# INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
# MA205 Complex Analysis Autumn 2012

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# Uniform Convergence

## RECALL

### Definition

Let $\{f_n\}$ be sequence of complex valued functions on a set $A$. We say that it is *uniformly convergent* on $A$ to a function $f(x)$ if for every $\epsilon > 0$ there exists $n_0$, *independent of* $x \in A$ such that for all $n \geq n_0$, we have, $|f_n(x) - f(x)| < \epsilon$, for all $x \in A$.

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- Uniform convergence implies pointwise convergence.
- Pointwise convergence does not imply uniform convergence when the domain is infinite.
- Uniform limit of continuous functions is continuous.

# Uniform Convergence: Geometric Series

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*Then for all* $|z| < r$ *and* $n \geq n_0$*, we have,*

$$\left|\frac{1 - z^n}{1 - z} - \frac{1}{1 - z}\right| = \left|\frac{z^n}{1 - z}\right| \leq \frac{|z^{n_0}|}{1 - |z|} \leq \frac{r^{n_0}}{1 - r} < \epsilon.$$

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## Theorem

**Weierstrass**[a] *M-test: Let* $\sum_n m_n$ *be a convergent series of non negative terms. Suppose there exists* $K > 0$ *and an integer* $N$ *such that* $|f_n(x)| \leq K m_n$ *for all* $n \geq N$ *and for all* $x \in A$. *Then* $\sum_n f_n$ *is uniformly and absolutely convergent in* $A$.

[a] Karl Weierstrass (1815-1897) a German mathematician is well known for his perfect rigor. He clarified any remaining ambiguities in the notion of a function, of derivatives, of minimum etc..

# Uniform Convergence:M-Test

**Proof:** Given $\epsilon > 0$ choose $n_0 > N$ such that $m_n + m_{n+1} + \cdots + m_{n+p} < \epsilon/K$, for all $n \geq n_0$.

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Then it follows that

$$|f_n(x)| + \cdots + |f_{n+p}(x)| \leq K(m_n + \cdots m_{n+p}) < \epsilon,$$

for all $n \geq n_0$ and for all $x \in A$.

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## Remark

The series $\sum_n m_n$ in the above theorem is called a 'majorant' for the series $\sum_n f_n$.

# Complex Differentiation

- The notions of continuity and differentiability of functions are discussed pointwise. Nevertheless, they require that the function be defined in a 'neighborhood' of the point under discussion. Therefore, in the case of real 1-variable functions, the domain of definition of functions are intervals. In the case of complex 1-variable, we have 'more' choices.

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$A \subset \mathbb{C}$ is said to be an open set if it is the union of open balls.

- Thus if $A$ is open and $z \in A$ then it follows that there is $r > 0$ such that $B_r(z) \subset A$.

# Definition And Basic Properties

## Definition

Let $z_0 \in \Omega \subset \mathbb{C}$, such that there exists $r > 0$ with $B_r(z_0) \subset \Omega$. Let $f : \Omega \longrightarrow \mathbb{C}$ be a map. Then $f$ is said to be **complex differentiable** (written $\mathbb{C}-$**differentiable** at $z_0$) if the limit on the right hand side of the following formula exists, and in that case we call this limit, the (**Cauchy**) derivative of $f$ at $z_0$ :

- $$\boxed{f'(z_0) := \frac{df}{dz}(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.} \qquad (1)$$

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**Basic Properties**

- **Sum of two $\mathbb{C}-$differentiable function is $\mathbb{C}$-differentiable,... etc..** Indeed, we have,

$$\boxed{(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0), \quad \alpha, \quad \beta \in \mathbb{C}.} \tag{2}$$

- Moreover, just like in the real case, it is also a **derivation,** i.e.,

$$\boxed{\textbf{Leibniz:} \quad (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).} \tag{3}$$

# Basic Properties

- We even have the same formula for the derivative of a quotient:

$$\boxed{\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}; \text{ if } g(z_0) \neq 0.} \tag{4}$$

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- **(Chain Rule :)** Let $f : A \to \mathbb{C},\quad g : B \to \mathbb{C},\quad f(A) \subset B$ and $z_0 \in A$. Suppose that $f'(z_0)$ and $g'(f(z_0))$ exist. Then $(g \circ f)'(z_0)$ exists and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

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**Theorem**

*If $f$ is differentiable at a point then it is continuous at that point.*

# Examples

- For $f(z) = z^n, n \geq 0$, $f'(z) = nz^{n-1}$, as in the case of real variable. (Use binomial expansion).

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- Are there more examples?

# Power series

$$\mathbb{K} = \mathbb{R}, \text{ or } \mathbb{C}.$$

## Definition

By a *formal power series in one variable t* over $\mathbb{K}$, we mean a sum of the form

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(2) Thus, all polynomials in $t$ are power series in $t$.
(3) The geometric series is a genuine example of a power series.

# Power series

- We can add two power series, by 'term-by-term' addition and we can also multiply them by scalars, just like polynomials, viz.,

$$\sum_n a_n t^n + \sum_n b_n t^n := \sum_n (a_n + b_n) t^n; \quad \alpha(\sum_n a_n t^n) := \sum_n \alpha a_n t^n.$$

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**Definition**

A formal power series $P(t) = \sum_n a_n t^n$ is said to be *convergent* if there exists a **non zero number** $z$ (real or complex) such that the series of complex numbers $\sum_n a_n z^n$ is convergent.

# Convergent Power series

**Theorem**

*Let $P(t)$ be a power series which is absolutely convergent for $t = z_0 \neq 0$. Then for all $|z| \leq |z_0|$, $P(z)$ is absolutely and uniformly convergent.*

**Proof:** Appeal to (Weierstrass' Majorant Criterion) M-test with $m_n = |a_n z_0^n|$ and $K = 1$. ♠

# Radius of Convergence:

## Definition

Let $P(t)$ be a power series. Let

$$r(P) = sup\{|z| \ : \ P(z) \text{ is convergent } \}$$

Then $r(P)$ is called radius of convergence of $P$. If $r(P) > 0$ we say $P$ is a convergent power series.

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- If $|z| < r(P)$ the $P(z)$ is absolutely convergent.
- If $|z| > r(P)$ then $P(z)$ is not absolutely convergent. Thus the collection of all points at which a given power series converges consists of an open disc centered at the origin and perhaps some points on the boundary of the disc.

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## Example

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- and the last one is convergent at all the point of the boundary. (Compare with $\sum_n \frac{1}{n^2}$.)

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- and the last one is convergent at all the point of the boundary. (Compare with $\sum_n \frac{1}{n^2}$.)
- These examples clearly illustrate that the boundary behavior of a power series needs to be studied more carefully.

# Term-by-Term Differentiation

## Definition

Given a power series $P(t) = \sum_{n \geq 0} a_n t^n$, the derived series $P'(t)$ is defined by taking term-by-term differentiation: $P'(t) = \sum_{n \geq 1} n a_n t^{n-1}$. The series $\sum_{n \geq 0} \frac{a_n}{n+1} t^{n+1}$ is called the integrated series.

## Theorem

*Any given power series, its derived series and its integrated series all have the same radius of convergence.*

# Analytic Functions

We shall call the sum function given by a convergent power series
An analytic function.
As seen above it follows that an analytic function is complex differentiable any number of times in the disc of convergence of the power series. Therefore, the derivative of an analytic function is also an analytic function.

- It is fairly obvious that the sum of two formal power series is convergent with radius of convergence at least the minimum of the two radii of convergence.

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- Therefore sum of two analytic functions is analytic. Similarly, the product of two analytic functions is also analytic.
- The identity function written $f(z) = z$ is clearly analytic in the entire plane (take $P(t) = z_0 + t$ to see that $f$ is analytic at $z_0$). Starting from this and using the above two observations we can deduce that any polynomial function is analytic throughout the plane.

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- We define

$$\boxed{\exp z := e^z := \sum_{n\geq 0} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .} \tag{5}$$

# Exponential Function

- If we denote by $u_n = \frac{z^n}{n!}$ the $n^{th}$ term of the series, then

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|z|}{n+1} < \frac{1}{2}$$

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- By comparison with the geometric series (**Ratio Test**), this means that the series is absolutely convergent for all $z$.
- Therefore the function exp makes sense in the entire of $\mathbb{C}$ and is differentiable throughout $\mathbb{C}$.
- Its derivative is given by

$$\boxed{\exp'(z) = \sum_{n \geq 1} \frac{n}{n!} z^{n-1} = \exp(z)} \qquad (6)$$

  for all $z$.

# Solution of a Differential Equation

- Also, $\exp(0) = 1$. Thus we see that exp is a solution of the initial value problem:

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- We can verify that

$$\exp(a+b) = \exp(a)\exp(b), a, b \in \mathbb{C} \tag{8}$$

directly by using the product formula for power series. (Use binomial expansion of $(a+b)^n$, and write down the details by yourself.)

# Solution of a Differential Equation

- Also, $\exp(0) = 1$. Thus we see that exp is a solution of the initial value problem:

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- This can also be proved by using the fact that any analytic solution of (7) has to be exp. This method is quite typical and educative and let us take this opportunity to learn this.

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- Now fix $w \in \mathbb{C}$ and consider the function $h(z) = \exp(z + w)\exp(-w)$. Clearly, $h$ is analytic, $h(0) = 1$ and $h'(z) = h(z)$.
- Therefore by the uniqueness of solution of (7), $h(z) = \exp(z)$ for all $z$, i.e., $\exp(z + w)\exp(-w) = \exp(z)$. This is nothing but the same as saying $\exp(z + w) = \exp(z)\exp(w)$.

# Exp is a homomorphism

Thus, we have shown that exp defines a homomorphism from the additive group $\mathbb{C}$ to the multiplicative group $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$.

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- This is the justification to have the notation $e^z$ for $\exp(z)$.

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- By comparing with the geometric series $\sum_n 2^{-n}$, it can be shown easily that $e < 3$.
- Also we have,

$$\boxed{e = \lim_{n \longrightarrow \infty} \left(1 + \frac{1}{n}\right)^n.} \qquad (9)$$

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- Hence,

$$\boxed{|e^{\imath y}| = 1, \quad y \in \mathbb{R}.} \tag{11}$$