INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

Anant R. Shastri

22^{nd} August 2012

Cauchy's Integral Formula

- Recall that we have proved the following:

Cauchy's Integral Formula

- Recall that we have proved the following:
- Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and γ is a simple closed curve enclosing a region R in Ω then every point $w \in R$ we have

$$\boxed{f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz.} \tag{1}$$

Cauchy's Integral Formula for derivatives

We also have seen that differentiation under integral sign is valid here repeatedly, and gives us

Theorem

Cauchy's Integral formula for Derivatives: *Let f be holomorphic in a domain Ω. Then f has derivatives of all order in Ω. Moreover, if C is a circle in Ω and z is a point inside the circle C then for all integers $n \geq 0$, we have,*

$$\boxed{f^{(n)}(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-w)^{n+1}}.} \qquad (2)$$

Cauchy's Integral Formula

Remark

Behold! We have shown that a function which is once complex differentiable is differentiable any number of times. Certainly this is something that we never bargained for while launching the theory of complex differentiation. There is more to come. It is time for us to reap the harvest:

Holomorphy implies Analyticity

- Let f be a holomorphic function in a domain Ω.

Holomorphy implies Analyticity

- Let f be a holomorphic function in a domain Ω.
- In (1), the curve γ can be chosen to be the oriented boundary of a closed disc D contained in Ω.

Holomorphy implies Analyticity

- Let f be a holomorphic function in a domain Ω.
- In (1), the curve γ can be chosen to be the oriented boundary of a closed disc D contained in Ω.
- Then the formula is valid for all points w in the interior of D. Fix one such w.

Holomorphy implies Analyticity

- Let f be a holomorphic function in a domain Ω.
- In (1), the curve γ can be chosen to be the oriented boundary of a closed disc D contained in Ω.
- Then the formula is valid for all points w in the interior of D. Fix one such w.
- Let a denote the center of this disc.

Holomorphy implies Analyticty

Then we have

$$\left|\frac{w-a}{z-a}\right| < 1, \; z \in \partial D.$$

and hence the geometric series

$$\frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = \frac{1}{z-a}\left(\sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n\right)$$

is uniformly convergent on the boundary of D.

Holomorphy implies Analyticty

Therefore, we can substitute (3) in (1) and interchange the order of integration and summation to obtain

$$f(w) = \frac{1}{2\pi i}\sum_0^\infty \left(\int_{\partial D} \frac{f(z)}{(z-a)^{n+1}}\right)(w-a)^n$$

$$i.e., \qquad f(w) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(w-a)^n \tag{4}$$

which is valid for all points w inside D.

Holomorphy implies Analyticity

- Thus, we have found a power series representation for f around each point of its domain.

Holomorphy implies Analyticity

- Thus, we have found a power series representation for f around each point of its domain.
- Note that the only condition on the size of the disc to be chosen is that it should be contained in the domain R. Therefore we have following theorem.

Holomorphy implies Analyticity

Theorem

Every holomorphic function on a domain R is analytic with a unique power series representation around each point $a \in R$; the power series so obtained has radius of convergence at least as big as $sup\{r \; : \; B_r(a) \subset R\}$.

Holomorphy implies Analyticity

Definition

The power series (4), obtained in the previous theorem is called the **Taylor's series** for f around the point $z = a$. When the point $a = 0$, it is also called the **Maclaurin's series**.

Cauchy's Estimates

Theorem

Cauchy's Estimate : *If f is holomorphic in an open set containing the closure of the disc $B_r(z)$ and let $M_r = Sup\{|f(w)| : |w - z| = r\}$. Then for all $n \geq 1$ we have,*

$$|f^{(n)}(z)| \leq \frac{n! M_r}{r^n}. \tag{5}$$

Cauchy's Estimates

Proof: Take C to be the circle of radius r around z.

Cauchy's Estimates

Proof: Take C to be the circle of radius r around z. Then we have,

$$\begin{aligned} |f^{(n)}(z)| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(w)\,dw}{(w-z)^{n+1}} \right| \\ &\leq \frac{n! M_r}{2\pi r^{n+1}} \int_C |dw| = \frac{n! M_r}{r^n}. \end{aligned}$$

This proves the theorem. ♠

Liouville's Theorem

- Definition

  A function that is holomorphic on the entire plane $\mathbb{C}$ is called an *entire function.*

Liouville's Theorem

- Definition

  A function that is holomorphic on the entire plane $\mathbb{C}$ is called an *entire function.*

- Theorem

  Liouville's Theorem : *A bounded entire function is a constant.*

Liouville's Theorem

Proof: Putting $n = 1$ in the Cauchy's estimate, we obtain that $|f'(z)| \leq M_r/r$.

Liouville's Theorem

Proof: Putting $n = 1$ in the Cauchy's estimate, we obtain that $|f'(z)| \leq M_r/r$.
Since f is bounded, let M be such that $M_r \leq M$ for all r. Now take the limit as $r \longrightarrow \infty$. Therefore $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore f is a constant.

The Fundamental Theorem of Algebra:

Theorem

The Fundamental Theorem of Algebra: *Let* $p(z) = a_n z^n + \cdots + a_1 z + a_0,\ a_i \in \mathbb{C}, a_n \neq 0$ *be a polynomial function in one variable of degree* $n \geq 1$ *over the complex numbers. Then the equation* $p(z) = 0$ *has at least one solution in* $\mathbb{C}$.

Proof of FTA

Proof: Assume that $p(z)$ is never zero. It follows that $f(z) = 1/p(z)$ is differentiable everywhere, i.e., $f(z)$ is an entire function.

Proof of FTA

Proof: Assume that $p(z)$ is never zero. It follows that $f(z) = 1/p(z)$ is differentiable everywhere, i.e., $f(z)$ is an entire function.
We shall show that $f(z)$ is bounded and then from Liouville's theorem it follows that f is a constant and hence p is a constant.

Proof of FTA

Proof: Assume that $p(z)$ is never zero. It follows that $f(z) = 1/p(z)$ is differentiable everywhere, i.e., $f(z)$ is an entire function.
We shall show that $f(z)$ is bounded and then from Liouville's theorem it follows that f is a constant and hence p is a constant.
But it is easily verified that any polynomial function of positive degree is not a constant. This contradiction will prove the theorem.

Proof of FTA continued

- Now to show that f is bounded, we first show that

$$\lim_{|z|\to\infty} |p(z)| = \infty$$

Proof of FTA continued

- Now to show that f is bounded, we first show that
$$\lim_{|z|\to\infty} |p(z)| = \infty$$
- Observe that $\left|\dfrac{p(z)}{z^n}\right| \to |a_n| \neq 0$ and hence

Proof of FTA continued

- Now to show that f is bounded, we first show that
$$\lim_{|z|\to\infty} |p(z)| = \infty$$
- Observe that $\left|\frac{p(z)}{z^n}\right| \to |a_n| \neq 0$ and hence
- $\lim_{|z|\to\infty} |p(z)| = \lim_{|z|\to\infty} |a_n||z^n| = \infty.$

The Fundamental Theorem of Algebra:

- Hence we can find large r such that

$$|z| > r \Longrightarrow |f(z)| < 1.$$

The Fundamental Theorem of Algebra:

- Hence we can find large r such that

$$|z| > r \Longrightarrow |f(z)| < 1.$$

- On the other hand, by continuity of f there exists K such that

$$|f(z)| \leq k \quad \forall \quad |z| \leq r.$$

The Fundamental Theorem of Algebra:

- Hence we can find large r such that

$$|z| > r \Longrightarrow |f(z)| < 1.$$

- On the other hand, by continuity of f there exists K such that

$$|f(z)| \leq k \quad \forall \quad |z| \leq r.$$

- If $M = max\{1, K\}$, then

$$|f(z)| \leq M, \quad \forall \quad z \in \mathbb{C}.$$

This completes the proof of FTA. ♠

Zeros of Holomorphic Functions

- Let f be holomorphic in a domain D, $a \in D$. Assume that all the derivatives of f vanish at a, and $f(a) = 0$. Then from the Taylor series representation, it follows that $f(z) = 0$ for all z in $B_r(a) \subset D$. In fact, we have,

Zeros of Holomorphic Functions

- Let f be holomorphic in a domain D, $a \in D$. Assume that all the derivatives of f vanish at a, and $f(a) = 0$. Then from the Taylor series representation, it follows that $f(z) = 0$ for all z in $B_r(a) \subset D$. In fact, we have,

- Theorem

  Let f be a holomorphic function in a domain Ω. Suppose there is a point $a \in \Omega$ such that $f^{(k)}(a) = 0$ for all $k \geq 0$. Then $f \equiv 0$ on Ω.

Zeros of Holomorphic Functions

- **Proof:** Given any point $b \in \Omega$ choose a continuous map

$$\gamma : [0,1] \to \Omega, \text{ such that } \gamma(0) = a, \gamma(1) = b.$$

Zeros of Holomorphic Functions

- **Proof:** Given any point $b \in \Omega$ choose a continuous map

$$\gamma : [0,1] \to \Omega, \text{ such that } \gamma(0) = a, \gamma(1) = b.$$

- Let

$$A = \{t \in [0,1] \ : \ f^{(k)}(\gamma(t)) = 0 \ \ \forall k \geq 0\}.$$

(Here, by $f^{(0)}$ we mean f itself.)

Zeros of Holomorphic Functions

- **Proof:** Given any point $b \in \Omega$ choose a continuous map

$$\gamma : [0,1] \to \Omega, \text{ such that } \gamma(0) = a, \gamma(1) = b.$$

- Let

$$A = \{t \in [0,1] \;:\; f^{(k)}(\gamma(t)) = 0 \;\; \forall k \geq 0\}.$$

(Here, by $f^{(0)}$ we mean f itself.)

- By the hypothesis, $0 \in A$. It is enough to prove that $1 \in A$.

Zeros of Holomorphic Functions

- Let $s = Sup\,A$. Then $0 \leq s \leq 1$.

Zeros of Holomorphic Functions

- Let $s = Sup\, A$. Then $0 \leq s \leq 1$.
- Also there exists $t_n \in A$ such that $t_n \to s$.

Zeros of Holomorphic Functions

- Let $s = Sup\, A$. Then $0 \leq s \leq 1$.
- Also there exists $t_n \in A$ such that $t_n \to s$.
- Since $f^{(k)}$ is continuous, and γ is also continuous, it follows that

$$f^{(k)}(\gamma(s)) = \lim_{n\to\infty} f^{(k)}(\gamma(t_n)) = 0.$$

Zeros of Holomorphic Functions

- Let $s = Sup\, A$. Then $0 \leq s \leq 1$.
- Also there exists $t_n \in A$ such that $t_n \to s$.
- Since $f^{(k)}$ is continuous, and γ is also continuous, it follows that

$$f^{(k)}(\gamma(s)) = \lim_{n\to\infty} f^{(k)}(\gamma(t_n)) = 0.$$

- Since this is true for all $k \geq 0$, this implies that $s \in A$.

Zeros of Holomorphic Functions

- Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which f is represented by its Taylor's series.

Zeros of Holomorphic Functions

- Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which f is represented by its Taylor's series.
- But all coefficients of Taylor series are zero, because $s \in A$.

Zeros of Holomorphic Functions

- Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which f is represented by its Taylor's series.
- But all coefficients of Taylor series are zero, because $s \in A$.

Zeros of Holomorphic Functions

- Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which f is represented by its Taylor's series.
- But all coefficients of Taylor series are zero, because $s \in A$.
- It follows that $f(z) = 0$ for all $z \in B_r(\gamma(s))$. But then it also follows that $f^{(k)}(z) = 0$ for all $z \in B_r(\gamma(s))$ and for all $k \geq 0$.

Zeros of Holomorphic Functions

- By continuity of γ, it follows that there exists $\epsilon > 0$ such that $\gamma(s-\epsilon, s+\epsilon) \subset B_r(\gamma(s))$. Therefore, $(s-\epsilon, s+\epsilon) \cap [0,1] \subset A$. Since s is the supremum of A, this is possible only if $s = 1$. Therefore, $1 \in A$ as claimed. ♠

Zeros of Holomorphic Functions

- The above theorem thus leads us to the following definition exclusively for holomorphic functions:

Zeros of Holomorphic Functions

- The above theorem thus leads us to the following definition exclusively for holomorphic functions:
- Let f be a holomorphic function which is not identically zero in a region Ω. Let $a \in \Omega$.

Zeros of Holomorphic Functions

- The above theorem thus leads us to the following definition exclusively for holomorphic functions:
- Let f be a holomorphic function which is not identically zero in a region Ω. Let $a \in \Omega$.
- Then by the above theorem there exists a non negative integer k such that $f^{(j)}(a) = 0$ for all $j < k$ and $f^{(k)}(a) \neq 0$.

Zeros of Holomorphic Functions

- The above theorem thus leads us to the following definition exclusively for holomorphic functions:
- Let f be a holomorphic function which is not identically zero in a region Ω. Let $a \in \Omega$.
- Then by the above theorem there exists a non negative integer k such that $f^{(j)}(a) = 0$ for all $j < k$ and $f^{(k)}(a) \neq 0$.
- We say that a is a *zero of order* k of f. Of course *a zero of order zero is not a zero at all!*

Zeros of Holomorphic Functions

- In terms of the above terminology, we may say that for a holomorphic function f, which is not identically zero, there are no zeros of infinite order.

Zeros of Holomorphic Functions

- In terms of the above terminology, we may say that for a holomorphic function f, which is not identically zero, there are no zeros of infinite order.
- Thus we have

$$f(z) = (z-a)^k g(z)$$

where $g(z)$ is some holomorphic function in Ω, and $g(a) \neq 0$. By continuity of $g(z)$, this implies that in a neighborhood of a, a is the only zero of $f(z)$.

Zeros of Holomorphic Functions

Let us make a formal definition, before summing up what we have seen just now.

Definition

A subset K of Ω is said to be *isolated in* Ω, if $\forall\, z \in \Omega$, we can find a disc Δ around z, such that either $\Delta \cap K = \{z\}$ or $= \emptyset$.

Zeros of Holomorphic Functions

Let us make a formal definition, before summing up what we have seen just now.

Definition

A subset K of Ω is said to be *isolated in* Ω, if $\forall\, z \in \Omega$, we can find a disc Δ around z, such that either $\Delta \cap K = \{z\}$ or $= \emptyset$.

Observe that an isolated set is a closed subset of Ω and does not have any limit points in Ω.

Zeros of Holomorphic Functions

Let us make a formal definition, before summing up what we have seen just now.

Definition

A subset K of Ω is said to be *isolated in* Ω, if $\forall\, z \in \Omega$, we can find a disc Δ around z, such that either $\Delta \cap K = \{z\}$ or $= \emptyset$.

Observe that an isolated set is a closed subset of Ω and does not have any limit points in Ω.
Also observe that every subset of an isolated set is isolated.

Isolated Sets

Example

Consider the following subsets of $\mathbb{C}$.

(1) Any finite subset is isolated.

Isolated Sets

Example

Consider the following subsets of $\mathbb{C}$.
(1) Any finite subset is isolated.
(2) The set of integers $\{m + \imath n \ : \ m, n \in \mathbb{Z}\}$ *is isolated.*
(3) The set $\{1/m \ : \ m \in \mathbb{Z}^+\}$ *is* **not** *isolated.*

Zeros of Holomorphic Functions

We can sum up our observations in the following theorem.

Theorem

Let f be a holomorphic function not identically zero, in a region Ω. Then the zero set of f

$$\mathcal{Z}_f := \{z \in \Omega \ : \ f(z) = 0\}$$

is an isolated subset of Ω.

Zeros of Holomorphic Functions

We can sum up our observations in the following theorem.

Theorem

Let f be a holomorphic function not identically zero, in a region Ω. Then the zero set of f

$$\mathcal{Z}_f := \{z \in \Omega \ : \ f(z) = 0\}$$

is an isolated subset of Ω.

As an immediate corollary we have:

Identity Theorem

- Theorem

  Identity Theorem: *Let f and g be holomorphic functions on a region Ω. Suppose $K \subset \Omega$ is such that for every $z \in K$, $f(z) = g(z)$ and K has a limit point in Ω. Then $f \equiv g$ on Ω.*

Identity Theorem

- Theorem

  Identity Theorem: *Let f and g be holomorphic functions on a region Ω. Suppose $K \subset \Omega$ is such that for every $z \in K$, $f(z) = g(z)$ and K has a limit point in Ω. Then $f \equiv g$ on Ω.*

  - **Proof:** For the function $f - g$, the set K happens to be a subset of the set of all zeros.

Identity Theorem

- Theorem

  Identity Theorem: *Let f and g be holomorphic functions on a region Ω. Suppose $K \subset \Omega$ is such that for every $z \in K$, $f(z) = g(z)$ and K has a limit point in Ω. Then $f \equiv g$ on Ω.*

  - **Proof:** For the function $f - g$, the set K happens to be a subset of the set of all zeros.
  - Since this set has a limit point, it follows that the set of all zeros of $f - g$ is not an isolated set. Hence, by the above theorem, $f - g \equiv 0$ on Ω. ♠

Zeros of Holomorphic Functions

- Typical instances of the above corollary are:

Zeros of Holomorphic Functions

- Typical instances of the above corollary are:
- (i) Two holomorphic functions agreeing on a non empty subregion, will have to agree on the whole region.

Zeros of Holomorphic Functions

- Typical instances of the above corollary are:
- (i) Two holomorphic functions agreeing on a non empty subregion, will have to agree on the whole region.
- (ii) Even if they agree on an arc which is not a single point then they agree on the whole region.

Zeros of Holomorphic Functions

Thus we see that a holomorphic function is well determined once we know its values on a set which has a limit point. This does not necessarily mean that we can effectively compute its value everywhere. In contrast, by Cauchy's integral formula, we could actually know the value of a holomorphic function inside a disc, the moment we know it on the boundary circle. However, both the results have their own importance and it is perhaps not wise to say that one result is superior to the other.

An Example

- Consider

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \tag{6}$$

An Example

- Consider

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \tag{6}$$

- It can be easily seen that f has continuous derivatives of all order.

An Example

- Consider

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \tag{6}$$

- It can be easily seen that f has continuous derivatives of all order.
- (The only point to be worried about is at the origin. Differentiate the function on the positive interval and take limit as $t \to 0^+$.)

An Example

- Consider
$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \tag{6}$$
- It can be easily seen that f has continuous derivatives of all order.
- (The only point to be worried about is at the origin. Differentiate the function on the positive interval and take limit as $t \to 0^+$.)
- All the derivatives at 0 vanish.

An Example

- Consider

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \tag{6}$$

- It can be easily seen that f has continuous derivatives of all order.
- (The only point to be worried about is at the origin. Differentiate the function on the positive interval and take limit as $t \to 0^+$.)
- All the derivatives at 0 vanish.
- Yet the function is not identically zero.