is holomorphic on the entire plane. The integral makes sense for all points a such that $|a| \neq 1$.
- For points $|a| < 1$, the curve γ defining the unit circle has the property $\eta(\gamma, a) = 1$ and for those points a such that $|a| > 1$ we have $\eta(\gamma, a) = 0$.

Winding Number

On the other hand, by Cauchy's theorem, the given integral is equal to $2\pi\imath e^{a^2}$ for $|a| < 1$ and 0 for $|a| > 1$.

Winding Number: Non-Existence of n^{th} root

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Winding Number: Non-Existence of n^{th} root

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- Assume that Ω is a domain which contains a closed contour $\gamma : [a, b] \longrightarrow \mathbb{C}$, such that $\eta(\gamma, 0)$ is odd. Then we claim that there does not exist any holomorphic function $g : \Omega \longrightarrow \mathbb{C}$ such that $g^2(z) = z, z \in \Omega$.

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- Let us assume on the contrary. Then by differentiating, we get, $2g(z)g'(z) = 1, z \in \Omega$.

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- Now,

$$2\pi\imath\eta(g\circ\gamma, 0) = \int_{g\circ\gamma}\frac{dw}{w}$$

$$= \int_a^b \frac{g'(\gamma(t))\gamma'(t)}{g(\gamma(t))}dt = \int_a^b \frac{\gamma'(t)}{2(g(\gamma(t)))^2}dt$$

$$= \frac{1}{2}\int_a^b \frac{\gamma'(t)}{\gamma(t)}dt = \pi\imath\eta(\gamma, 0).$$

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- This means that $\eta(\gamma, 0)$ is even which is absurd.

Winding Number: An application

Example

Let us now consider the function $f(z) = 1 - z^2$ *and study the question when and where there is a holomorphic single valued branch* g *of the square root of* f *i.e.,* $g^2 = f$. *Observe that* $z = \pm 1$ *are the zeros of* f *and hence if these points are included in the region then there would be trouble.*

Winding Number: An application

By differentiating the identity $g^2 = f$, we obtain $2g(z)g'(z) = f'(z) = -2z$. This is impossible since, at $z = \pm 1$, the L.H.S.$=0$ and R.H.S. $= \mp 2$. So the region on which we expect to find g should not contain ± 1.

Winding Number: An application

Next assume that Ω contains a small circle C around 1, say, contained in a punctured disc $\Delta' := B_\epsilon(1) \setminus \{1\}$ around 1. Restricting our attention to Δ', observe that there is a holomorphic branch of the square root of $1+z$ say h defined all over $B_\epsilon(1)$. Clearly $h(z) \neq 0$ here and hence $\phi = g/h$ will then be a holomorphic function on $\Delta' \cap \Omega$ such that $\phi^2 = 1 - z$. This contradicts our observation in the previous example.

Winding Number

By symmetry, we conclude that Ω cannot contain any circle which encloses only one of the points $-1, 1$.
Finally, suppose that both ± 1 are in the same connected component of $\mathbb{C} \setminus \Omega$. Then for all closed contours γ in Ω, both ± 1 will be in the same connected component of $\mathbb{C} \setminus Im(\gamma)$ and hence $\eta(\gamma, 1) = \eta(\gamma, -1)$. For instance, take $\Omega = \mathbb{C} \setminus [-1, 1]$. Then for any circle C with center 0 and radius > 1, $\eta(C, 1) = \eta(C, -1) = 1$.

Winding Number

We shall now see that the square root of f exists. Consider the flt $T(z) = \dfrac{1-z}{1+z}$. This maps $\mathbb{C} \setminus [-1, 1]$ onto $\mathbb{C} \setminus \{x \in \mathbb{R} \ : \ x \leq 0\}$, on which we can choose a well defined branch of the square root function. This amounts to say that we have a holomorphic function $h : \mathbb{C} \setminus [-1, 1] \longrightarrow \mathbb{C}$ such that $h(z)^2 = \dfrac{1-z}{1+z}$. Now consider $g(z) = h(z)(1 + z)$. Then $g(z)^2 = f(z)$ as required. ♠

Winding Number

In fact, $\Omega(= \mathbb{C} \setminus [-1, 1])$ happens to be a maximal domain on which $1 - z^2$ has a well defined square root. This follows from our earlier observation that any such domain on which g exists cannot contain a circle which encloses only one of the two points $-1, 1$.

Winding Number

Finally, observe that, in place of $[-1, 1]$, if we had any arc joining -1 and 1, the image of such an arc under T would be an arc from 0 to ∞ and hence on the complement of it, square-root would still exist. Also, the above discussion holds verbatim to the function $(z-a)(z-b)$ for any $a \neq b \in \mathbb{C}$. You can also modify this argument to construct other roots. Now it is time for you to at a look at the exercise below.

Winding Number Examples

Evaluate $\displaystyle\int_\gamma (e^z - e^{-z})z^{-4}\,dz$, where γ is one of the closed contours drawn below: