INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

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Lecture 8: Cauchy's theorem

Simply connected domains

Intergal formula

Jordan curve Theorem

Definition

By a Jordan curve **Jordan path** we mean a continuous function $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(t_1) \neq \gamma(t_2)$ for any two $t_1 \neq t_2$ except when they are end points of the interval. If in addition $\gamma(a) = \gamma(b)$ then we call it a **Jordan loop.** This is also known as **simple closed curve.** Especially in this context, it is beneficial to 'confuse' the path with its image.

Jordan curve theorem

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This just means that if C is a simple closed curve then

$$\mathbb{C} \setminus C = U_1 \sqcup U_2$$

where each U_i is path connected and moreover, there is **NO** (continuous) path starting from a point in U_1 and ending in a point in U_2.

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This is also intuitively clear. But rigorous proofs of these facts are not easily obtained.

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We then say U is **simply connected.**
We can take the following slightly different wording as the definition:

Jordan curve theorem

Definition

Let U be a domain (open connected set) in $\mathbb{C}$. We say U is **simply connected**, if the following property holds: no simple closed curve γ in U encloses any point of $\mathbb{C}$ which is not in U.

Examples

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- The **inside** of a simple close curve in $\mathbb{C}$ is simply connected.

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- So, $U \setminus \{a\}$ is **not simply connected.**
- Of course we can even punch larger round holes also to make a domain not simply connected.
- On the other hand, by filling all 'holes' in a domain you can make it simply connected.

Simply connected domains

Remark

An alternative and modern description of a simply connected domain is that every closed curve in it can be continuously shrunk to a single point. We have no time to discuss this property deeper.

Next we need to recall Green's theorem which relates an area integral to a line integral on the boundary, that you have learnt in your calculus course.

Green's Theorem

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Green's Theorem for Multi-Connected Domains: Let R be a closed and bounded domain in xy-plane whose boundary ∂R consists of finitely many piecewise smooth curves. Let $f(x,y)$ and $g(x,y)$ be functions which are continuous and have continuous partial derivatives f_y and g_x everywhere in some domain containing R. Then

$$\int\!\!\int (g_x - f_y)dxdy = \int_{\partial R} (fdx + gdy)$$

Green's Theorem

Remember that the integration on the right is being taken along the **entire** boundary curve C of R, parametrerised in such a way that the area of R lies on the left as one traces the curve in the in positve direction.

Cauchy's Theorem on a Simply Connected Domain

Theorem

Cauchy's Theorem (Version-I) *Let U be a simply connected domain in $\mathbb{C}$ and f be a holomorphic function on it. Then for any simple closed curve γ in U, we have,*

$$\int_\gamma f(z)dz = 0.$$

Proof of Cauchy's Theorem

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- Therefore, $f = u + \imath v$ is complex differentiable at every point of S and hence its real and imaginary parts u, v satisfy CR-equations.
- Also $f(z)dz = (u + \imath v)(dx + \imath dy)$ $= (udx - vdy) + \imath(vdx + udy)$.

Proof of Cauchy's Theorem continued

By Green's theorem, we have,

$$\int_\gamma f(z)dz = \int_{\partial S} f(z)dz$$

$$= \int\int_S (u_y + v_x)dxdy + \imath \int\int_R (v_y - u_x)dxdy = 0.$$

Cauchy's Theorem

Using Green's theorem for multi-connected domains, allowing curves to be a finite union of simple closed curves, and arguing as before, we obtain the following:

Cauchy's theorem version-II

Theorem

Cauchy's Theorem: version-II *Let R be a domain in $\mathbb{C}$ bounded by the oriented path ∂R, (the boundary not necessarily connected). Suppose f is holomorphic on an open set U containing $R \cup \partial R$. Then*

$$\int_{\partial R} f(z)dz = 0.$$

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- **Cauchy's Theorem III-version**
- Let U be a simply connected domain, $A \subset U$, a finite subset and $f : U \to \mathbb{C}$ be a continuous function such that $f : U \setminus A \to \mathbb{C}$ is holomorphic.
- Then for any closed contour γ in U, we have

$$\int_\gamma f(z)dz = 0. \qquad (1)$$

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- Observe that as seen before in Primitive Existence Theorem, it is enough to prove (1) for simple closed contours γ.
- We shall first prove this for the case when γ is a simple closed contour not passing through any points of A.
- Let R be the domain enclosed by γ. Then $R \subset U$. This is precisely where simple connectivity of U is used.

Proof of Cauchy's III- version

- Let $A \cap R = \{a_1, \ldots, a_k\}$. Given $\epsilon > 0$, we must show that $\left|\int_\gamma f(z)dz\right| \leq \epsilon$.

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- Choose sufficiently small $r > 0$ such that $B_r(a_j) \cap \gamma = \emptyset, 1 \leq j \leq k$ and such that $r|f(z)| < \epsilon/2\pi k$ for all z on the boundary of $B_r(a_i)$ and for $i = 1, 2, \ldots, k$.

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- (This is possible by the continuity of f at a_j's.)

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- Since $\partial S = \gamma \cup (C_1)^{-1} \cup (C_2)^{-1} \cup \cdots \cup (C_k)^{-1}$, we get

$$\int_{\gamma} f(z)dz = \sum_{i=1}^{k} \int_{C_i} f(z)dz. \tag{2}$$

Proof III-version

- Let C_j be the oriented boundary of $B_r(a_j)$ and $M_j = sup\{|f(z)| \ : \ z \in C_j\}$. Then $rM_j \leq \epsilon/2\pi k$.

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- Now by M-L inequality, it follows that

$$\left|\int_\gamma f(z)dz\right| = \left|\sum_{j=1}^{k}\int_{C_j} f(z)dz\right|$$
$$\leq \sum_{j=1}^{k} M_j L(C_j) = 2\pi r \sum_{j=1}^{k} M_j \leq \epsilon.$$

That completes the proof when $\gamma \cap A = \emptyset$.

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- Given $\epsilon > 0$ we can find a curve γ_ϵ in U, not passing through any point in A and such that
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- Given $\epsilon > 0$ we can find a curve γ_ϵ in U, not passing through any point in A and such that
$$\left| \int_{\gamma_\epsilon} f(z)dz - \int_{\gamma} f(z)dz \right| < \epsilon.$$
- This is a direct consequence of M-L inequality, the details are left to you as an exercise. By the first part, the first integral vanishes and hence the second one also should vanish. ♠

An Example

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- To see this, choose a disc D with center z_0 and contained in S.

An Example

Apply Cauchy's theorem (Version-II) to $f(z) = \frac{1}{z-z_0}$ in $R = S \setminus D$. Since

$$\partial R = \gamma \cdot (\partial D)^{-1}, \text{ we get}$$

$$\int_\gamma \frac{dz}{z - z_0} = \int_{\partial D} \frac{dz}{z - z_0} = 2\pi\imath.$$

Cauchy Integral Formula:

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- Then for every point $w \in U$, not lying on γ, we have

$$\boxed{\int_\gamma \frac{f(z)}{z-w} dz. = f(w) \int_\gamma \frac{dz}{z-w}} \tag{4}$$

Cauchy Integral Formula:

In particular, if γ is a simple closed curve enclosing a domain R in U then every point $w \in R$ we have

$$\boxed{f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz.} \tag{5}$$

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$$F(z) = \frac{f(z) - f(w)}{z - w}, \quad z \neq w.$$

  Then F is holomorphic in $U \setminus \{w\}$.
- Also, since f is holomorphic at w, $\lim_{z \to w} F(z)$ exists and is equal to $f'(w)$.
- Therefore by taking $F(w) = f'(w)$, F will be continuous at w.

Integral formula

Therefore, we can apply Cauchy's theorem (III version) to F, to conclude that

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Therefore, we can apply Cauchy's theorem (III version) to F, to conclude that

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This means that

$$\int_\gamma \frac{f(z)}{z-w}dz = f(w)\int_\gamma \frac{dz}{z-w} = 2\pi\imath f(w).$$

The latter part follows easy, from the previous example.

Integral formula for derivatives

Remark

In (5), observe that the integrand is a function of two complex variables, where w varies over the interior of R and z varies over the boundary. It is a continuous function of these variable and for each fixed z, it is a holomorphic function. Therefore, differentiation under the integral sign wrt to w is valid and we have

$$f'(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-w)^2}.$$

Integral formula for derivatives

By repeated application of this theme we obtain:

Theorem

Cauchy's Integral formula for Derivatives: *Let f be holomorphic in a domain U. Then f has derivatives of all order in U. Moreover, if C is a circle in U and z is a point inside the circle C then for all integers $n \geq 0$, we have,*

$$\boxed{f^{(n)}(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-w)^{n+1}}.} \qquad (6)$$