INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

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Lecture 7: Line integrals

- Path Connectivity
- Path Independence
- Simply connected domains

Differentiation Under Integral Sign: An application

In the last lecture, we have computed the integral

$$\int_{|z|=r} \frac{dz}{z} = 2\pi\imath. \tag{1}$$

The same computation goes through to give you the formula

$$\int_{|z-a|=r} \frac{dz}{z-a} = 2\pi\imath$$

also. We shall now go a step furhter and generalize this identity.

Differentiation Under Integral Sign: An application

Theorem

For all points w such that $|w - a| < r$ we have

$$\boxed{\int_{|z-a|=r} \frac{dz}{z - w} = 2\pi\imath.} \tag{2}$$

Differentiation Under Integral Sign

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$$-\int_{|z-a|=r} \frac{dz}{(z-w)^2} = \int_{|z-a|=r} \frac{d}{dz}\left(\frac{1}{z-w}\right) dz = 0.$$

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- This is true for all points w in the open disc $|z-a| < r$ which is convex. Therefore, h is a constant. ♠

Paths

The convexity is too fragile as we have seen and does not take us very far. So, we need to have a little more robust notion viz., path connectivity.

Definition

By a *path* in U we mean a continuous function $\gamma : [a, b] \to U$.
The two points $\gamma(a), \gamma(b)$ are called *initial point* and *terminal point* respectively or together called *end points*.
Often people confuse a path for its image but so far as this confusion is good let us use it.

Paths

Definition

We say a subset $A \subset \mathbb{C}$ is path connected if every pair of points in A are end-points of a path in A.

Definition

A path connected open subset U of $\mathbb{C}$ is called a **domain**

Paths

Example

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(4) $\mathbb{C} \setminus L$ *where* L *is an entire line, is* **not** *path connected.*

Paths

Remark

It can be shown that in any domain, every pair of points can be joined by a smooth parameterised curve. As an immediate consequence we have:

Theorem

Let U be a domain. If $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable and $f'(z) = 0$ for all $z \in U$. Then f is a constant on U.

Existence of Primitives

Recall from calculus of two real variables that a differential $pdx + qdy$ is called **exact** if there exist a function u with continuous partial derivatives such that $u_x = p$ and $u_y = q$; i.e.,

$$du = pdx + qdy.$$

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In that case, u is a *primitive* of $pdx + qdy$.
[More generally, we say a function g is a primitive of another function f if $g' = f$.]

Existence of Primitives

Theorem

(Path Independence) *Let U be a domain in $\mathbb{C}$, and p, q be continuous maps on U taking real or complex values. Then the following two conditions are equivalent.*

(a) The differential $pdx + qdy$ is exact on U.

(b) For all closed continuous contours γ in U we have,

$$\int_\gamma pdx + qdy = 0. \tag{3}$$

Path Independence

Proof: [By taking real and imaginary parts separately, the statement of the theorem for complex valued functions follows from that for real valued functions. Therefore, you can assume that only real valued functions appear in the proof below. However, such an assumption is not a logical necessity.]

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Let $\gamma : [a, b] \longrightarrow U$ be a contour joining z_0 and z say, given by $\gamma(t) = (x(t), y(t))$. Suppose $du = pdx + qdy$ and that (a) holds.

Path Independence

Then by definition,

$$
\begin{aligned}
&= \int_\gamma (pdx + qdy) \\
&= \int_a^b [p(\gamma(t))x'(t) + q(\gamma(t))y'(t)]\, dt \\
&= \int_a^b (u_x x'(t) + u_u y'(t))\, dt \\
&= \int_a^b \frac{d}{dt} u(x(t), y(t))\, dt \\
&= u(x(b), y(b)) - u(x(a), y(a)) \\
&= u(\gamma(b)) - u(\gamma(a)) = u(z) - u(z_0).
\end{aligned}
$$

Path Independence

Observe that we have used the fundamental theorem of integral calculus of 1-variable above. Now, if γ is closed, then $z_0 = z$ and hence $\int_\gamma (pdx + qdy) = 0$. This proves (a) $\implies$ (b).
To prove (b) $\implies$(a), fix any point $z_0 \in U$. Then for every point $z \in U$, choose a contour γ_z from z_0 to z in U.

Path Independence

Define

$$u(z) := \int_{\gamma_z} (pdx + qdy). \tag{4}$$

Let us proceed to prove that $du = pdx + qdy$, i.e., $\frac{\partial u}{\partial x} = p, \frac{\partial u}{\partial y} = q$. Given $z = (x, y) \in U$, choose sufficiently small $\epsilon > 0$, so that $(x + h, y) \in U$ for all $|h| < \epsilon$.

Path Independence

Now restrict h further, to be a real number. We have two specific ways of approaching the point $z+h$ from z_0. One is along the chosen path γ_{z+h}. The other one is to first trace γ_z and then trace the line segment $[z, z+h]$. Condition (b) implies that

$$\begin{aligned} u(z+h) &:= \int_{\gamma_{z+h}} pdx + qdy \\ &= \int_{\gamma_z} pdx + qdy + \int_{[z,z+h]} pdx + qdy. \end{aligned}$$

Path Independence

Therefore,

$$u(z+h) - u(z) = \int_{[z,z+h]} (pdx + qdy). \qquad (5)$$

Now recall that the segment $[z, z+h]$ is parameterized by

$$t \mapsto (x + th, y), 0 \leq t \leq 1.$$

Therefore, $dx = hdt$ and $dy = 0$.

Path Independence

Thus

$$\begin{aligned} & u(z+h) - u(z) \\ = \ & \int_{[z,z+h]} (pdx + qdy) = \int_0^1 p(x+th, y) h dt \\ = \ & p(x + t_0 h, y) h \end{aligned}$$

for some $0 \le t_0 \le 1$, by the mean value theorem of integral calculus of 1-real variable.

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Thereore

$$\lim_{h \to 0} \frac{u(z+h) - u(z)}{h} = \lim_{h \to 0} p(x + t_0 h, y) = p(x, y).$$

Path Independence

The proof that $\dfrac{\partial u}{\partial y} = q$ is similar, by taking $\imath h$ in place of h. ♠

Path Independence

Corollary

In the situation of theorem 3, assume further that U is a convex domain. Then (a), (b) are equivalent to the following:

(c) For all triangles T contained in U

$$\int_{\partial T} pdx + qdy = 0. \tag{6}$$

Path Independence

Proof: The implication (b) $\implies$ (c) is obvious. To prove (c) $\implies$ (a) we imitate the proof of (b) $\implies$ (a) except that we now take γ_z to be the line segment $[z_0, z]$ from z_0 to z. (This is where convexity of U is used.) Then the hypothesis (c) is enough to arrive at (5) since the closed path $\gamma_z \cdot [z, z+h] \cdot \gamma_{z+h}^{-1} = \partial T$ is the boundary of a triangle $T = \Delta(z_0, z, z+h)$ contained in U. The rest of the proof is as before. ♠

Path Independence

Remark

The function u in theorem 3, if it exists, is unique up to an additive constant. (Why?) The ambiguity in the additive constant is a cheap price we pay for the freedom we enjoy in the choice of the base point z_0.

Primitive Existence

Corollary

Primitive existence theorem: For a continuous complex valued function f defined in a domain U, the integral $\int_{\omega} f\,dz = 0$ for all closed contours ω iff f is the derivative of a holomorphic function on U.

Primitive Existence

Proof: Suppose there is a holomorphic function g such that $g' = f$. By CR equations, we have $f = g' = g_x = g_y/\imath$ and hence

$$f(z)dz = f(z)(dx + \imath dy) = g_x dx + g_y dy.$$

Therefore, from theorem 3, it follows that $\int_\omega f\,dz = 0$ for all closed contours in U.

Primitive Existence

Conversely, suppose $\int_\omega f\,dz = 0$ for all closed contours in U, then by taking $p = f$ and $q = \imath f$ in the above theorem, it follows that there exists $F : U \longrightarrow \mathbb{C}$, such that $F_x = p = f; \;\; F_y = q = \imath f.$

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Primitive Existence

Example

As seen in the example 1, in the previous section, $\displaystyle\int_{|z-a|=r} \frac{dz}{(z-a)} \neq 0$. *It follows that* $\dfrac{1}{z-a}$ *does not have a primitive in any punctured neighbourhood of* $z = a$. *Equivalently, this means that we cannot define* $\log(z-a)$ *in any punctured neighbourhood of* $z = a$, *as a single valued function. (Of course, in a small neighbourhood of any other point, it is the derivative of a holomorphic function.*

Path Independence

Remark

The function u in the above theorem is indeed obtained by definite integrals along arbitrarily fixed contours starting at an arbitrarily fixed initial point. The value of the integral depends only on the choice of this initial point. Hence or otherwise, the function u is unique up to any additive constant.

An Example

- Consider function $f(z) =$ the principle value of $\sqrt{z}$. Let us integrate this on the upper semi-circle:

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$$C_1 : e^{i\theta}, 0 \leq \theta \leq \pi.$$

- By definition, we have

$$\int_{C_1} f(z)dz = \int_0^{\pi} e^{\imath\theta/2} d(e^{\imath\theta}) = -2(\imath + 1)/3.$$

An Example

- But on the lower semi-circle $C_2 : e^{-\imath\theta}, 0 \leq \theta \leq \pi$, observe that $f(z)$ has a discontinuity at end point of the circle.

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- But on the lower semi-circle $C_2 : e^{-\imath\theta}, 0 \leq \theta \leq \pi$, observe that $f(z)$ has a discontinuity at end point of the circle.
- Therefore

$$\begin{aligned}\int_{C_2} f(z)dz &= \lim_{s\to\pi}\int_0^s e^{-\imath\theta/2}d(e^{-\imath\theta})\\ &= -\imath\lim_{s\to\pi}\int_0^s e^{-3\imath\theta/2}d\theta\\ &= \lim_{s\to\pi}\frac{2}{3}(e^{-3\imath s/2}-1) = \frac{2}{3}(\imath-1).\end{aligned}$$

An Example

Thus the two integrals are different which shows that the integral is path-dependant. This phenomenon is explained by the fact there is no continuously defined anti-derivative of $\sqrt{z}$ in a domain which 'encircles' the origin.

Jordan curves

Definition

By a Jordan curve **Jordan path** we mean a continuous function $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(t_1) \neq \gamma(t_2)$ for any two $t_1 \neq t_2$ except when they are end points of the interval. If in addition $\gamma(a) = \gamma(b)$ then we call it a **Jordan loop.** This is also known as **simple closed curve.** Especially in this context, it is beneficial to 'confuse' the path with its image.

Jordan curve theorem

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Theorem

Any simple closed curve in $\mathbb{C}$ separates $\mathbb{C}$ into two components one bounded and another unbounded.

This just means that if C is a simple closed curve then

$$\mathbb{C} \setminus C = U_1 \sqcup U_2$$

where each U_i is path connected and moreover, there is **NO** (continuous) path starting from a point in U_1 and ending in a point in U_2.

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This is also intuitively clear. But rigorous proofs of these facts are not easily obtained.