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Review of Sequences and Series

This is a brief summary of the theory of convergence of sequences and series. We assume that you are already familiar with the general theory of convergence of sequences and series, such as elementary properties, various convergence tests etc.. So, here we recall them very briefly to the extent that is needed for immediate purpose. There are several good books from which you can learn this topic better. One such reference is [K]. For the time being, going through the material here should be enough.

Review of Sequences and Series

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Definition

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- It is customary to denote a sequence f by $\{s(n)\}$ or $\{s_n\}, \{a_n\}, \{b_n\}$ etc. $\{s_0, s_1, s_2, s_3, \ldots,\}$ etc.

Review of Sequences and Series

Definition

A sequence $\{z_n\}$ of numbers is said to be *convergent* to the *limit* w if for every $\epsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$, we have,

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It follows that the limit of a sequence if it exists, is unique.

For if w_1 and w_2 are two limits of a sequence $\{z_n\}$ then, given $\epsilon > 0$ we can choose n_0 as above, so that for $n \geq n_0$ we have, $|z_n - w_1| < \epsilon$, and $|z_n - w_2| < \epsilon$. Hence, $|w_1 - w_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we must have $w_1 = w_2$.

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  to represent the limit of the sequence $\{z_n\}$.
- If a sequence is not convergent then it is said to be *divergent.*
- If $\{a_n\}, \{b_n\}$ are convergent sequences, then the sequences $\{a_n + b_n\}$ and $\{za_n\}$ are also so with limits given by

$$\lim_{n \longrightarrow \infty}(a_n + b_n) = \lim_{n \longrightarrow \infty} a_n + \lim_{n \longrightarrow \infty} b_n; \quad \lim_{n \longrightarrow \infty} za_n = z \lim_{n \longrightarrow \infty} a_n.$$

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- It is fairly obvious that if $z_n = a_n + \imath b_n$ where, $a_n, b_n \in \mathbb{R}$, then $z_n \longrightarrow w$ iff $a_n \longrightarrow \Re(w)$ and $b_n \longrightarrow \Im(w)$.

Cauchy's Criterion

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Theorem

A sequence $\{z_n\}$ *of numbers is convergent iff it satifies the following* **Cauchy's Criterion:**
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- A sequence satisfying Cauchy's criterion is called a *Cauchy sequence.* The theorem can then be stated as: *a sequence of real or complex numbers is Cauchy iff it is convergent.*

Cauchy's Criterion

As an important application of sequences, we have:

Theorem

Let $f : X \longrightarrow \mathbb{C}$ be any function defined on a subset of $\mathbb{C}$. For any $z \in X$, f is continuous at z iff for every sequence $z_n \longrightarrow z$, we have, $f(z_n) \longrightarrow f(z)$.

Example

Recall that we define the principal value ARG of a complex number to be the value of θ where $z = \cos\theta + \imath \sin\theta, -\pi < \theta \leq pi$.

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Example

Recall that we define the principal value ARG of a complex number to be the value of θ where $z = \cos\theta + \imath \sin\theta, -\pi < \theta \leq pi$. Put $\theta_n = -\pi + 1/n$, and $z_n = \cos\theta_n + \imath \sin\theta_n$. Let $f(z) = Arg\, z$. Then at $z = -1$ you will see that f is not continuous. Because $f(z_n) \longrightarrow -\pi$ whereas $z_n \longrightarrow -1$ and $f(-1) = \pi$.

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Definition

The *sequence of partial sums* associated to the above series is defined to be $s_n := \sum_{k=1}^{n} z_k$. We say the series is *convergent* if the associated sequence $\{s_n\}$ of partial sums is convergent. In that case, if s is the limit of this sequence, then we say s is the *sum* of the series and write

$$\sum_n z_n := s.$$

Review of Sequences and Series

Basic Properties: If $\sum_n z_n$ and $\sum_n w_n$ are convergent series then:

- for any complex number λ, we have, $\sum_n \lambda z_n$ is convergent and

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- $\sum_n (z_n + w_n)$ is convergent and

$$\sum_n (z_n + w_n) = \sum_n z_n + \sum_n w_n.$$

Review of Sequences and Series

Cauchy's convergence criterion can be applied to series also. This yields:

Theorem

A series $\sum_n z_n$ of real or complex numbers is convergent iff for every $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$ and for all $p \geq 0$, we have,

$$\left|z_n + z_{n+1} + \cdots + z_{n+p}\right| < \epsilon.$$

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It follows that if a series is convergent, then its n^{th} term z_n tends to 0. However, this is not a sufficient condition for convergence of the series, as illustrated by the series $\displaystyle\sum_n \frac{1}{n}$.

Review of Sequences and Series

- More generally, all notions and results that we have for sequences have corresponding notions and results for series also, via the sequence of partial sums of the series. Thus, once a result is established for a sequence, the corresponding result is available for series as well and vice versa, without specifically mentioning it.

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- An absolutely convergent series is convergent.
- Converse is not true: $\displaystyle\sum_n (-1)^n \frac{1}{n}$.
- The notion of absolute convergence plays a very important role throughout the study of convergence of series. As an illustration we shall obtain the following useful result about the convergence of the product series.

Sequences and Series

Definition

Given two series $\sum_n a_n, \sum_n b_n$ the *Cauchy product* of these two series is defined to be $\sum_n c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

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Theorem

If $\sum_n a_n, \sum_n b_n$ are two absolutely convergent series then their product series is absolutely convergent and its sum is equal to the product of the sums of the two series:

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right). \tag{1}$$

Sequences and Series

Proof: (Under the extra assumption that both the series consist of non negative real numbers only, the conclusion of the theorem is obvious. Though we need not use this result, it gives us a useful hint how to go about with the proof in the general case.)

Sequences and Series

Put

$$R_n = \sum_{k \geq n} |a_k|; \quad T_n = \sum_{k \geq n} |b_k|.$$

Clearly,

$$\sum_0^n |c_n| \leq \sum_0^n \sum_0^n |a_k||b_l| \leq \left(\sum_0^n |a_k|\right)\left(\sum_0^n |b_l|\right) \leq R_0 T_0, \quad \text{for all } n.$$

This means that $t_n = \sum_0^n |c_n|$ is a monotonically increasing sequence of real numbers which is bounded above and hence is convergent. Therefore the series $\sum_n c_n$ is absolutely convergent. Further,

$$\left| \sum_{k \leq 2n} c_k - \left(\sum_{k \leq n} a_k\right)\left(\sum_{k \leq n} b_k\right) \right| \leq R_0 T_{n+1} + T_0 R_{n+1},$$

since the terms that remain on the LHS after cancellation are of the form $a_k b_l$ where either $k \geq n+1$ or $l \geq n+1$. Upon taking the limit as $n \longrightarrow \infty$, we obtain (1). ♠

Review of Sequences and Series

An important property of an absolutely convergent series is the following:

Theorem

Let $\sum_n z_n$ be an absolutely convergent series. Then every rearrangement $\sum_n z_{\sigma_n}$ of the series is also absolutely convergent. Moreover, we have $\sum_n z_n = \sum_n z_{\sigma_n}$.

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Recall that a rearrangement $\sum_n z_{\sigma_n}$ of $\sum_n z_n$ is obtained by taking a bijection $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$.

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- Then $a_n^\pm \geq 0$, and $a_n = a_n^+ - a_n^{-1}$
- And $|a_n^\pm| \leq |a_n|$. Therefore both the series $\sum_n a_n^+, \sum_n a_n^-$ are convergent. By addition theorem so is the series

$$\sum_n a_n = \sum_n (a_n^+ - a_n^-) = \sum_m a_n^+ - \sum_n a_n^-.$$

Review of Sequences and Series

- This is then true for any re-arrangement series $\sum_n a_{\sigma_n}$ as well. Therefore, $\sum_n a_{\sigma_n} = \sum_n a^+_{\sigma_n} - \sum_n a^-_{\sigma_n}$

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- But each of the series $\sum_n a^\pm_{\sigma_n}$ be a convegent series of non negative terms, is equal to $\sum_n a^\pm_n$ respectively.
- Therefore $\sum_n a_{\sigma_n} = \sum_n a^+_n - \sum_n a^-_n = \sum_n a_n$.

Uniform Convergence

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In the definition of the convergence, it is obvious that the largeness of n_0 depends on the smallness of ϵ. Naturally, it would also vary from sequence to sequence.
If we want the function f to behave well, it is clearly necessary that there must be some control on the variation of n_0 from sequence to sequence.
One way to do this leads us to the notion of **uniform convergence.**

Uniform Convergence

Definition

Let $\{f_n\}$ be sequence of complex valued functions on a set A. We say that it is *uniformly convergent* on A to a function $f(x)$ if for every $\epsilon > 0$ there exists n_0, *independent of* $x \in A$ such that for all $n \geq n_0$, we have, $|f_n(x) - f(x)| < \epsilon$, for all $x \in A$.

Uniform Convergence

Remark

Observe that if $\{f_n\}$ is uniformly convergent, then for each $x \in A$, we have, $f_n(x) \longrightarrow f(x)$. This is called *point-wise convergence* of the sequence of functions. It is fairly easy to see that if A is a finite set then pointwise convergence at all the points of A implies uniform convergence. Thus the interesting case of uniform convergence occurs only when A itself is an infinite set. Then pointwise convergence does not imply uniform convergence as seen in the following example.

Pointwise does not imply uniform

Example

A simple example of a sequence which is point-wise convergent but not uniformly convergent is $f_n : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *given by* $f_n(x) = 1/nx$*. It is uniformly convergent in* $[\alpha, \infty)$ *for all* $\alpha > 0$ *but not so in* $(0, \alpha)$*.*

Uniform Convergence

Theorem

A sequence of complex valued functions $\{f_n\}$ *is uniformly convergent iff it is* uniformly Cauchy *i.e., given* $\epsilon > 0$ *there exists* n_0, *such that for all* $n \geq n_0$ *and for all* $x \in A$, *we have,*

$$|f_{n+p}(x) - f_n(x)| < \epsilon.$$

Remark

Likewise, there are other theorems, such as sum of uniform convergent sequence is uniformly convergent etc.

Uniform Convergence

Whatever we have done so far for sequences applies to series of functions as well by merely making the corresponding statements about the sequence of partial sums. For examples a series $\sum_n f_n$ of functions is uniformly Cauchy iff the sequence $\{s_n(x)\}$ of partial sums is uniformly Cauchy etc.