INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

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Lecture 6: Line Integrals

Riemann Integral of complex valued functions

Parameterized curve

Basic Properties

Integrals of complex valued functions

Definition

Given $f : [a, b] \longrightarrow \mathbb{C}$ a continuous. We define

$$\boxed{\int_a^b f(t)\,dt := \int_a^b \text{Re}\,(f(t))\,dt + \imath \int_a^b \text{Im}\,(f(t))\,dt.} \quad (1$$

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Standard properties of Riemann integrals of real valued functions all hold for the above integral of a complex valued function. For instance, linearity properties are easy to verify.

Integrals of complex functions

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$$\boxed{\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|dt} \qquad (2)$$

Put $w = re^{\imath\theta} = \int_a^b f(t)dt$.

Integrals of complex functions

Then $|w| = r = e^{-\imath\theta}w$. That is,

$$\left|\int_a^b f(t)\,dt\right| = r = e^{-i\theta}\int_a^b f(t)\,dt = \int_a^b e^{-i\theta}f(t)\,dt$$

$$= \int_a^b \text{Re}\,(e^{-i\theta}f(t))\,dt \leq \int_a^b |f(t)|dt.$$

Parameterized curve

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Here the dot on the top denotes differentiation with respect to t. This just means that $\gamma(t) = (x(t), y(t)) \in U$ and $\dot{x}, \dot{y}$ exist and are continuous, and $(\dot{x}(t), \dot{y}(t)) \neq (0, 0)$.

Parameterized curve

Example

The curve $\gamma : t \mapsto (t^2, t^3)$ *(OR* $t \mapsto t + \iota t^3$*) is given by a function which has continuous derivative. However, at* $t = 0$, $\dot{\gamma} = (0, 0)$. *Therefore, if the domain of the function is allowed to include the point* 0 *then it is not a smooth curve. Otherwise it is a smooth curve.*

Parameterized curve

Example

Consider the curves

$$C_1(t) = e^{2\pi \imath t}, \;\; C_2(t) = e^{4\pi \imath t}, \;\; C_3(t) = e^{-2\pi \imath t}, \;\; 0 \le t \le 1$$

Each of them have its image equal to the unit circle. However, they are all different curves as 'parametrized curves.'

Sense in a parameterized curve

Remark

Geometrically, by a curve we often mean the image set of a curve as given above. A parametrized curve is much refined notion than that. For instance, observe that the parametrization automatically defines a sense of orientation on the curve, the ‘way’ in which the ‘geometric curve’ is being traced.

Parameterized curve

We shall fix the following notation for certain parameterized curves:

- Given $z_1, z_2 \in \mathbb{C}$, write $[z_1, z_2]$ for the curve given by

$$t \mapsto (1-t)z_1 + tz_2, \quad 0 \leq t \leq 1.\}$$

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- Given $z_1, z_2 \in \mathbb{C}$, write $[z_1, z_2]$ for the curve given by
$$t \mapsto (1-t)z_1 + tz_2, \quad 0 \leq t \leq 1.\}$$
- The circle with centre w and radius r traced exactly once in the counterclockwise sense will be denoted by
$$|z - w| = r := \{t \mapsto w + re^{2\pi\imath t}, 0 \leq t \leq 1.\}$$

Contour integration

Let γ be a smooth curve in U. Then for any continuous function $f : U \to \mathbb{C}$ we define the *contour integral*, or *line integral* of f along γ to be

$$\boxed{\int_\gamma f\,dz := \int_a^b f(\gamma(t))\dot{\gamma}(t)\,dt.} \tag{3}$$

Contour Integration

Observe that $\dot{\gamma}(t)$ is a complex number for each t, say, $\gamma(t) = x(t) + \imath y(t)$, then $\dot{\gamma}(t) = \dot{x}(t) + \imath \dot{y}(t)$.

The expressions dx, dy etc.

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- Hence the of the above definition can also be expressed as

$$\int_\gamma f(z)dz := \int_a^b (u(\gamma(t))\dot{x}(t) - v(\gamma(t))\dot{y}(t))\, dt$$
$$+\imath \int_a^b (u(\gamma(t))\dot{y}(t) + v(\gamma(t))\dot{x}(t))\, dt.$$

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- $$= \left(\int_a^b udx - vdy, \quad \int_a^b udy + vdx\right).$$

The expressions dx, dy etc.

Therefore it follows that $dx + \imath dy = dz$. There expressions are called **1-forms.** For us they are good for carrying out integration: indicators of which variable is being integrated. This is the only justification for the name 'complex integrals' which many authors use.

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Examples

- (1) Compute the value of $\displaystyle\int_{[0,1+i]} x\,dz$.
- **Sol:** Here the curve γ is the line segment from 0 to $1+i$.
- Recall that this curve is given by: $\gamma(t) = (1+\imath)t, \quad 0 \le t \le 1.$
- Then $\dot\gamma(t) = 1+i$ for all t and hence by definition

$$\int_\gamma x\,dz = \int_0^1 x(\gamma(t))\dot\gamma(t)\,dt = \int_0^1 t(1+\imath)\,dt = \frac{1+\imath}{2}.$$

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- (2) Let us compute $\int_C z^n dz$, where C is any circle with origin as centre and oriented counter-clockwise.
- **Sol:** We have $C : \gamma(t) = re^{\imath 2\pi t}, 0 \leq t \leq 1.$
- By definition, we have,

$$\int_C z^n dz = \int_0^1 r^n e^{2n\pi\imath t}(2\pi\imath)re^{2\pi\imath t}dt$$

$$= \quad r^{n+1}\int_0^1 e^{2\pi\imath(n+1)t}dt = 2\pi\imath r^{n+1}\int_0^1 e^{2\pi\imath(n+1)}dt.$$

Examples

This is easily seen to be $=0$ if $n \neq -1$ and $= 2\pi\imath$ if $n = -1$.

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Shifting the origin to $z = a$, taking $n = -1$ we obtain

$$\boxed{\int_{|z-a|=r} \frac{dz}{z-a} = 2\pi\imath.} \tag{4}$$

Some basic properties of the integral:

- 1. **Invariance Under Change of Parameterization**
  Let $\tau : [\alpha, \beta] \longrightarrow [a, b]$ be a continuously differentiable function with $\tau(\alpha) = a,\ \tau(\beta) = b,\ \dot{\tau}(t) > 0,\ \forall\ t$. Then

$$\boxed{\int_{\gamma\circ\tau} f(z)\,dz \int_{\gamma} f(z)\,dz} \tag{5}$$

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- This follows by chain rule and the Law of substitution for Riemann integration.

Change of parameterization:

$$LHS := \int_{\alpha}^{\beta} f(\gamma \circ \tau(t)) \frac{d(\gamma \circ \tau)}{dt}(t)\, dt.$$

$$RHS = \int_{a}^{b} f(\gamma(s)) \frac{\gamma}{ds} ds$$

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Now make the substitution $s = \tau(t)$ and use the fact $ds = \dot{\tau} dt$.

Basic Properties

(2) **Linearity**

For all $\alpha, \ \beta \in \mathbb{C}$

$$\boxed{\int_\gamma (\alpha f + \beta g)(z)\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz} \quad (6)$$

Basic Properties

(3) **Additivity Under Sub-division** or **Concatenation**

If $a < c < b$ and $\gamma_1 = \gamma|_{[a,c]},\ \gamma_2 = \gamma|_{[c,b]}$, are the restrictions to the respective sub-intervals of a parameterized curve $\gamma : [a, b] \to \mathbb{C}$, then

$$\boxed{\int_\gamma f(z)\,dz := \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz} \tag{7}$$

Basic Properties

(4) **Orientation Respecting**

We also have,

$$\boxed{\int_{\gamma^{-1}} f(z)\,dz = -\int_{\gamma} f(z)\,dz} \tag{8}$$

where γ^{-1} is the curve γ itself traced in the opposite direction, viz., $\gamma^{-1}(t) = \gamma(a+b-t)$.

Basic Properties

To see this, put $t = a + b - s$. Then,

$$\begin{aligned} L.H.S. &= \int_a^b f(\gamma^{-1}(s)) \frac{d\gamma^{-1}}{ds}(s)\, ds \\ &= \int_b^a f(\gamma(t))(-\dot{\gamma}(t))(-dt) \\ &= -\int_a^b f(z)\, dz = R.H.S. \end{aligned}$$

Basic Properties

(5) **Interchange of order of integration and limit**

If $\{f_n\}$ is a sequence of continuous functions **uniformly convergent** to f then the limit and integration can be interchanged viz.,

$$\boxed{\lim_{n\to\infty}\int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.} \tag{9}$$

This follows from the corresponding property of Riemann integration.

Basic Properties

(6) **Term-by-term Integration** From (5) it also follows that whenever we have a uniformly convergent series of functions then *term-by-term* integration is valid.

$$\boxed{\int_\gamma \left(\sum_n f_n(z) \right) dz = \sum_n \left(\int_\gamma f_n(z)\, dz \right)} \quad (10)$$

Basic Properties

(7)**Fundamental Theorem of integral calculus**
Suppose g is complex differentiable in U. Then for all smooth curves $\gamma : [a, b] \to U$ we have

$$\int_\gamma g'(z)dz = g(\gamma(b)) - g(\gamma(a)). \tag{11}$$

For the composite function $g \circ \gamma$ is differentiable in $[a, b]$. Therefore

$$\int_\gamma g'(z)dz = \int_a^b \frac{d}{dt}(g \circ \gamma(t))dt = g(\gamma(b)) - g(\gamma(a)).$$

♠

Contours

- Definition

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  By a *contour*, we mean the concatenation (composite) $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_k$ of a finite number of smooth parameterized curves γ_i taken in a fixed order.

  - Observe that γ is continuously differentiable except at finitely many points of the interval, where even the continuity also may break.

Contours

- Property (7) comes to our help and says that the only natural way to define the integrals over arbitrary contours is by the formula

$$\boxed{\int_{\gamma} f(z)\,dz := \sum_{j=1}^{k} \int_{\gamma_j} f(z)\,dz.} \qquad (12)$$

Contours

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- Verify directly that all the basic properties mentioned above for line integrals is valid for contour integrals as well.

Length of a countor

Definition

Length of a contour: Let $\gamma : [a, b] \longrightarrow \mathbb{R}^2, \ \ \gamma(t) = (x(t), y(t))$ be a continuously differentiable arc. Then the *arc-length* of γ is obtained by the integral

$$\boxed{L(\gamma) := \int_a^b |\dot\gamma(t)| dt = \int_a^b [(\dot x(t))^2 + (\dot y(t))^2]^{1/2}\, dt} \tag{13}$$

Length of a countor

It is easily checked that $L(\gamma)$ is independent of the choice of parameterization of γ as discussed earlier.

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It is easily checked that $L(\gamma)$ is independent of the choice of parameterization of γ as discussed earlier. Sometimes we use the following complex notation for (13): If $\gamma(t) = z(t) = x(t) + \imath y(t)$, this becomes

$$\boxed{L(\gamma) := \int_\gamma |dz|} \tag{14}$$

Examples

As a simple exercise, let us compute the length of the circle

$C_r := z(\theta) == (r\cos\theta, r\sin\theta)\ \ 0 \leq \theta \leq 2\pi.$

$$\begin{aligned} L(C_r) &= \int_{C_r} |dz| \\ &= \int_0^{2\pi} (r^2\sin^2\theta + r^2\cos^2\theta)^{1/2} d\theta \\ &= r\int_0^{2\pi} d\theta = 2\pi r. \end{aligned}$$

A notation and a consequence

- We now introduce the notation:

$$\boxed{\int_\gamma |f(z)\,dz| \; := \int_a^b |f(\gamma(t))\dot\gamma(t)|\,dt.} \tag{15}$$

for any continuous function f and any contour γ.

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$$\boxed{\int_\gamma |f(z)\,dz| \;:= \int_a^b |f(\gamma(t))\dot{\gamma}(t)|\,dt.} \qquad (15)$$

  for any continuous function f and any contour γ.
- Note that as a consequence of (2), it follows that

$$\boxed{\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)dz|} \qquad (16)$$

M-L Inequality

Theorem

M-L Inequality *Let U be an open set in $\mathbb{C}$, f be a continuous function on U and $\gamma : [a, b] \longrightarrow U$ be a contour in U. Let*
$M = sup\{|f(\gamma(t))| \ : \ a \leq t \leq b\}$. *Then*

$$\boxed{\left|\int_\gamma f(z)\,dz\right| \ \leq \ ML(\gamma).} \tag{17}$$

M-L Inequality

Proof: This is an immediate consequence of (2)

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\dot{\gamma}(t)\,dt\right|$$

$$\leq\ M\int_a^b |\dot{\gamma}(t)|\,dt = ML(\gamma).$$

Continuity of the Integrals

Theorem

Let Ω *be an open set in* $\mathbb{R}^n$ *and* $g : \Omega \times [a, b] \longrightarrow \mathbb{C}$ *be a continuous function. Put*

$$\phi(P) = \int_a^b g(P, t)\, dt, \quad P \in \Omega.$$

Then $\phi : \Omega \longrightarrow \mathbb{C}$ *is a continuous function.*

Continuity of Integrals

- **Proof:** Let B be a closed ball of radius, say $\delta_1 > 0$, around a point $P_0 \in \Omega$ such that $B \subset \Omega$.

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- **Proof:** Let B be a closed ball of radius, say $\delta_1 > 0$, around a point $P_0 \in \Omega$ such that $B \subset \Omega$.
- Then $B \times [a, b]$, is a closed and bounded subset of a Eucidean space. Hence, g restricted to this set is uniformly continuous.

Continuity of Integrals

This means that given $\epsilon > 0$, we can find a $\delta_2 > 0$ such that

$$|g(P_1, t_1) - g(P_2, , t_2)| < \epsilon/(b-a)$$

for all $(P_i, t_i) \in B \times [a, b]$ whenever $\|(P_1, t_1) - (P_2, t_2)\| < \delta_2$. Now let $\delta = min\{\delta_1, \delta_2\}$ and $|P - P_0| < \delta$.

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for all $(P_i, t_i) \in B \times [a, b]$ whenever $\|(P_1, t_1) - (P_2, t_2)\| < \delta_2$. Now let $\delta = min\{\delta_1, \delta_2\}$ and $|P - P_0| < \delta$. Then

$$|\phi(P) - \phi(P_0)| = \left| \int_a^b (g(P,t) - g(P_0,t))\, dt \right| \leq \epsilon.$$

This proves the continuity of ϕ at P_0.

Differentiation Under Integral Sign

Theorem

Differentiation Under the Integral Sign *Let U be an open subset of $\mathbb{C}$ and $g : U \times [a, b] \longrightarrow \mathbb{C}$ be a continuous functions such that for each $t \in [a, b]$, the function $z \mapsto g(z, t)$ is complex differentiable and the map $\dfrac{\partial g}{\partial z} : U \times [a, b] \longrightarrow \mathbb{C}$ is continuous. Then $f(z) = \int_a^b g(z, t)dt$ is complex differentiable in U and*

$$f'(z) = \int_a^b \frac{\partial g}{\partial z}(z, t)\, dt.$$

Differentiation Under Integral Sign

- **Proof:** Given $z_0 \in U$, we need to show that

$$\lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} - \int_a^b \frac{\partial}{\partial z} g(z_0, t) dt \right] = 0.$$

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- Put
$h(z, t) = g(z, t) - f(z_0, t)) - (z - z_0)\frac{\partial}{\partial z} g(z_0, t).$

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$h(z,t) = g(z,t) - f(z_0,t)) - (z-z_0)\frac{\partial}{\partial z}g(z_0,t).$
- Then we have to show

$$\lim_{z\to z_0}\left[\frac{1}{z-z_0}\int_a^b h(z,t)dt\right]=0.$$

Differentiation Under Integral Sign

- Let $r > 0$ be such that $B = \bar{B}_r(z_0) \subset U$. Then $B \times [a, b]$ is closed and bounded and hence $\dfrac{\partial g}{\partial z}$ is uniformly continuous on it.

Differentiation Under Integral Sign

- Let $r > 0$ be such that $B = \bar{B}_r(z_0) \subset U$. Then $B \times [a, b]$ is closed and bounded and hence $\dfrac{\partial g}{\partial z}$ is uniformly continuous on it.
- Hence, given $\epsilon > 0$ we can choose $0 < \delta < r$ such that

$$\left|\frac{\partial g}{\partial z}(z_1, t) - \frac{\partial g}{\partial z}(z_2, t)\right| < \frac{\epsilon}{b - a} \qquad (18)$$

for all $t \in [a, b]$ and $z_1, z_2 \in B$ such that $|z_1 - z_2| < \delta$.

Differentiation Under Integral Sign

- Now, let $0 < |z - z_0| < \delta$. Then $|h(z,t)|$ is equal to

$$\left|\int_{[z_0,z]} \left(\frac{\partial g}{\partial w}(w,t) - \frac{\partial g}{\partial z}(z_0,t)\right) dw\right| \leq \epsilon |z - z_0|,$$

by M-L inequality.

Differentiation Under Integral Sign

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by M-L inequality.

- $$\left|\frac{1}{z - z_0}\int_a^b h(z,t)dt\right| < \epsilon.$$

This proves the theorem. ♠

Vanishing derivative.

Theorem

Let U be a convex open set, $f : U \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function such that $f'(z) = 0$ for all $z \in U$. Then f is a constant function on U.

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Theorem

Let U be a convex open set, $f : U \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function such that $f'(z) = 0$ for all $z \in U$. Then f is a constant function on U.

Proof: Fix $z_0 \in U$ and for every point $z \in U$ define $g(t) = f((1-t)z_0 + tz)$. Then $g : [0,1] \to \mathbb{C}$ is a differentiable function and $g'(t) = 0$ by chain rule. This implies that $g(1) = g(0)$ That is the same as saying $f(z) = f(z_0)$. ♠