INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

Department of Mathematics

MA 205 - Complex Analysis

Exercise Bank

SET I

Q.1-S Prove that three distinct points z_1, z_2, z_3 in the plane form the vertices of an equilateral triangle iff $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$. Deduce that if w_1, w_2, w_3 are points dividing the three sides of triangle $\Delta(z_1, z_2, z_3)$ in the same ratio, then the triangle $\Delta(w_1, w_2, w_3)$ is equilateral iff the triangle $\Delta(z_1, z_2, z_3)$ is so.

Q.2-T If z_1, z_2, z_3 are three distinct complex numbers of equal moduli, prove that

$$2 \arg \frac{z_2 - z_1}{z_3 - z_1} = \arg \frac{z_2}{z_3}.$$

Which theorem in school geometry does this correspond to?

Q.3-S Prove that the centres of the squares described outwardly on the sides of a plane quadtrilateral are the vertices of a quadrilateral whose diagonals are equal in length and are perpendicular to each other.

Q.4-T Let $A_1, A_2, \ldots, A_n$ be the vertices of a regular n-gon. Assume that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$. Determine n. (JEE 1994)

Q.5-L An **isometry** of the complex plane is a mapping $T : \mathbb{C} \longrightarrow \mathbb{C}$ which preserves all distances. All translations, rotations and reflections (into straight lines) are isometires. The first two preserve while the last ones reverse orientations. Prove that:

(a) a straight line L in $\mathbb{C}$ has an equation of the form $b\overline{z} + \overline{b}z = c$ where b is a complex number with $|b| = 1$ and c is real and further that the reflection, say z^*, of a point z into L is given by $\overline{z}b + z^*\overline{b} = c$.

(b) every orientation preserving isometry is given by a function of the form $f(z) = az + t$ where a, t are complex numbers with $|a| = 1$. [Hint: First consider the case where the origin is fixed.]

(c) every orientation reversing isometry of the plane is given by a function of the form $f(z) = a\overline{z} + t$ with $|a| = 1$.

(d) By a **glide reflection,** we mean a reflection in a line followed by a non zero translation along the line of reflection. Show that a rigid motion is a glide reflection iff it is given by $f(z) = a\bar{z} + t$ with $|a| = 1$ and $\Im(t/\sqrt{a}) \neq 0$.

(This gives a complete classification of all isometries of the plane and also makes it easier to decide, for example, when the composite of two rotations is a rotation, and when it is a translation.)

Q.6-S For every positive integer n, prove that

(i) $\left[1-\binom{n}{3}+\binom{n}{4}-\binom{n}{6}+\cdots\right]^2+\left[\binom{n}{1}-\binom{n}{3}+\binom{n}{5}+\cdots\right]^2=2^n$

(ii) $1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\dfrac{1}{2}+\dfrac{\sin(\frac{2n+1}{2})\theta}{2\sin\frac{\theta}{2}}(0<\theta<2\pi)$

(iii) $\sin\theta+\sin 2\theta+\cdots+\sin n\theta=\dfrac{1}{2}\cos\dfrac{\theta}{2}-\dfrac{\cos(\frac{2n+1}{2})\theta}{2\sin\frac{\theta}{2}}(0<\theta<2\pi)$

(iv) $(1-z_1)(1-z_2)\cdots(1-z_{n-1})=n$ where $z_1,z_2,\ldots,z_{n-1}$ are the n^{th} roots of unity other than 1.

(v) $\sin\dfrac{\pi}{n}\sin\dfrac{2\pi}{n}\cdots\sin\dfrac{(n-1)\pi}{n}=\dfrac{n}{2^{n-1}}$. (JEE favourite) [Hint: Use (iv).]

Q.7-L Using $|z|^2=z\bar{z}$, prove that
(i) $|z_1+z_2|\le|z_1+|z_2|$,
(ii) $|z_1w_1+z_2w_2|\le\sqrt{|z_1|^2+|z_2|^2}\sqrt{|w_1|^2+|w_2|^2}$ and
(iii) $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$.

Interpret (i) and (iii) geometrically.

Q.8-L By an example show that, in general, $\text{Arg }z_1+\text{Arg }z_2\neq\text{Arg }(z_1z_2)$ where $\text{Arg}\,z$ is the principal value of the argument of z.

Q.9-T If $z_1z_2\neq 0$, prove that $\text{Re }(z_1\overline{z}_2)=|z_1|\,|z_2|$ if and only if $\arg z_2=2n\pi$ for some integer n. When this happens show further that

(i) $|z_1+z_2|=|z_1|+|z_2|$ (ii) $|z_1-z_2|=|\,|z_1|-|z_2|\,|$.

Q.10-L For two complex numbers z_1,z_2 regarded as vectors in the plane, show that the dot and cross product are respectively given by

(i) $z_1\cdot z_2=\Re(z_1\bar{z}_2)$; (ii) $z_1\times z_2=\Im(\bar{z}_1z_2)\mathbf{k}$.

Q.11-L **Give an example of a sequence of complex numbers $\{z_n\}$ which converges to a complex number $z\neq 0$, yet $\{\text{Arg }z_n\}$ does not converge to $\text{Arg }z$. However, if $z_n\to z(\neq 0)$ then show that there exist $\theta_n\in\arg z_n$ such that $\theta_n\to\theta\in\arg z$.**

Set II

1. (L) Show that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is multiplication by a complex number iff $a = d$ and $c = -b$.

2. (L) Establish the following generalization of Cauchy-Reimann equations. "If $f(z) = u + iv$ is differentiable at a point $z_0 = z_0 + iy_0$ of a domain G, then

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}, \quad \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s} \quad (*)$$

at (x_0, y_0) where $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial n}$ denote directional differentiation in two orthogonal directions s and n at (x_0, y_0), such that n is obtained from s by making a counter-clockwise rotation."

3. (T) Let $f = u + iv$ be a holomorphic function around z_0 and suppose $f'(z_0) \neq 0$. Consider the curves

$$C_1 : u = \text{Re } (f(z_0)); C_2 : v = \Im(f(z_0))$$

passing through z_0. Show that C_1 and C_2 are perpendicular to each other at z_0.

4. (L) If $f(z) = \frac{\bar{z}^2}{z}, z \neq 0$ and $f(0) = 0$, show that Cauchy-Riemann equations are satisfied at $z = 0$, but $f'(0)$ does not exist.

5. (T) Determine points at which the following functions are complex differentiable.
(i) $f(z) = xy + iy$ (ii) $g(z) = e^y e^{ix}$.

6. (T) If $f(z)$ is holomorphic and $|f(z)|$ is a constant in a domain D, then show that $f(z) = c$, a constant in D.

7. (S) Show that the following functions are harmonic and find a harmonic conjugate for each of them:
(i) $u_1(x, y) = 2x(1 - y)$. (ii) $u_2(x, y) = \sinh x \sin y$.

8. (S) If $f(z)$ is holomorphic in a domain D, show that $|f(z)|^2$ is not harmonic unless $f(z)$ is constant.

9. (L) Show that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ at $z_0 = (x_0, y_0) \neq (0,0)$ are equivalent to $u_r = \frac{1}{r}v_\theta$ and $v_r = \frac{1}{r}u_\theta = -v_r$ at $x_0 + iy_0 = r_0 e^{i\theta_0}$. Hence show that if $f(z) = u + iv$

$$f'(z_0) = e^{-i\theta_0}(u_r + iv_r) = -\frac{i}{z_0}(u_\theta + iv_\theta)$$

10. (T) Let f be a holomorphic function and $\phi : \mathbb{C} \to \mathbb{R}$ be a function with continuous second order partial derivatives. Show that

$$\nabla^2(\phi \circ f) = |f'|^2(\nabla^2(\phi) \circ f.$$

11. (S) Let f be a holomorphic function and ϕ be a harmonic function. Show that $\phi \circ f$ is harmonic.

12. (T) Show that three points in the plane representing z_1, z_2, z_3 are collinear iff

$$\begin{vmatrix} 1 & z_1 & \bar{z_1} \\ 1 & z_2 & \bar{z_2} \\ 1 & z_3 & \bar{z_3} \end{vmatrix} = 0.$$

13. (S) Show that four points in the plane representing $z_1, z_2, z_3, z_4 \in \mathbb{C}$ are either collinear or concyclic iff

$$\begin{vmatrix} 1 & z_1 & \bar{z_1} & z_1\bar{z_1} \\ 1 & z_2 & \bar{z_2} & z_2\bar{z_2} \\ 1 & z_3 & \bar{z_3} & z_3\bar{z_3} \\ 1 & z_4 & \bar{z_4} & z_4\bar{z_4} \end{vmatrix} = 0.$$

14. (0) (**Gauss-Lucas**) Show that the roots of the derivative $p'(z)$ of a polynomial $p(z)$ all lie in the convex hull of the roots of $p(z)$.

SET III

1. (L) Give an example of a sequence of complex numbers $\{z_n\}$ which converges to a complex number $z \neq 0$, yet $\{\text{Arg } z_n\}$ does not converge to Arg z. However, if $z_n \to z(\neq 0)$ then show that there exist $\theta_n \in \arg z_n$ such that $\theta_n \to \theta \in \arg z$.

2. (L) **Hemachandra Numbers** For any positive integer n, let H_n denote the number of poetic meters (patterns) having the fixed duration n counting short syllables as one beat and long syllables as two beats. For example, in the names 'Amitabh' and Gangooli', there are a total of 5 syllables each and the patterns are $1-1-2-1$

and $2-2-1$. Clearly $H_1 = 1$ and $H_2 = 2$. Hemachandra [1] noted that since the last syllable is either of one beat or two beats it follows that $H_n = H_{n-1} + H_{n-2}$ for all $n \geq 3$. These numbers were known to Indian poets, musicians and percussionists as **Hemachandra numbers.**

Define $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 2$. Note that $F_n = H_{n-2}, n \geq 2$. These F_n are called **Fibinacci numbers.** [2] (Thus the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.)

Using power series, obtain a closed form expression for the n-th Fibonacci number F_n. These numbers are defined recursively by the relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial values $F_0 = 0$ and $F_1 = 1$. (Thus the first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, ...)

3. (S) Let r be a fixed positive integer. Suppose we have an unlimited supply of r types of objects. For a positive integer n, let a_n be the number of ways to choose n objects (repetitions being allowed freely). Show that a_n is precisely the coefficient of z^n in the expansion of $(1-z)^{-r}$. Hence find a closed formula for a_n.

4. (L) Let $p(z)$ be a polynomial of degree > 0. Show that for every $M > 0$ there exists $R > 0$ such that for all $z \in \mathbb{C}$ with $|z| > R$ we have $|p(z)| > M$. (This is precisely what we mean by saying: $p(z) \to \infty$ as $z \to \infty$.)

5. (T) Find the behavior of e^z as $|z| \to \infty$ along $\arg z = 0, \pi/2, \pi$. In particular, verify that e^z does not have the property mentioned in the previous exercise.

6. (T) Prove $|\cos z| \geq \sinh |y|$, where $z = x + iy$.

7. (T) Find all values of z for which (a) $\cos z$ (b) $\sin z$ are real.

8. (S) Show that all solutions of (a) $\cos z = 0$ (b) $\sin z = 0$ are real.

9. (S) Solve $\ln z = \frac{1}{2}\pi i$.

10. Find the principal value of (i) (L) $(1+i)^i$; (ii) (T) 3^{3-i}.

[1] Hemachandra Suri (1089-1175) was born in Dhandhuka, Gujarat. He was a Jain monk and was an adviser to king Kumarapala. His work in early 11 century is already based on even earlier works of Gopala.

[2] Leonardo Pisano (Fibonacci) was born in Pisa, Italy (1175-1250) whose book Liber abbaci introduced the Hindu-Arabic decimal system to the western world. He discovered these numbers at least 50 years later than Hemachandra's record.

11. Show that :
   (i) (S) $\sin^{-1} z = -i \ln(iz \pm \sqrt{1-z^2})$; (ii) (L) $\cos^{-1} z = -i \ln(z \pm \sqrt{z^2-1})$;
   (iii) (T) $\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)$; (iv) (S) $\sinh^{-1} z = \ln(z \pm \sqrt{z^2+1})$;
   (v) (S) $\cosh^{-1} z = \ln(z \pm \sqrt{z^2-1})$; (vi) (S) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$
   (vii) (S) $2\cot 2z = \cot z + \cot(z + \frac{\pi}{2})$.

Set IV

1. (T) Suppose $f(t) = u(t) + iv(t)$ is a continuous complex-valued function on an interval $[a, b]$. Prove that $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$.

2. (L) Using the last exercise, prove a stronger version of the $M - L$ inequality, viz. $|\int_C f(z)dz| \leq \int_C |f(z)||dz|$.

3. Integrate z^2 along (a) (T) the line segment from 0 to i, (b) (S) the arc of the parabola $y = x^2$ from 0 to $2 + 4i$, both directly and by finding a primitive of the integrand.

4. (L) Let $\sqrt{z}$ be the principal value of the square root of z. Evaluate $\int \frac{dz}{\sqrt{z}}$ along (a) the upper semicircle $|z| = 1$, (b) the lower semicircle $|z| = 1$. Why do they differ?

5. (S) Evaluate $\int_C |z|\bar{z}dz$ where C is the closed contour consisting of the line segment from -1 to 1 and the semicircle $|z| = 1, y \geq 0$, taken in the counterclockwise direction.

6. (T) Suppose $f(z)$ is holomorphic and $f'(z)$ is continuous in a domain containing a closed curve C. (The hypothesis about continuity of f' is redundant but we are not in a position to prove this.) Prove that $\int_C \overline{f(z)} f'(z)dz$ is purely imaginary.

7. (S) Assume $f(z)$ is holomorphic (with $f'(z)$ continuous) and satisfies the inequality $|f(z)-1| < 1$, thropughout a domain D, Prove that $\int_C \frac{f'(z)}{f(z)}dz = 0$ for every closed curve C in D.

8. (L) Prove that a domain D is simply connected if and only $\int_C f(z)dz$ is path-independent for *every* function $f(z)$ which is holomorphic in D.

SET-V

1. (T) For each of the following functions, examine whether Cauchy's theorem can be applied to evaluate the integrals around the unit circle taken counterclockwise. Hence or otherwise evaluate the integrals.
   (a) $\text{Ln}\,(z+2)$ (b) $\dfrac{1}{|z|^3}$ (c) $|z|$ (d) e^{-z^2} (e) $\tanh z$ (f) $\bar{z}$ (g) $\dfrac{1}{z^3}$

2. Find :
   (a) (T) $\displaystyle\int_C \frac{z^2-z+2}{z^3-2z^2}dz$, where C is the boundary of the rectangle with vertices $3 \pm i, -1 \pm i$ traversed clockwise.
   (b) (S) $\displaystyle\int_C \frac{\sin z}{z+3i}dz$, $C : |z-2+3i| = 1$. (counterclockwise)

3. Evaluate $\displaystyle\int_B f(z)dz$ where $f(z)$ is (a) (T) $\dfrac{z+2}{\sin \frac{z}{2}}$ (b) (S) $\dfrac{z}{1-e^z}$ where B is the boundary of the domain between $|z| = 4$ and the square with sides along $x = \pm 1, y = \pm 1$, oriented in such a way that the domain always lies to its left.

4. C is the unit circle traversed counterclockwise. Integrate over C,
   (a)(T) $\dfrac{e^z-1}{z}$ (b)(S) $\dfrac{z^3}{2z-i}$ (c) (S) $\dfrac{\cos z}{z-\pi}$ (d)(S) $\dfrac{\sin z}{2z}$.

5. (S) Integrate $\dfrac{1}{z^4-1}$ over (a) $|z+1| = 1$, (b) $|z-i| = 1$, each curve being taken counterclockwise. [Hint: Resolve into partial fractions.]

6. (T) Let C be $|z| = 3$ in the counterclockwise sense. For any z with $|z| \neq 3$, let $g(z) = \displaystyle\int_C \frac{2w^2-w-2}{w-z}dw$. Prove that $g(2) = 8\pi i$. Find $g(4)$.

7. Let γ be any smooth closed curve, not passing through $z_0 \in \mathbb{C}$.
   (a) (T) Show that the integral $\omega = \displaystyle\int_\gamma \frac{dz}{z-z_0}$ is an integer multiple of $2\pi i$.
   (The integer $\eta(\gamma; z_0) := \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{dz}{z-z_0}$ is called the *winding number* of the closed path γ around the point z_0.)
   (b) (T) Let C be any circle in $\mathbb{C}$. Compute $\eta(C, z)$ for any point not on C.
   (c) (S) Let γ be any closed curve contained in the interior of the upper half plane. Compute $\eta(\gamma, 0)$.
   (d) (S) Let γ be closed curve contained in a disc. Compute $\eta(\gamma, z)$ for any point z outside the disc.

(e) (T) For any integer n, define $\gamma_n : [0,1] \to \mathbb{C}$ by $\gamma_n(t) = e^{2\pi int}$. Compute $\eta(\gamma_n; 0)$. (Observe that γ_n actually goes around the unit circle n times.)

(f) (S) Given any closed curve γ in $\mathbb{C}$ show that there exist points $z \in \mathbb{C}$ such that $\eta(\gamma; z) = 0$.

(g) (S) Generalize the above result to any piecewise smooth closed curve.

8. Obtain the power series expansion of the following functions around the origin and compute their radius of convergence.
(a) Ln $(1+z)$ (b) arctan z

9. Let $f(z) = \sum_n a_n z^n$ in $|z| < r$. The function f is said to be **even** (or (**odd**) if $f(-z) = f(z)$ (respectively, if $f(-z) - f(z)$ for all $|z| < r$. Show that $a_{2n-1} = 0$ (respectively $a_{2n} = 0$) for all n.

10. (a) Show that $g(z) = \frac{z}{e^z - 1}$ has power series representation around 0 valid in $|z| < 2\pi$. Write $g(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$. The numbers B_n are called **Bernoulli numbers**.
(b) Verify that $h(z) := g(z) + \frac{z}{2}$ is an even function and use it to conclude that $B_1 = -1/2$ and $B_{2n+1} = 0$ for $n \geq 1$.
(c) Comparing the coefficients of the identity

$$1 = \left(\frac{e^z - 1}{z}\right)\left(\frac{z}{e^z - 1}\right) = \left(\sum_1^{\infty} \frac{z^{n-1}}{n!}\right)\left(\sum_0^{\infty} \frac{B_n z^n}{n!}\right)$$

prove the identity

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0.$$

(d) Compute $B_k, k \leq 16$.
(e) Compute $\limsup_n \sqrt[n]{\frac{B_n}{n!}}$.
(f) Obtain power series representation for *tan* z and z*cot* z around 0.

11. **Taylor's Expansion:** Let $f(z)$ be holomorphic in Ω, containing a. Show that, for any positive integer n, we can write

$$\begin{aligned} f(z) = f(a) + \frac{f^{(1)}(a)}{1!}(z-a) \quad &+ \quad \frac{f^{(2)}(a)}{2!}(z-a)^2 + \cdots \\ &+ \quad \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{(n-1)} + f_n(z)(z-a)^n \end{aligned}$$

where $f_n(z)$ is also holomorphic in Ω. Moreover, show that for each closed disc D containing a in its interior and contained in Ω and for all points z in the interior of D, $f_n(z)$ can be represented by a line integral,

$$f_n(z) = \frac{1}{2\pi\imath}\int_{\partial D}\frac{f(w)}{(w-a)^n(w-z)}dw$$

where, ∂D is the boundary circle of D traced in the counter clockwise sense.

SET VI

1. (L) **Fundamental Theorem of Algebra** Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_i \in \mathbb{C}, a_n \neq 0$ be a polynomial function in one variable of degree $n \geq 1$ over the complex numbers. Then show that the equation $p(z) = 0$ has at least one solution in $\mathbb{C}$.

2. Given a real number α not equal to a non negative integer, use Newton's binomial series of $(1+z)^\alpha$ to show that

$$\limsup_{n\longrightarrow\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\right|^{1/n} = 1$$

provided it exists.

3. Obtain the power series expansion of the following functions around the origin and compute their radius of convergence.
(a) (T) $\frac{1}{\sqrt{1-z^2}}$ (b) (S) $\arcsin\, z$.

4. (L) **Identity Theorem:** Let f and g be holomorphic functions on a domain Ω. Suppose there exists a sequence of **distinct points** $z_n \in \Omega$ such that $z_n \to w \in \Omega$ and $f(z_n) = g(z_n), \quad n \geq 1$. Then $f(z) = g(z)$ for all $z \in \Omega$.

5. (T) Let $f, g : \Omega \to \mathbb{C}$ be holomorphic functions on a non empty domain Ω. Suppose $fg \equiv 0$. Show that $f(z) = 0$ for all $z \in \Omega$ OR $g(z) = 0$ for all $z \in \Omega$. Give an example to show that this property does not hold for C^∞-functions.

6. Let f be an entire functon. Suppose there exist constants $M, K > 0$ and a positive integer n such that $|f(w)| \leq K|w|^n$ for all $|w| > M$. Then show that f is a polynomial of degree $\leq n$.

7. Let f, g be entire functions $g(z)$ being never equal to 0. Suppose $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Then show that $f(z) = cg(z)$ for all $z \in \mathbb{C}$ where c is some constant.

8. (L) Let now $z = a$ be an isolated singularity of f. We say a is a pole of f if

$$\lim_{z \to a} |f(z)| = \infty.$$

Show that if $z = a$ is a pole of f then there exists a positive integer k such that in a disc around a, $\lim_{z \to a}(z-a)^{k+1} f(z) = 0$. The least such integer k satisfying the above condition is called the **order of the pole** of f at $z = a$. If the order $k = 1$ then the pole is called a **simple pole.** If $k > 1$, then the pole is called a **multiple pole.**

9. (S) Let f have a simple pole at z_0 and g be holomorphic. Then $R_{z_0}(fg) = g(z_0)R_{z_0}(f)$.

10. Determine all the singularities, type of singularities, order of poles and residues at such poles, if any, for the following functions.
(a)(L) $\tan z$ (b)(L) $\cot z$ (c)(S) $\frac{3}{2-z}$ (d)(T) $\frac{z}{z^3-1}$
(e)(S) $\frac{1}{(z^2-1)^2}$ (f)(T) $\frac{1}{(1-\cos z)^2}$ (g)(S) $\frac{\sinh z}{z^4}$ (h)(T) $z \sin\left(\frac{1}{z}\right)$.

11. The discussion of isolated singularity can be carried out for the point $z = \infty$ as well. To begin with we need that the function is defined and holomorphic in a neighborhood of infinity, i.e., $f(z)$ is holomorphic in $|z| > M$ for some sufficiently large M. We say that ∞ is a removable singularity or a pole of f iff

$$\lim_{z \to \infty} \left| \frac{f(z)}{z^{n+1}} \right| = 0$$

for some integer $n \geq 0$.
(a)(L) Show that ∞ is a removable singularity or a pole of f iff 0 is a removable singularity or a pole for $g(z) = f(1/z)$. Accordingly, we shall assign the order of the pole at infinity also, viz., the order of the pole at ∞ for f is the order of the pole of $g(w) = f(1/w)$ at 0.
(b)(L) What value(s) of n will tell you that ∞ is a removable singularity?
(c)(L) Show that for a polynomial function of degree d, ∞ is a pole of order d.
(d)(T) Show that an entire function has a pole of order d at ∞ iff it is a polynomial of degree d.
(e)(T) Show that a meromorphic function has a removable singularity or a pole at ∞ iff it is a rational function.

12. Determine the location and type of singularities of the following functions in the extended complex plane. Also find their principal parts near the singularities and residue at the singularity.
(a) (T) $\frac{1}{(z+a)^3}$ (b) $z^2 + z^{-1}$ (c) $\cos z$ (d) $\frac{\sin z}{z}$. (e) $\exp\left(\frac{1}{z-1}\right)$.

13. (S) Try compute the residue of $\exp(1/z-1)/\exp z - 1$ at $z=1$; try computing the principal part. Why the methods employed in the previous exercise fail?

SET VII

Q.1-L Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ where $-1 < a < 1$. Do the problem both without and with the method of residues. Can the method of residues be used to evaluate $\displaystyle\int_0^{\pi} \frac{d\theta}{1+a\sin\theta}$?

Q.2-S A similar problem about the integrals $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ and $\displaystyle\int_0^{\pi} \frac{d\theta}{1+a\cos\theta}$.

Q.3-S Prove that $\displaystyle\int_0^{\pi} \sin^{2n}\theta \, d\theta = \frac{(2n)!\pi}{2^{2n}(n!)^2}$. Do the problem without and with the method of residues. [*Hint*: For the latter, first obtain a reduction formula for the integral, in terms of the discrete parameter n.]

Q.4-L Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$ both with and without residues. Compare the two methods.

Q.5-T Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1}$ using residues.

Q.6-S Evaluate $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$ using residues.

Q.7-T What difficulty do you encounter in evaluating $\displaystyle\int_0^{\infty} \frac{dx}{x^3+1}$ by the method of the last two problems? Prove that this integral can be evaluated by applying the residue theorem to an integral over the boundary of a sector bounded by the circle $\{z : |z| = R\}$ where $R > 1$ and the the rays $\{z : z = t,\ t \geq 0\}$ and $\{z : z = t\omega,\ t \geq 0\}$ and then letting $R \to \infty$. (Here ω denotes a cube root of unity $\neq 1$.)

Q.8-L Evaluate $\displaystyle\int_{\infty}^{\infty} \frac{\cos x\,dx}{x^2+1}$.

Q.9 Prove that :

(i) (T) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right), \quad (a > b > 0)$

(ii) (S) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax\,dx}{(x^2+b^2)^2} = \frac{\pi}{2b^3}(1+ab)e^{-ab}$ where $a > 0$ and $b > 0$. Can this be obtained from (i)?

Q.10-L Prove Jordan's inequality $\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R}$ for $R > 0$ and use it to evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx$. Also obtain the same result without Jordan's inequality by integrating along the suitably indented boundary of a rectangle with vertices at $-X_1, X_2, -X_1+iY$ and X_2+iY where X_1, X_2, Y tend to ∞ independently of each other.

Q.11-S Use Jordan's inequality to show that the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{x\sin x\, dx}{x^2+2x+2}$ is $\frac{\pi}{e}(\sin 1 + \cos 1)$.

Q.12 Some improper integrals can be evaluated by combining the method of residues with some other known improper integrals (which may have been obtained by very different methods). For example, using the well-known result $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ prove that:

(i)-T $\int_0^{\infty} e^{-x^2} \cos(2bx)\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$ $(b > 0)$ by integrating the function e^{-z^2} around the boundary of the rectangle with vertices at $\pm a$ and $ib \pm a$ where $a > 0$ and then letting a tend to ∞. Also derive it without residues by treating the integral as a function of the continuous parameter b and obtaining a differential equation for it by differentiating (w.r.t. b) under the integral sign. Note the conceptual similarity with the alternate solution to Q.3.

(ii)-S $\int_0^{\infty} \sin(x^2)\, dx = \int_0^{\infty} \cos(x^2)\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$ (**Fresnel's integrals**) by integrating e^{iz^2} along the boundary of the sector $\{(x,y) : x \geq 0, 0 \leq y \leq x, x^2+y^2 \leq R^2\}$ and then letting $R \to \infty$.

SET VIII

Q.1-S Let $a \geq 0, b \geq 0$. Following the same contour as used in the evaluation of $\int_0^{\infty} \frac{\sin x}{x}\, dx$, evaluate $\int_0^{\infty} \frac{\cos(ax)-\cos(bx)}{x^2}\, dx$ and deduce that $\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$. Can the first integral be evaluated by separately evaluating $\int_0^{\infty} \frac{\cos(ax)}{x^2}\, dx$ and $\int_0^{\infty} \frac{\cos(bx)}{x^2}\, dx$?

Q.2-T Let $a > 0, b > 0$. Using the formula

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\, dx = \ln(b/a)$$

(which can be obtained by reversing the order of integration in the double integral $\int_a^b \int_0^{\infty} e^{-xy} dx\, dy$) evaluate $\int_0^{\infty} \frac{\cos(ax)-\cos(bx)}{x}\, dx$. Why can't this be done by taking the same contour as in the last problem?

Q.3-S Prove that $\int_0^\pi \ln(\sin\theta)\, d\theta = -\pi \ln 2$ by integrating the principal logarithm of $1 - e^{2iz} = -2ie^{iz}\sin z$ along the boundary of a square with vertices at $0, \pi, iR$ and $\pi + iR$ (suitably indented at the corners 0 and π) and letting $R \to \infty$. (The integral can also be evaluated by observing that

$$\begin{aligned}\int_0^\pi \ln(\sin\theta)d\theta &= 2\int_0^{\pi/2} \ln(\sin\theta)d\theta = \int_0^{\pi/2} \ln(\sin\theta)d\theta + + \int_0^{\pi/2} \ln(\cos\theta)d\theta \\ &= = \int_0^{\pi/2} \ln(\sin 2\theta)d\theta - \frac{\pi}{2}\ln 2 = \frac{1}{2}\int_0^\pi \ln(\sin\theta)d\theta - \frac{\pi}{2}\ln 2\end{aligned}$$

if we assume that certain formulas for definite integrals continue to hold even for improper integrals.)

Q.4-L A **keyhole contour** is the boundary of the portion of an annulus (with the inner circle having a very small radius) obtained by deleting the part lying between two radii which are inclined at a very small angle to each other. (As a result, these two radii are often replaced by line segments parallel to the acute angle bisector between them.) Evaluate the integral $\int_0^\infty \frac{dx}{1+x+x^2}$ by integrating the function $f(z) = \frac{\log z}{1+z+z^2}$ along a keyhole contour around the origin, where $\log z$ is to have an argument lying between 0 and 2π.

Q.5-S Evaluate $\int_0^\infty \frac{(\ln x)^2}{1+x^2}\, dx$ by integrating $f(z) = \frac{(\log z)^3}{1+z^2}$ around a keyhole contour around the origin.

Q.6-L Prove that conformality (along with continuity of the partial derivatives implies holomorphicity). Prove, in fact, that prservation of angles between curves alone is sufficient, while constancy of the scaling factor in all directions is almost sufficient in that, a function with this property is either holomorphic or conjugate holomorphic.

Q.7-S Determine the angle through which the tangents to all curves passing through the point $2 + i$ are rotated under the transformation $w = z^2$.

Q.8-T Under the transformation $w = z^2$, determine (i) the image of a sector of a circle of radius r centred at 0, (ii) the image of the semi-infinite strip $\{x + iy : 0 \le x \le k, y \ge 0\}$ and (iii) the inverse image of the rectangle $\{u+iv : a \le u \le b, c \le v \le d\}$. Sketch.

Q.9-S Find the image of the strip $\{x + iy : x > 0, 0 < y < 2\}$ under $w = iz + 1$. Sketch the strip and the image.

Q.10-S Prove that the transformation $w = 1/z$ maps a straight line or a circle depending upon whether the line passes through 0 or not. What about the image of a circle? Find and sketch the images of the discs of unit radii centred at (i) 0, (ii) 1 and (iii) $1 + i$.

SET IX

Q.1-L Prove that every fractional linear transformation (FLT) is a composite of a translation, a rotation, a dilation (or a contraction) and an inversion.

Q.2-L Let z_1, z_2, z_3, z_4 be four distinct, extended complex numbers. Define their **cross ratio** to be $[z_1, z_2, z_3, z_4] = \dfrac{z_1 - z_3}{z_1 - z_4}\dfrac{z_2 - z_4}{z_2 - z_3}$. Modify this definition suitably if any of the points is ∞. Note that the order of the points matters. Prove that the cross ratio is preserved under fractional linear transformations.

Q.3-S For every complex number w other than $1, 0$, and ∞, prove that $w = (w, 1, 0, \infty)$. Hence show that given any three distinct complex numbers z_2, z_3, z_4, the unique L.F.T. which takes them to $1, 0$ and ∞ respectively is given by $T(z) = (z_1, z_2, z_3, z_4)$. View Theorem 2 on p. 694 in Kreyszig in this light.

Q.4-T Prove that every F.L.T. maps a circle onto a circle, where a 'circle' means either a straight line or a circle. [*Hint:* Use Problem 1 above with Problem 10 in Tutorial 8.]

Q.5-S Prove that four distinct complex numbers z_1, z_2, z_3, z_4 lie on a 'circle' if and only if the cross ratio (z_1, z_2, z_3, z_4) is a real number.

Q.6-T Let z_2, z_3, z_4 be distinct points on a 'circle' C in $\hat{\mathbb{C}}$. Two points z and z^* are said to be **symmetric** w.r.t. C if $(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$. Prove that if C is a straight line then z and z^* are symmetric w.r.t. C if and only if they are reflections of each other into C while if C is an (ordinary) circle with centre M and radius r and P, Q are points represented by z, z^* respectively then z, z^* are symmetric w.r.t. C if and only if P, Q lie on the same ray from M and $MP.MQ = r^2$. (As a consequence, it follows that symmetry is independent of the choice of the three points z_2, z_3, z_4 on the 'circle'.)

Q.7-S Prove that symmetry is preserved under F.L.T.'s.

Q.8-L Prove that every F.L.T. which maps the (open) unit disc onto itself is of the form $T(z) = c\dfrac{z - a}{1 - \bar{a}z}$ for some complex numbers a, c with $|a| < 1$ and $|c| = 1$.

Q.9-T Map the region between the circles $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ conformally onto an infinite strip and then onto a half plane.

Q.10-S Prove that an entire function whose real part is bounded below is a constant function. How will you generalise this result?

Q.11-S Map the region in the first quadrant bounded by the coordinate axes and the hyperbola $y = 1/x$ conformally onto the upper half plane.

Q.12-O Besides the Riemann sphere, there is another interpretation of the extended complex plane $co\hat{m}x$ called the **complex projective line**. It makes it easier to see what is really 'linear' in an L.F.T. Consider an ordered pair (z_1, z_2) of (ordinary) complex numbers z_1, z_2 at least one of which is non-zero. If $z_2 \neq 0$, we associate the complex number z_1/z_2 to this pair. Otherwise we associate ∞. Note that the same complex number may be associated to many different pairs. Now suppose an extended complex number z corresponds to the pair (z_1, z_2). Then for any complex numbers a, b, c, d with $ad \neq bc$, we have $\frac{az+b}{cz+d} = \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{w_1}{w_2}$ where $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. This is a well-defined, non-singular linear transformation of the complex two dimensional vector space into itself.

Q.13-O Let D be a domain, $w \notin D$ be any point. Suppose that $f(z) = \sqrt{z - w}$ is a well defined holomorphic function. Put $f(D) = D_1$.
(i) Show that $f : D \to D_1$ is a biholomorphic mapping. (ii) Show that if $z \in D_1$ then $-z \notin D_1$.
(iii) Show that there is an open ball $B_r(w)$ such that $B_r(w) \cap D_1 = \emptyset$.
(iv) Put $f_2(z) = \frac{1}{z-w}$. and $f_2(D_1) = D_2$. Show that $f_2 : D_1 \to D_2$ is a biholomorphic mapping.
(v) Show that $D_2 \subset B_{1/r}(0)$.
(vi) Conclude that D is biholomorphic to a bounded domain. (vi) Obtain a generalization of the statement in Exercise 10 based on this.

Indeed, it can be proved that any domain such as D above is biholomorphic with the unit disc itself. But the proof is not easy and beyond the present course.

SET X

Q.2-T Prove that the transformation $w = \sin z$ maps the semi-infinite strip $-\pi/2 \leq x \leq \pi/2$ bijectively onto the upper half plane $v \geq 0$. Identify the points where this transformation is conformal. Show that the images of the horizontal segments lie along confocal ellipses. Similarly study the transformations $w = \cos z$, $w = \sinh z$ and $w = e^z$.

Q.3-S Obtain transformations that are bijective and conformal (except possibly at the boundary points) which map the upper half plane to (i) the unit disc, (ii) an infinite strip (iii) a semi-infinite strip, (iv) an infinite sector of a given angular width and (v) the region $\{(u, v) : u > 0, v > 0, uv \geq 1\}$.

Q.4-S Example 7 on p. 801 along with Problem 9 on p. 803 of Kreyszig.

Q.5-L Find the steady state temperature $T(x, y)$ in a thin semi-infinite plate $y \geq 0$ whose faces are insulated and whose edge $y = 0$ is kept at temperature 0 except the segment $-1 < x < 1$ which is kept at temperature 1.

Q.6-T A similar problem for the semi-infinite strip $y \geq 0, -\pi/2 \leq x \leq \pi/2$ whose vertical sides are kept at a constant temperature 0 and the horizontal side at a constant temperature 1.

Q.7-S Same problem for a plate in the form of an infinite quadrant if the segment of unit length at the end of one edge is insulated, the rest of that edge is kept at a temperature T_1 and the other edge is kept at a temperature T_2.

Q.8-T Prove the **Mean Value Property** for harmonic functions which says that if $u(z)$ is harmonic in a domain D containing the closed disc $\{z : |z - z_0| \leq R\}$, then $u(z_0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(z_0 + Re^{i\alpha})d\alpha$. Deduce the **maximum principle** for harmonic functions which says that unless u is a constant, it has neither a maximum nor a minimum in D.

Q.9-L The formula in Problem 8 is a very special case of Poisson's formula. Prove, however, that Poisson's formula can be derived from it.

Q.10-S If $f(z)$ is holomorphic and non-constant in a domain D, prove that $\max\{|f(z)| : z \in D\}$ does not exist while $\min\{|f(z)| : z \in D\}$, if it exists, must be 0. (The first part is called the **maximum modulus principle** for holomorphic functions.)

Q.11-T If $f(z)$ is holomorphic in $|z| < 1$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $|z| < 1$, prove that $|f(z)| \leq |z|$ for all $|z| < 1$. (**Schwarz's Lemma**)