INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

Anant R. Shastri

September 5, 2012

Lecture 13

We shall once again return to the study of general behaviour of holomorphic functions both local and global.

Gauss's Mean Value Theorem

Let us begin with:

Theorem

(Gauss's mean value theorem) *Let f be a complex differentiable function on a disc $B_R(z_0)$. Then for $0 < r < R$,*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{\imath\theta})d\theta. \tag{1}$$

Gauss's Mean Value Theorem

Proof: This is just going back to the definition of the right hand side of the formula

$$f(z_0) = \frac{1}{2\pi\imath}\int_{|z-z_0|=r}\frac{f(z)}{z-z_0}dz.$$

The parameterization of the circle is $z(\theta) = z_0 + re^{\imath\theta}, 0 \leq \theta \leq 2\pi$, and therefore the RHS is given by

$$\frac{1}{2\pi\imath}\int_0^{2\pi}\frac{f(z_0+re^{\imath\theta})}{re^{\imath\theta}}re^{\imath\theta}\imath d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{\imath\theta})d\theta.$$

Gauss's Mean Value Theorem

This is obviously the *arithmetic mean* (continuous version!) of the function f on the circle. This is in some sense much more stronger than Lagrange's mean value theorem of real 1-variable calculus, which only says that the mean value is attained at some point in the interval. By taking real and imaginary parts we get mean value property of harmonic functions (more about this in next section). Any kind of mean value theorems are going to be useful. Here is an illustration.

Maximum Modulus Principle

Example

Suppose f is a holomorphic function in a domain U such that $|f|$ is a constant function. Then f is a constant function.

Maximum Modulus Principle

Proof: We know if $f' \equiv 0$ then f is a constant.

Maximum Modulus Principle

Proof: We know if $f' \equiv 0$ then f is a constant. Suppose $f' \not\equiv 0$. This means that there is a disc $D \subset U$ on which $f'(z) \neq 0$.

Maximum Modulus Principle

Proof: We know if $f' \equiv 0$ then f is a constant. Suppose $f' \not\equiv 0$. This means that there is a disc $D \subset U$ on which $f'(z) \neq 0$.
Now $|f(z)|^2 \equiv k$ implies $f(z)\bar{f}(z) \equiv k$. This mean

$$0 = \frac{\partial(f\bar{f})}{\partial z} = f'(z)\bar{f}(z).$$

Thereofore $f(z) = 0$ on D. But then $f'(z) = 0$ on D which is a contradiction.

Maximum Modulus Principle

Theorem

(Maximum modulus principle) *Let* $f : \Omega \to \mathbb{C}$ *be a* **non-constant** *complex differentiable function on a domain* Ω*. Then there does not exist any point* $w \in \Omega$ *such that* $|f(z)| \leq |f(w)|$ *for all* $z \in \Omega$.

Maximum Modulus Principle

Proof: If possible, let there be such a point say $a \in U$. Choose $r > 0$ such that $B_r(a) \subset \Omega$. Then for $0 < r' < r$, by GMT we have,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + r'e^{\imath\theta})d\theta.$$

Maximum Modulus

Therefore

$$k = |f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a + r'e^{\imath\theta})| d\theta.$$

Therefore

$$\int_0^{2\pi} (k - |f(a + r'e^{\imath\theta})|) d\theta \leq 0.$$

Maximum Modulus

Since the integrand is a continuous non-negative real function, this means it is identically zero, i.e., $k = |f(a + r'e^{\imath\theta})|$ for all θ. Since this is true for $0 < r' < r$, we have shown that $|f(z)| = k$ for all $z \in B_r(a)$. Therefore, f itself is a constant on $B_r(a)$.

Maximum Modulus

Since the integrand is a continuous non-negative real function, this means it is identically zero, i.e., $k = |f(a + r'e^{\imath\theta})|$ for all θ. Since this is true for $0 < r' < r$, we have shown that $|f(z)| = k$ for all $z \in B_r(a)$. Therefore, f itself is a constant on $B_r(a)$. But then by **Identity theorem,** it follows that f is a constant on U. ♠

Maximum Modulus

Remark

Suppose f is a complex differentiable function which never vanishes. Then $1/f$ is holomorphic and by maximum modulus principle applied to this, it follows that $|f|$ does not attain its minimum in the interior of the domain.

Maximum Modulus

Remark

There are several equivalent versions of the maximum modulus principle. Here is one such. Suppose f is a non-constant holomorphic function on a closed, connected and bounded set K of $\mathbb{C}$, then the maximum of $|f(z)|$ occurs only on the boundary of K. To see this, observe that since K is assumed to be closed and bounded, $|f(z)|$ being continuous, attains its maximum at some $z_0 \in K$. However, $z_0 \notin \operatorname{int} K$, by the above theorem. Hence $z_0 \in K \setminus \operatorname{int}(K) = \partial K$.

Open mapping theorem

The maximum principle has lot of applications. Here is just one example.

Open mapping theorem

The maximum principle has lot of applications. Here is just one example.

Theorem

(Open mapping theorem) *A non-constant holomorphic function on an open set is an* open mapping.

Open mapping theorem

Proof: Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function and let U be any open subset of Ω. We must show that $f(U)$ is an open set in $\mathbb{C}$. Let $w_0 \in f(U)$ be any arbitrary point, say, $w_0 = f(z_0)$, $z_0 \in U$.

Open mapping theorem

Proof: Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function and let U be any open subset of Ω. We must show that $f(U)$ is an open set in $\mathbb{C}$. Let $w_0 \in f(U)$ be any arbitrary point, say, $w_0 = f(z_0)$, $z_0 \in U$. For simplicity, we may assume that $z_0 = 0 = w_0$ and then we have to show that $f(U)$ contains a neighbourhood of 0.

Open mapping theorem

Let $\epsilon > 0$ be such that $V := B_\epsilon(0) \subset U$ and $f(z) \neq 0$ for $|z| = \epsilon$. Such a choice is possible because the set of zeros of f is isolated.

Open mapping theorem

Let $\epsilon > 0$ be such that $V := B_\epsilon(0) \subset U$ and $f(z) \neq 0$ for $|z| = \epsilon$. Such a choice is possible because the set of zeros of f is isolated.
It follows that $0 < \delta := \inf\{|f(z)| \ : \ |z| = \epsilon\}$.

Open mapping theorem

Let $\epsilon > 0$ be such that $V := B_\epsilon(0) \subset U$ and $f(z) \neq 0$ for $|z| = \epsilon$. Such a choice is possible because the set of zeros of f is isolated.
It follows that $0 < \delta := \inf\{|f(z)| \ : \ |z| = \epsilon\}$.
Suppose $w \in \mathbb{C}$ is such that $|w| < \delta$ and $w \notin f(V)$.

Open mapping theorem

Let $\epsilon > 0$ be such that $V := B_\epsilon(0) \subset U$ and $f(z) \neq 0$ for $|z| = \epsilon$. Such a choice is possible because the set of zeros of f is isolated.
It follows that $0 < \delta := \inf\{|f(z)| \ : \ |z| = \epsilon\}$.
Suppose $w \in \mathbb{C}$ is such that $|w| < \delta$ and $w \notin f(V)$.
Put $g(z) = f(z) - w, z \in V$. Then g is holomorphic on V and does not vanish. Therefore, $1/g$ is holomorphic in V.

Open mapping theorem

By maximum principle, it follows that

$$\frac{1}{|w|} = \frac{1}{|g(0)|} < \sup\left\{\frac{1}{|f(z)-w|} \;:\; z \in \partial V\right\} = \frac{1}{\delta-|w|}.$$

It follows that $|w| > \delta/2$. Hence, $B_{\delta/2}(0) \subset f(V)$.
♠

Open mapping theorem

Remark

Thus, it is clear that if $f : \Omega \longrightarrow \mathbb{C}$ is a non-constant holomorphic mapping on a non-empty open set, then $f(\Omega)$ cannot be contained in any contour.

Conformal mapping

Definition

Let $f : U \to V$ be a continuous mapping with continuous partial derivatives where U, V are some open sets. We say f is a **conformal** mapping iff
(i) f is a bijection (one to one)
and

Conformal mapping

Definition

Let $f : U \to V$ be a continuous mapping with continuous partial derivatives where U, V are some open sets. We say f is a **conformal** mapping iff
(i) f is a bijection (one to one)
and (ii) at each point $z \in U$ the map f is angle preserving.

Conformal mapping

Condition (ii) is equivalent to say that if $f = u + \imath v$ then u, v satisfy CR equations. Therefore it follows that f is complex differentiable.

Conformal mapping

Condition (ii) is equivalent to say that if $f = u + \imath v$ then u, v satisfy CR equations. Therefore it follows that f is complex differentiable.

Thus we are lead to consider holomorphic mappings which are bijections. Many familiar holomorphic mapping are not bijective. What do we do then? Well! Restrict their domain so that they become injective and then take the codomain to be the image.

Conformal mapping

Example

(i) Take the exponential function. If we restrict it to any infinite strip $\{x + \imath y \ : \ a < y < a + 2\pi\}$ *then we know it is injective. What is the image?*

Conformal mapping

Example

(ii) Take $f(z) = \sin z$. *Now of course, we need to cut down the size of the srrip above. That is* $\sin z$ *is injective on strips of the form*

$$\{x + \imath y \ : \ a < x < a + \pi\},$$

OR

$$\{x + \imath y \ : \ a < x < a + 2\pi; y > 0\}.$$

Conformal mapping

Example

(ii) Take $f(z) = \sin z$. *Now of course, we need to cut down the size of the srrip above. That is* $\sin z$ *is injective on strips of the form*

$$\{x + \imath y \;:\; a < x < a + \pi\},$$

OR

$$\{x + \imath y \;:\; a < x < a + 2\pi; y > 0\}.$$

Put $z = x + \imath y$, $\sin z = u + \imath v$. *Then*

$$u = \sin x \cosh y; \quad v = \cos x \sinh y.$$

Conformal mapping

Example

(ii) Take $f(z) = \sin z$*. Now of course, we need to cut down the size of the srrip above. That is* $\sin z$ *is injective on strips of the form*

$$\{x + \imath y \ : \ a < x < a + \pi\},$$

OR

$$\{x + \imath y \ : \ a < x < a + 2\pi; y > 0\}.$$

Put $z = x + \imath y, \ \sin z = u + \imath v$*. Then*

$$u = \sin x \cosh y; \qquad v = \cos x \sinh y.$$

Conformal mapping

The vertical lines $x = c \neq 0$ are mapped onto hyperbolas

$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1.$$

Likewise the horizontal lines $y = k$ are mapped onto ellipses

$$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 c} = 1.$$

Möbius Transformations

Definition

A fractional linear transformation (flt) (also called Möbius transformation) is a *non-constant* rational function in which both numerator and denominator are at most of degree one in z.

Möbius Transformations

Definition

A fractional linear transformation (flt) (also called Möbius transformation) is a *non-constant* rational function in which both numerator and denominator are at most of degree one in z.

They are given by the formula

$$z \mapsto \frac{az+b}{cz+d} \tag{2}$$

Fractional Linear Transformations

Remark

Of course, at least c or d has to be non-zero in order to make any sense out of this formula. Also for $c = 0$, this defines an affine linear map $z \mapsto \frac{a}{d}z + \frac{b}{d}$, and whatever we are going to say about fractional linear transformation is easily verified in that case. So, throughout, we shall assume that $c \neq 0$, the case $c = 0$ being easily understood.

Fractional Linear Transformations

Definition

By the *extended complex plane,* we mean $\mathbb{C} \cup \{\infty\}$ and denote it by $\widehat{\mathbb{C}}$.

Fractional Linear Transformations

We can then define

$$f(\infty) = \lim_{z\infty} f(z) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c}.$$

It follows that each non-constant fractional linear transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a one-one and onto mapping.

Fractional Linear Transformations

We next observe that

$$\frac{az+b}{cz+d} = \frac{a}{c} + \left(\frac{-ad+bc}{c^2}\right)\left(\frac{1}{z+d/c}\right).$$

Thus it is clear that we can express each FLT as a composite of a few very simple maps.

Fractional Linear Transformations

Let T_α denote translation by α, viz., $z \mapsto z + \alpha$. Similarly, let μ_α denote the multiplication by α. Finally, let η denote the inversion $z \mapsto z^{-1}$. Let us denote by $\lambda_1 = d/c, \lambda_2 = (bc - ad)/c^2$ and $\lambda_3 = a/c$. Then we see that

$$h_A = T_{\lambda_3} \circ \mu_{\lambda_2} \circ \eta \circ T_{\lambda_1}. \tag{3}$$

Fractional Linear Transformations

Since, the geometric behaviour of translations rotations and scaling are easily understood, in order to understand the geometric properties of any FLT, we have to study the geometric properties of the inversion map η alone. As an illustration, let us prove the following.

Fractional Linear Transformations

Theorem

The set of all circles and straight lines in the plane is preserved by any fractional linear transformation.

Fractional Linear Transformations

Theorem

The set of all circles and straight lines in the plane is preserved by any fractional linear transformation.

(Observe that the theorem does not assert that each circle is mapped to a circle. Nor does it say that each line is mapped to a line.)

Fractional Linear Transformations

Proof: From the decomposition (3) for a fractional linear transformation, it is clear that we need to verify this property only for the inversion map η. Because any way the other maps involved in the composition are orthogonal transformations, translations or scaling, which map circles to circles and lines to lines.

Fractional Linear Transformations

Now recall from your high school geometry that an equation of the form

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0 \tag{4}$$

(where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$) represents a circle (or a straight line) if $\alpha \neq 0$ (respectively, if $\alpha = 0$.)

Fractional Linear Transformations

If $z := x + \imath y \neq 0$ and $w := z^{-1} = u + \imath v$, then we have

$$u = \frac{x}{x^2 + y^2}; \;\; v = \frac{-y}{x^2 + y^2}.$$

$$x = \frac{u}{u^2 + v^2}; \;\; y = \frac{-v}{u^2 + v^2}.$$

Therefore, $z = x + \imath y$ satisfies (4) iff $w = u + \imath v$ satisfies

$$\delta(u^2 + v^2) + \beta u - \gamma v + \alpha = 0. \tag{5}$$

This last equation represents a circle or a straight line according as $\delta \neq 0$ or $= 0$. ♠

Fractional Linear Transformations

QUESTION: When a circle is mapped onto a straight line and vice versa by a fractional linear transformation $\frac{az+b}{cz+d}$?

Fractional Linear Transformations

QUESTION: When a circle is mapped onto a straight line and vice versa by a fractional linear transformation $\frac{az+b}{cz+d}$?
Answer circles which pass through $-d/c$ and straight lines which do not pass through $-d/c$.

Bilinearity

Putting $w = \frac{az+b}{cz+d}$ and clearing the denominator, we obtain the following

$$cwz + dw - az - b = 0 \tag{6}$$

where $ad - bc \neq 0$.
The formula (6) can be used to define both the transformation and its inverse: $z \mapsto w; \quad w \mapsto z$.
Observe that (6) is a polynomial equation in two variables $z, \; w$; it is a linear polynomial in each of the variables. That is the reason why a fractional linear transformation is also called a *bilinear*

Bilinearity

We know that a real linear map on $\mathbb{R}^2$ is completely determined by its value on any two independent vectors. In the case of fractional linear transformations, the situation is similar. For, suppose a fractional linear transformation T given by (2) fixes a point w.

Bilinearity

From (6) it follows that w satisfies the equation $cX^2 + (d + a)X + b = 0$ which is a polynomial equation of degree ≤ 2. Now assume that T fixes three distinct points. Since any polynomial of degree less than or equal to 2 with three distinct roots has to be identically zero, we get $c = 0 = b$ and $a = d$. This is the same as saying that T is the identity map. Thus we have proved:

Bilinearity

Theorem

Every fractional linear transformation, which fixes three distinct points of $\widehat{\mathbb{C}}$, is necessarily the identity map.