INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

Anant R. Shastri

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Mapping Properties

FLT continued
- Cross Ratio
- Some problems from Tutorial Sheet
- Applications to Physics

Lecture 14:

Bilinearity

Theorem

Every fractional linear transformation, which fixes three distinct points of $\widehat{\mathbb{C}}$*, is necessarily the identity map.*

Bilinearity

We can now conclude that if two fractional linear transformations agree on any three distinct points, then they must be the same. For, if $T_1(z_j) = T_2(z_j), j = 1, 2, 3$, then it follows that $T_1^{-1} \circ T_2(z_j) = z_j, i = 1, 2, 3$. Therefore, $T_1^{-1} \circ T_2 = Id$.

Symmetric form of FLT

Another interesting way of putting the fractional linear transformation is the following:

$$\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)}. \tag{1}$$

To see this, all that you have to do is to clear the denominators and simplify and see that you get a biliear form in z, w. This is very useful in obtaining actual form of FLT.

Symmetric form of FLT

Theorem

Given two sets $\{z_j\}$ and $\{w_j\}$ of three distinct elements each in $\widehat{\mathbb{C}}$, there is a unique fractional linear transformation f such that $f(z_j) = w_j, j = 1, 2, 3.$

The proof is obvious.

Symmetric form of FLT

Example

To illustrate the algorithmic nature of (1), let us consider the problem of determining the FLT that maps $0 \mapsto -1,\ 1 \mapsto -\imath$ *and* $-1 \mapsto \imath$. *We simply write*

$$\frac{(z-0)(-1-1)}{(z-1)(-1-0)} = \frac{(w+1)(\imath+\imath)}{(w+\imath)(\imath+1)}.$$

Upon simplification, this turns out to be

$$w = \frac{z-\imath}{z+\imath}.$$

Cayley Map

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1. It maps the entire upper half plane $\{z \; : \; \Im(z) > 0\}$ onto the open unit disc $|z| < 1$.

2 It maps $-i$ to ∞.

3. It maps the real axis onto the unit circle minus the point point $(1, 0)$.

Cross Ratio

The symmetric format (1) of a fractional linear transformation also relates it to another classical geometric notion.

Definition

Given four distinct points z_1, z_2, z_3, z_4 of $\widehat{\mathbb{C}}$, we define their **cross ratio** to be

$$[z_1, z_2, z_3, z_4] := \left(\frac{z_1 - z_3}{z_1 - z_4}\right) \Big/ \left(\frac{z_2 - z_3}{z_2 - z_4}\right)$$

$$= \left(\frac{z_1 - z_3}{z_2 - z_3}\right)\left(\frac{z_2 - z_4}{z_1 - z_4}\right).$$

Cross Ratio

Here if one of the points is ∞, then the meaning assigned to $[z_1, z_2, z_3, z_4]$ is to replace ∞ by a complex number z and take the limit as $z \to \infty$. For example,

$$[\infty, z_2, z_3, z_4] = \lim_{z\to\infty} \left(\frac{z - z_3}{z - z_4}\right) / \left(\frac{z_2 - z_3}{z_2 - z_4}\right) = \frac{z_2 - z_4}{z_2 - z_3}.$$

Cross Ratio

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Fixing z_2, z_3, z_4, the map $z \mapsto [z, z_2, z_3, z_4]$ is a fractional linear transformation. Conversely, given z_2, z_3, z_4 the fractional linear transformation, which takes

$$z_2 \mapsto 1; \quad z_3 \mapsto 0; \quad z_4 \mapsto \infty$$

is nothing but $[z, z_2, z_3, z_4]$.

Cross Ratio

Thus if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a fractional linear transformation, then $f(z) = [z, f^{-1}(1), f^{-1}(0), f^{-1}(\infty)]$. Of course, each cross ratio gives rise to different fractional linear transformations depending upon which one of the four slots is treated as a free variable and the other three fixed.

Cross Ratio

Theorem

Let T be a fractional linear transformation. Then

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = [z_1, z_2, z_3, z_4].$$

Proof: Since T is the composite of translation, rotation, dilation and inversion, it is enough to prove this statement when T itself is one of these. The case when $T(z) = 1/z$ is the one which is non-trivial. Even this is routine and hence we shall leave this as an exercise to you. ♠

An Application of Cayley Map

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Proof: *If $\chi(z) = \frac{z-\imath}{z+\imath}$ is the Cayley map we know that $g(z) = \chi \circ f(z)$ is an entrie function and takes values inside $B_1(0)$. That means g is an entire function which is bounded. By Liouville's theorem, g is a constant. But $f = \chi^{-1} \circ g$ and hence f is a constant.*

An Application of Cayley Map

The method employed above itself has lot of applications. For instance try to prove the following statements:

Q.1 If f is an entire function such that $f(z)$ is never hits the negative real axis then f is a constant.

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Q.2 Let f be holomorphic in the punctured disc $B_1(0) \setminus \{0\}$ and taking values in the upper-half plane. Then 0 is a removable singularity of f.

Q.3 Let f be as in Q.2, except that we only know that $f(z)$ never hits the line segment $[-1, 1]$. Then show that 0 is a removable singuarity or a pole of f.

Problems from Tut Set VIII

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Solution: The desired angle is simply an argument of the derivative of $f'(2+i)$, i.e. of $4+2i$. Taking principal argument, this angle is $\tan^{-1}(\frac{1}{2})$.

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Sketch.

Problems from Tut Set VIII

Solution: (i) Write $w = z^2$ in the polar form. Then the image is a sector of a disc centred at O and having radius r^2. If the angular width of the original sector is $\phi < \pi$, then that of the image is 2ϕ. But if $\phi > \pi$, then the image is the entire disc of radius r^2.

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Take note of the fact when is the mapping one-to-one.

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(iii) Put $z = x + \imath y$ and $z^2 = w = u + \imath v$. Then $u = x^2 - y^2, v = 2xy$. Putting $x = 0$ we get $u = -y^2, v = 0$. This means that the axis $x = 0$ goes to $v = 0$. But we also have $u = -y^2 \leq 0$. So, the side $\{iy : y \geq 0\}$ goes to the negative u-axis. Similarly the segment $0 \leq x \leq k, y = 0$ goes to the segment $v = 0, 0 \leq u \leq k^2$. The side $\{k + iy : y \geq 0\}$ goes to $\{(u, v) : u = k^2 - y^2, v = 2ky, \ y \geq 0\}$ which is an arc of a parabola.

Problems from Set X

[Q.8-T] **(a) Prove the Mean Value Property for harmonic functions which says that if $u(z)$ is harmonic in a domain U containing the closed disc $D = \{z : |z - z_0| \leq R\}$, then**

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\alpha})d\alpha.$$

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(b) Deduce the **maximum principle** for harmonic functions which says that unless u is a constant, it has neither a maximum nor a minimum in U.

Problems from Set X

Solution: (a) Choose $R' > R$ so that the closed disc $D' = \{z : |z - z_0| \leq R'\} \subset U$. Then we know that u has a harmonic conjugate, say v in D'. Let $f(z) = u(z) + \imath v(z)$. Then $f(z)$ is holomorphic in D' and hence we have Gauss Mean Value Theorem. Take the real part to get the result.

Problems from Set X

(b) (Here, we shall prove (b) only for the case when U is convex. The general case, needs a little more topological arguments which we shall skip.)

Problems from Set X

To prove the maximum principle, suppose first that U is convex and $u(z)$ attains its maximum value, say M, at some point $z_0 \in U$. Take R sufficiently small so that the closed disc $\{z : |z - z_0| = R\}$ is contained in U. Then

$$2\pi u(z_0) = \int_0^{2\pi} u(z_0 + Re^{i\alpha})\, d\alpha \qquad (2)$$

Rewriting the L.H.S. as $\displaystyle\int_0^{2\pi} u(z_0)\, d\alpha$ and subtracting, we get

Problems from Set X

$$\int_0^{2\pi} [u(z_0) - u(z_0 + Re^{i\alpha})] \, d\alpha = 0 \tag{3}$$

By the assumption of maximality of $u(z)$ at z_0, the integrand is non-negative for all $\alpha \in [0, 2\pi]$. Moreover it is continuous. So vanishing of its integral implies that it is identically zero, i.e. $u(z_0 + Re^{i\alpha}) = u(z_0)$ for every $\alpha \in [0, 2\pi]$. As this holds for all sufficiently small R, we see that $u(z) = M$ for all z in some neighbourhood W of z_0.

Problems from Set X

Since U is convex, we know u has a harmonic conjugate v on U. Since u is a constant on W so is v. That means the holomorphic function $f = u + \imath v$ is a constant W. Therefore f is a constant on whole of U. This means that u is a constant on U which is a contradiction.

Problems from Set X

A similar argument will work to show that $u(z)$ cannot attain its minimum on D unless it is a constant. But that is hardly necessary, because we can apply the earlier part to $-u$ which is also harmonic in U.

Problems from Tut Set X

[Q.11-T] **If** $f(z)$ **is holomorphic in** $|z| < 1$ **with** $f(0) = 0$ **and** $|f(z)| \leq 1$ **for all** $|z| < 1$**, prove that** $|f(z)| \leq |z|$ **for all** $|z| < 1$. **(Schwarz's Lemma)**

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Problems from Tut Set X

Solution: Let D denote the open unit disc and for $0 < r < 1$, let D_r be the closed disc $\{z : |z| \leq r\}$ and C_r denote its boundary circle. Define $g(z)$ on D by

$$g(z) = \begin{cases} f(z)/z, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \tag{1}$$

Problems from Tut Set X

Because of the hypothesis $f(0) = 0$, $g(z)$ is holomorphic on the whole disc D (including the point 0). Also for $0 < r < 1$,

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r} \text{ for all } z \in C_r \tag{2}$$

By continuity, $|g(z)|$ attains its maximum on D_r. But by the maximum modulus principle it cannot do so at an interior point. Hence by (2), we have

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r} \text{ for all } z \neq 0 \in D_r \tag{3}$$

Problems from Tut Set X

Letting $r \to 1$ we get

$$|g(z)| \leq 1, \quad |z| < 1.$$

This just means for $z \neq 0$,

$$|f(z)| < |z|, \; |z| < 1 \tag{4}$$

Problems from Tut Set X

Remark

(More is true than what is asked in the problem. For example, along with (2), we also have $|g(0)| = |f'(0)| \leq 1/r$ for every $r \in (0, 1)$. So by taking limit as $r \to 1$, we get

$$|f'(0)| \leq 1 \tag{5}$$

Further, if equality holds in (5) or even for one value of z in (4), then we get that $g(z)$ is a constant on D, i.e. $f(z) = cz$ for some c.

Problems from Tut Set X

Remark

The essence of Schwarz's lemma is that even though the hypothesis only gives a uniform upper bound on $|f(z)|$ for $z \in D$, the conclusion gives you a sharper upper bound which is tailor made, i.e. which depends on $|z|$. By elementary normalisations of the domain and the codomain, the lemma can be extended to functions that map a disc of radius R to a disc of radius M.

Problems from Tut Set X

Further, using suitable L.F.T.s (as in the solution to Problem 9), we can replace the condition $f(0) = 0$ by a condition of the form $f(z_0) = w_0$ where $|z_0| < R$ and $|w_0| < M$.

Solving for Harmonic functions

[Q.5-L] **Find the steady state temperature $T(x, y)$ in a thin semi-infinite plate $y \geq 0$ whose faces are insulated and whose edge $y = 0$ is kept at temperature 0 except the segment $-1 < x < 1$ which is kept at temperature 1.**

Solving for Harmonic functions

Solution: The problem amounts to finding a harmonic function $T(x, y)$ on the upper half plane H satisfying the boundary conditions

$$T(x,0) = \begin{cases} 0, & x < -1 \\ 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases} \quad (1)$$

There is no obvious solution to this problem. In fact, the familiar functions do not have such step function like behaviour on the boundary. As long as these two portions of the boundary sit next to each other we can't think of any solution.

Solving for Harmonic functions

So we first transform the domain conformally to an infinite strip whose two sides correspond to the two portions of the boundary of H. Such a transformation is not obvious either. But we note that the principal logarithm converts the upper half plane to the strip $\{(u,v) : 0 \leq v \leq \pi\}$ with the lower and upper boundaries being the images of the positive and the negative real axis respectively.

Solving for Harmonic functions

So, we introduce an intermediate variable z_1 and look for a transformation which converts H to the upper half plane, say H_1 in the z_1-plane with the portion $\{(x,0) : -1 < x < 1\}$ of the boundary of H going to the positive x_1- axis of the z_1-plane and the other portion of the boundary of H going to the negative x_1-axis of the z_1-plane (or vice versa). This is best achieved by a F.L.T. which sends one of the two points 1 and -1 to 0 and the other to ∞.

Solving for Harmonic functions

So we take

$$z_1 = \frac{z-1}{z+1} \tag{2}$$

This transformation does map H conformally onto H_1. The boundary conditions now reduce to

$$T(x_1, 0) = \begin{cases} 1, & x_1 < 0 \\ 1, & x_1 > 0 \end{cases} \tag{3}$$

Solving for Harmonic functions

This is a slight improvement over (1) because now there is only one jump discontinuity for T on the boundary. As noted before, the principal logarithm will separate the two portions of the boundary of H_1. So, we now let

$$w = \mathrm{Ln}\ z_1 = \mathrm{Ln}\ \frac{z-1}{z+1} \tag{4}$$

This transforms H in the z-plane to the infinite strip, say D, in the w-plane given by

$$D = \{(u, v) : 0 \le v \le \pi\} \tag{5}$$

Solving for Harmonic functions

The boundary conditions now become

$$T(u,v) = \begin{cases} 0, & v = 0 \\ 1, & v = \pi \end{cases} \tag{6}$$

This simplified boundary value problem has an obvious solution, viz.

$$T(u,v) = \frac{v}{\pi} \tag{7}$$

All that remains now is to translate this solution in terms of the original variables x and y.

Solving for Harmonic functions

From (4) we have

$$\begin{aligned} v &= \operatorname{Arg}\left(\frac{z-1}{z+1}\right) \\ &= \operatorname{Arg}\left(\frac{(z-1)(\overline{z}+1)}{|z+1|^2}\right) \\ &= \operatorname{Arg}\left[(z-1)(\overline{z}+1)\right] \\ &= \operatorname{Arg}\left[((x-1)+iy)(x+1)-iy)\right] \\ &= \operatorname{Arg}\left[(x^2+y^2-1)+2iy\right] \\ &= \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) \qquad (8) \end{aligned}$$

Solving for Harmonic functions

So, ultimately, the temperature function $T(x, y)$ for the original problem is

$$T(x, y) = \frac{1}{\pi} \tan^{-1}\left(\frac{2y}{x^2 + y^2 - 1}\right) \tag{9}$$

with the understanding that for points of H outside the circle $= x^2 + y^2 = 1$ by $\tan^{-1}$ we take the (unique) value in $[0, \pi/2)$, for those on C we take it as $\pi/2$ and for those inside C, it is a value lying between $\pi/2$ to π. In particular, for all points on the segment from -1 to 1 (without the end points) it is π.

Solving for Harmonic functions

[Q.6-T] **A similar problem for the semi-infinite strip** $y \geq 0, -\pi/2 \leq x \leq \pi/2$ **whose vertical sides are kept at a constant temperature 0 and the horizontal side at a constant temperature 1.**

Solving for Harmonic functions

Solution: This problem can be converted to the previous one using a suitable transformation. The region, say R, here is precisely the semi-infinite strip in Problem 2. There it was shown that the transformation $w = \sin z$ transforms R into the upper half plane, say H, of the w-plane. Moreover the boundary conditions given here transform to precisely those in the last problem. So, we already have a solution, viz.

$$T(u, v) = \frac{1}{\pi} \tan^{-1}\left(\frac{2v}{u^2 + v^2 - 1}\right) \tag{1}$$

Solving for Harmonic functions

We only need to express this in terms of the original variables x and y. Since $w = u + iv = \sin z$ we have $u = \sin x \cosh y$ and $v = \cos x \sinh y$. Substituting these into (1) gives

$$\begin{aligned} & T(x, y) \\ = \; & \frac{1}{\pi} \tan^{-1}\left(\frac{2 \cos x \sinh y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y - 1} \right) \end{aligned} \tag{2}$$

Solving for Harmonic functions

This can be expressed in a more compact form. The identity $\cosh^2 y = \sinh^2 y + 1$ reduces the denominator to $\sinh^2 y - \cos^2 x$. Hence the quotient in(4) becomes

$$\frac{2\cos x \sinh y}{\sinh^2 y - \cos^2 x} = \frac{2(\cos x/\sinh y)}{1 - (\cos x/\sinh y)^2} \tag{3}$$

Solving for Harmonic functions

Further simplification is possible using a little trigonometry. We let $\tan\alpha = \cos x/\sinh y$ Then the R.H.S. of (3) is simply $\tan 2\alpha$ and (2) becomes

$$T(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{\cos x}{\sinh y}\right) \tag{4}$$

This is not only simpler than (2) but also has the additional advantage that since in the given region R we have $\cos x \geq 0$ and $\sinh \geq 0$ at all points, $\tan^{-1}$ is to take values only in $[0, \pi/2]$ everywhere.

Solving for Harmonic functions

[Q.7-S] **Same problem for a plate in the form of an infinite quadrant if the segment of unit length at the end of one edge is insulated, the rest of that edge is kept at a temperature T_1 and the other edge is kept at a temperature T_2.**

Solving for Harmonic functions

Solution: Let D be the given quadrant in the xy-plane. Let A and B denote, respectively, the portions of the x-axis from 0 to 1 and the portion from 1 to ∞. Let C denote the other arm of the quadrant, viz. the non-negative y-axis. Insulation of A means that no heat can flow across it, or equivalently, the normal derivative of T along it is zero. That is, $\partial T/\partial y \equiv 0$ on A.

Solving for Harmonic functions

On the other hand, on B the temperature is constant and hence $\partial T/\partial x \equiv 0$ on B. But A and B are parts of the same straight line and it is difficult to think of a function whose partial derivatives behave so differently on these two parts. (There is no such clash between A and C because the conditions $\partial T/\partial y \equiv 0$ on A and $\partial T/\partial y \equiv 0$ on C are simultaneously satisfied by many functions.)

Solving for Harmonic functions

To rectify the situation, we transform the region to another region D^* so that the images of A and B are at right angles to each other (and, at the same time, ensuring the mutual orthogonality of the images of A and C). From Problem 2, the quadrant D is the image, under the sine function, of a semi-infinite strip. So we take the inverse transformation $w = \sin^{-1} z$.