INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

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Department of Mathematics
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Course Name MA205 Complex Analysis
Instructor: Anant R Shastri
Office Location: Math Building I Floor 102C:
Phone: 7477
Consultation Time: 6-00pm to 7-00pm on TUESDAYS

The Field $\mathbb{R}$ of Real numbers

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- V *The law of additive inverse*: Given any $a \in \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that $a + x = 0$.
- VI *The law of multiplicative inverse*: Given $a \in \mathbb{R}, a \neq 0$, there exists a unique $x \in \mathbb{R}$ such that $ax = 1$.

Furthermore, there is a *total ordering* '$<$' on $\mathbb{R}$, compatible with the above arithmetic operations, which makes $\mathbb{R}$ into an *ordered field.* Recall that $<$ is a total ordering means that:

- VII given any two real numbers a, b, either $a = b$ or $a < b$ or $b < a$. The ordering $<$ is compatible with the arithmetic operations means the following:

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- VII given any two real numbers a, b, either $a = b$ or $a < b$ or $b < a$. The ordering $<$ is compatible with the arithmetic operations means the following:
- VIII $a < b \Longrightarrow a + c < b + c$ and $ad < bd$ for all $a, b, c \in \mathbb{R}$ and $d > 0$.

Definition of Complex Numbers

We define the algebra of complex numbers $\mathbb{C}$ to be the set of formal symbols $x + \imath y, \quad x, y \in \mathbb{R}$ together with the addition and multiplication defined as follows:

$$(x_1 + \imath y_1) + (x_2 + \imath y_2) = (x_1 + x_2) + \imath(y_1 + y_2);$$

$$(x_1 + \imath y_1)(x_2 + \imath y_2) = (x_1 x_2 - y_1 y_2) + \imath(x_1 y_2 + y_1 x_2).$$

$$\boxed{\imath^2 + 1 = 0; \quad \text{i.e., } \imath^2 = -1.}$$

Observe that a complex number is well-determined by the two real numbers, x, y viz., $z := x + \imath y$. These are respectively called the *real part* and *imaginary part* of z. We write:

$$\boxed{\boxed{\Re z = x; \quad \Im z = y.}} \tag{1}$$

If $\Re(z) = 0$, we say z is *(purely) imaginary* and similarly if $\Im(z) = 0$, then we say z is *real*. The only complex number which is both real and purely imaginary is 0. Observe that, according to our definition, every real number is also a complex number.

```
equating the real and the imaginary parts of the two sides
of an equation
```

is indeed a part of the definition of complex numbers and will play a very important role.

Theorem

There is no total ordering $<$ *on* $\mathbb{C}$ *such that*

$$a < b \Longrightarrow a + c < b + c, a, b, c \in \mathbb{C}$$

and

$$a < b, 0 < c \Longrightarrow ac < bc, a, b, c \in \mathbb{C}.$$

Proof: If so, either $0 < \imath$ or $\imath < 0$. Consider the first case. By multiplying both sides by $\imath$, we get $0 < -1$. Therefore, upon multiplying both sides of this inequality by -1, we get $0 < 1$. Now adding the two inequalities we get $0 < 0$, which is absurd. Similarly, you can verify that the assumption $\imath < 0$ would lead to a contradiction.

Conjugation

Definition

Following common practice, for $z = x + \imath y$ we denote by $\overline{z} = x - \imath y$ and call it the (complex) *conjugate* of z. and call it
the *conjugate* of z.

$$\boxed{\Re(z) = \frac{z + \overline{z}}{2}; \qquad \Im(z) = \frac{z - \overline{z}}{2\imath}.} \tag{2}$$

$$\boxed{\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}, \quad \overline{\overline{z}} = z.} \tag{3}$$

The Absolute Value

Definition

Given $z \in \mathbb{C}$, $\quad z = a + \imath b$, we define its *absolute value* (*length*) $|z|$ to be the non-negative square root of $a^2 + b^2$, i.e.,

$$|z| := \sqrt{(a^2 + b^2)}.$$

Remark

$|z|^2 = z\bar{z}$. Therefore
$z \in \mathbb{C}, \quad |z| \neq 0 \Longleftrightarrow z \neq 0$.
Also, for $z \neq 0$,

$$z^{-1} = \bar{z}|z|^{-2}.$$

Basic Identities and Inequalities

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- (B3) $|\Re(z)| \leq |z|$ (resp. $|\Im(z)| \leq |z|$); equality holds iff $\Im(z) = 0$ (resp. $\Re(z) = 0$).
- (B4) *Cosine Rule:*

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\Re(z_1\overline{z_2}).$$

Basic Identities: Continued

- (B5) **Parallelogram Law :**

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

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- (B6) **Triangle inequality :** $|z_1 + z_2| \le |z_1| + |z_2|$ and equality holds iff one of the z_j is a non-negative multiple of the other.
- (B7) **Cauchy-Scwartz Inequality**

$$\left|\sum_{j=1}^{n} z_j w_j\right|^2 \le \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right).$$

- Cartesian Coordinate Form

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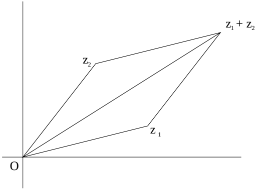
- Polar form

How to add Complex numbers Geometrically

The picture below illustrates how to add two complex numbers geometrically.



The **'parallelogram law'** (B5) now becomes:

```
The sum of the squares of the lengths of the diagonals of
a parallelogram is equal to the sum of the squares of the
lengths of the sides.
```

Given $(x, y) = z \neq 0$, the angle θ, measured in counter-clockwise sense, made by the line segment $[0, z]$ with the positive real axis is called the *argument* or *amplitude* of z :

$$\theta = arg\ z.$$

$$\boxed{\boxed{x = r\cos\theta; \qquad y = r\sin\theta}} \tag{4}$$

Let us temporarily set-up the notation

$$\boxed{E(\theta) := \cos\theta + \imath\sin\theta.} \tag{5}$$

Then the complex number $z = x + \imath y$ takes the form

$$z = r(\cos\theta + \imath\sin\theta) =: rE(\theta).$$

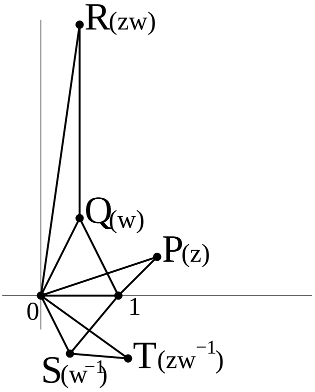
Observe $|z| = r$. Now let $z_1 = r_1 E(\theta_1), \quad z_2 = r_2 E(\theta_2)$. Using additive identities for sine and cosine viz.,

$$\boxed{\begin{aligned} \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2, \\ \cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \end{aligned}} \tag{6}$$

we obtain

$$\boxed{z_1 z_2 = r_1 r_2 E(\theta_1 + \theta_2).} \tag{7}$$

Geometric Multiplication of complex numbers:



In the picture above, various triagles are similar. It tells you how to multiply two complex numbers. For instance triangles $01P$ and $0QR$ are similar

If we further remind ourselves that the argument can take values (in radians) between 0 and 2π, then the above identity tells us that $\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$ provided $z_1 \neq 0, \quad z_2 \neq 0$.
Put $z_j = r_j E(\theta_j)$ for $j = 1, 2$, and let θ be the angle between the vectors represented by these points. Then $z_1 \bar{z}_2 = r_1 r_2 E(\theta_1 - \theta_2)$ and hence $\Re(z_1 \bar{z}_2) = r_1 r_2 \cos\theta$. Thus,

$$\boxed{\cos\theta = \frac{\Re(z_1 \bar{z}_2)}{|z_1 z_2|}.} \tag{8}$$

Now, we can rewrite the cosine rule as:

$$|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2 r_1 r_2 \cos\theta. \tag{9}$$

Note that by putting $\theta = \pi/2$ in (9), we get Pythagoras theorem.

Remark

Observe that given $z \neq 0$, *arg z* is a multi-valued function. Indeed, if θ is one such value then all other values are given by $\theta + 2\pi n$, where $n \in \mathbb{Z}$. Thus to be precise, we have

$$arg\ z = \{\theta + 2\pi n\ :\ n \in \mathbb{Z}\}$$

This is the first natural example of a ' multi-valued function'. We shall come across many multi-valued functions in complex analysis, all due to this nature of arg z. However, while carrying out arithmetic operations we must 'select' a suitable value for arg from this set. One of these values of *arg z* which satisfies $-\pi < arg\ z \leq \pi$ is singled out and is called the *principal value* of arg z and is denoted by *Arg z*.

Equation of a line:

- Let $ax + by + c = 0$ represent a line in cartesian coordinates, $a, b, c \in \mathbb{R}, \quad (a, b) \neq (0, 0)$.

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- Thus, we see that the general equation of a line in the plane can be given by complex numbers as:

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- This line is perpendicular to w.

Equation of a circle:

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- Equivalently

$$\boxed{\boxed{|z-w| = r.}} \quad (12)$$

Rigid Motions

Definition

By a **rigid motion** or an **isometry** of the plane, we mean a mapping $f : \mathbb{C} \to \mathbb{C}$ which preserves distances, i.e.,

$$|f(z) - f(w)| = |z - w| \text{ for all } z, w \in \mathbb{C}.$$

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- (i) all points fixed. (ii) has no fixed points (iii) fixes exactly one point.
- All the three of them preserve orientation.
- The last one changes orientation and fixes precisely a line.

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- Composite two rigid motions is again a rigid motion.

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- Therefore, we obtain,

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- Therefore, we obtain,

$$z^* - z = sw, s \in \mathbb{R}; \quad \& \quad w(\overline{z^* + z}) + \bar{w}(z^* + z) = 2t.$$

- Substitute $z^* = z + sw$ in the latter and use the fact $w\bar{w} = 1$ to obtain $s = t - (w\bar{z} + \bar{w}z)$. Simply to get

$$\boxed{z^* = wt - w^2\bar{z}.} \tag{13}$$

Equivalently

$$\boxed{w\bar{z} + \bar{w}z^* = t.} \tag{14}$$

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- Let $f : \mathbb{C} \to \mathbb{C}$ be a RM.
- Put $b = -f(0)$ and define $g(z) = f(z) + b$. Then g is also a RM. and $g(0) = 0$.
- Now $|g(1)| = 1$. So, put $a = g(1)$ and define $h(z) = a^{-1}g(z)$. Then h is RM and $h(0) = 0, h(1) = 1$, and $h(\imath) = \pm\imath$.

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  Then we claim that $f(z) = z$ for all z.
- **Case 2:** Assume that $h(\imath) = -\imath$. Put $\bar{h}(z) = \overline{h(z)}$. Then $\bar{h}$ is a RM and $\bar{h}(0) = 0, \bar{h}(1) = 1, \bar{h}(\imath) = \imath$. So, we are in case 1. ♠.

We know all of them

Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a rigid motion such that $f(0) = 0$. Then there exist unique $a, b \in \mathbb{C}$ with $|a| = 1$ such that

- $$f(z) = az + b, \forall z \in \mathbb{C}$$

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- If $g(z) = z$ this means $f(z) = z + b$ and hence f is a translation.
- If $g(z) = az, |a| = 1, a \neq 1$ then $f(z) = az + b$ and we can solve for $az + b = z$ which means f has a fixed point. So, this case does not occur.

- Finally if g is a reflection in a line passing through the origin, then $g(z) = a\bar{z}$ which can be expressed as

$$w\bar{z} + \bar{w}z^* = 0$$

where $z^* = g(z) = a\bar{z}$. Here $w^2 = -a$.

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- Now let us resolve $b = b_1 + b_2$ in the direction of w and perpedicular to it. It follows that adding b_1 to g is the same as taking reflection in the line parallel to L and passing through b_1. Whereas adding b_2 moves the point parellel to L. Since f has no fixed points, it follows that $b_2 \neq 0$.

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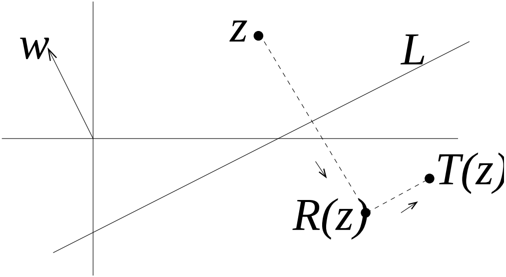
- This is the same as saying $b \neq \imath s\sqrt{a}$, for some $s \in \mathbb{R}$ i.e., $\Re(b/\sqrt{a}) \neq 0$.

Glide Reflection



Definition

By a glide-reflection we mean a RM which is a reflection in a line followed by a translation by a non zero vector in the direction of L.

It is easy to see that a glide-reflection does not have any fixed point and does not preserve the orientation. The converse follows from what we have seen above.

Glide Reflection

Theorem

Let f be RM of the plane. If it fixes one point then it a rotation about that point (and hence preserves orientation). If it fixes no points then f is a glide reflection (and reverses orientation).

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- " Sail to $\cdots$ North latitude and $\cdots$ West longitude where thou wilt find a deserted island. There lieth a large medow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine.

STORY TIME

- There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn *right* by a right angle and take the same number of steps. Put here a spike in the ground.

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- Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn *left* by a right angle and see that thou takest the same number of steps, and put another spike into the ground.
- Dig half-way between the spikes; the treasure is there."