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Properties of Exp

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$$\exp(a+b) = \exp(a)\exp(b),\, a, b \in \mathbb{C} \tag{1}$$

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  Thus, we have shown that $\exp$ **defines a homomorphism from the additive group** $\mathbb{C}$ **to the multiplicative group** $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$.
- Also

$$\exp(nz) = \exp(z)^n$$

  for all integers $n \geq 0$.

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- This is the justification to have the notation e^z for $\exp(z)$ for all z.

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- Also we have,

$$e = \lim_{n \longrightarrow \infty} \left(1 + \frac{1}{n}\right)^n. \tag{2}$$

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- Hence,

$$|e^{\imath y}| = 1, \quad y \in \mathbb{R}. \tag{4}$$

Trigonometric Functions.

- Writing $e^{\imath y} = u + \imath v = rE(\theta)$ it follows that $r = 1$ and $u^2 + v^2 = 1$. Therefore $u = \cos\theta$ and $v = \sin\theta$.

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- On the other hand, by taking term-by-term real and imaginary parts of the series $\sum_n \frac{(\imath y)^n}{n!}$, we obtain

$$\cos\theta = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - + \cdots ;$$

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- The power series on the RHS have radius of convergence ∞.

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- Motivated by this, we can define the complex trigonometric functions by

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- All standard properties of sin and cos can be derived using the above power series definitions.

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- It follows that $e^{2\pi\imath} = 1$. (For a rigorous definition of π you may refer to the optional problem 13 in set III of your tutorial sheets.)

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- Indeed, we shall prove that

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- Since $e^x \geq 0$ for all $x \in \mathbb{R}$, and $\cos m\pi = \pm 1$, it follows that $\cos m\pi = 1$ and $e^x = 1$. Therefore $x = 0$ and $m = 2n$, as desired.

Exponential Function

Finally, we shall state the following without proof.

$$\exp(\mathbb{C}) = \mathbb{C}^{\star}. \tag{11}$$

Exponential Function

Remark

One of the most beautiful equations:

$$\boxed{\boxed{\textbf{Euler:} \quad e^{\pi \imath} + 1 = 0}} \tag{12}$$

which relates in a simple arithmetic way, *five* of the most fundamental numbers, made Euler[a] believe in the existence of God!

[a] See E.T. Bell's book 'Men of Mathematics', for some juicy stories

Mapping Properties of Exp and Trigonometric functions

You are familiar with the real limit

$$\lim_{x \longrightarrow \infty} exp(x) = \infty.$$

However, such a result is not true when we replace the real x by a complex z. In fact, given any complex number $w \neq 0$, we have seen that there exists z such that $\exp(z) = w$. But then $\exp(z + 2n\pi\imath) = w$ for all n. Hence we can get z' having arbitrarily large modulus such that $\exp(z') = w$.

Mapping Properties of Exp and Trigonometric functions

As a consequence, it follows that $\lim_{z \longrightarrow \infty} \exp(z)$ does not exist. Indeed we know

$$e^x \to \infty \text{ as } x \to \infty$$

$$e^x \to 0 \text{ as } x \to -\infty$$

and

$$\left| e^{\imath y} \right| = 1, y \in \mathbb{R}.$$

The last formula means that under exp the imaginary axis is mapped over the unit circle.

Mapping Properties

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- Putting $exp(\imath z) = T$ we have $\sin z = \frac{T - T^{-1}}{2\imath} = w$. This gives a quadratic equation in T :

$$T^2 - 2\imath w T - 1 = 0.$$

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$$T^2 - 2\imath w T - 1 = 0.$$

- If T_1 is a solution then $T_1 \neq 0$.
- There are many z such that $\exp \imath z = T_1$. If z_1 is one such then $\sin z_1 = w$.

Hyperbolic Functions

Likewise, the complex hyperbolic functions are defined by

$$\boxed{\sinh z = \frac{e^z - e^{-z}}{2}; \quad \cosh z = \frac{e^z + e^{-z}}{2}.} \tag{13}$$

It is easy to see that these functions are all analytic. Moreover, all the usual identities which hold in the real case amongst these functions also hold in the complex case and can be verified directly. One can study the mapping properties of these functions as well, which have wide range of applications.

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- In the real case, this was very easy since the mapping exp : $\mathbb{R} \to \mathbb{R}^+$ is a one-one and onto mapping and therefore has a well defined inverse viz. the logarithm ln .
- However, as we have observed, unlike in the real case, the complex exponential function e^z is not one-one, and hence its inverse is going to be a multi-valued function, or rather a set valued function.

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- Pick up any $w \neq 0$. (this is needed!) Let us define $\log w$ (or $\ln w$) to be equal to the set of all $z \in \mathbb{C}$ satisfying the equation $e^z = w$.
- Thus

$$\ln w := \ln |w| + \imath \arg w.$$

Observe that the multi-valuedness of $\ln w$ is caused by that of $\arg w$:

$$arg\ \ w = \{\theta + 2n\pi \ \ : \ \ n \in \mathbb{Z}\}$$

The Logarithm:

If θ is chosen such that $-\pi < \theta \leq \pi$ (some authors choose $0 \leq \theta < 2\pi$) then we call it **Principle value of arg** *w* and denote it by *Arg w*. Accordingly we get **Principle value of** log and denote it by *Log w*.

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It follows that

$$\log w = \{Log\ w + 2\pi n\imath, n \in \mathbb{Z}\}.$$

The Logarithm:

- We have the identity

$$\boxed{\boxed{\ln(w_1 w_2) = \ln w_1 + \ln w_2,}} \tag{14}$$

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- Here, we have to interpret this identity 'set-theoretically'.
- *Caution:* When z is a positive real number, $\ln z$ has two meanings! Unless mentioned otherwise one should stick to the older meaning, viz., $\ln z = \operatorname{Ln} z$ in that case.

Exponents of complex numbers

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- For any two complex numbers $z, w \in \mathbb{C} \setminus \{0\}$, define

$$z^w := e^{w \ln z}. \tag{15}$$

Complex exponents

- Observe that on the rhs the term $\ln z$ is a multi-valued function. Therefore, in general, this makes z^w a set of complex numbers rather than a single number. For instance, $2^{1/2}$ is a two element set viz., $\{\sqrt{2}, -\sqrt{2}\}$.

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- First, let us take the simplest case, viz., $w = n \geq 1$. Then irrespective of the value of z (15) gives the single value which is equal to z multiplied with itself n times. For negative integer exponents also, the story is the same except that, we need to have $z \neq 0$. But as soon as w is not an integer, we can no longer say that this is single-valued.

Complex exponents

- Does this definition follow the familiar laws of exponents:

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- Yes indeed. The only caution is that these formulae tell you that the two terms on either side of the equality sign are equal as sets. This is essentially a consequence of the property (14):

$$\ln(ab) = \ln(a) + \ln(b)$$

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- Such a function should be continuous to begin with.
- If we restrict the domain suitably, then we see that the 'argument' can be defined continuously.
- In fact for this to hold, we must be careful about a few things.

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- First of all, in our domain of definition of ln, 0 should never be there.
- Secondly, in the domain of l, we **should not able to 'go around' the origin.**
- One way to ensure this is to throw away an entire ray emerging from the origin, from the complex plane, then for each point of the remaining domain a continuous value of the argument can be chosen. This in turn, defines a continuous value of the logarithmic function also. We make a formal definition.

The Logarithm as a function:

Definition

Given a multi-valued function f, on an open set Ω, by a branch of f we mean a specific continuous function $g : \Omega \longrightarrow \mathbb{C}$ such that $g(z) \in f(z)$ for all $z \in \Omega$.

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For instance, if h is a function which is not one-one, then its inverse is a multi-valued function. Then any continuous function g such that $g \circ h = Id$ over a suitable domain will be called a branch of h^{-1}.

Branch of a Multi-valued Function

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In particular, branches of the inverse of the exponential function are called branches of the logarithmic function. Over domains such as $\mathbb{C} \setminus L$ where L is an infinite half-ray from the origin, we easily see that ln has countably infinite number of branches.

Branch Lemma

Here is the justification for the definition of 'branch'.

Lemma

Let $h : \Omega_1 \longrightarrow \Omega_2$ be a complex differentiable function. $g : \Omega_2 \longrightarrow \Omega_1$ be a continuous function such that $h \circ g(w) = w, \quad \forall \quad w \in \Omega_2$. Suppose $w_0 \in \Omega_2$ is such that $h'(z_0) \neq 0$, where $z_0 = g(w_0)$. Then g is $\mathbb{C}$-differentiable at w_0, with $g'(w_0) = (h'(z_0))^{-1}$.

Branches of Multi-values functions

- Observe that as a corollary, we have obtained complex differentiable branches of the logarithmic function. For instance, $Ln(z) := ln\, r + \imath\theta, \ \ -\pi < \theta < 2\pi$, is one such branch defined over the entire of $\mathbb{C}$ minus the negative real axis. The question of the nature of domains on which ln has well defined branches will be discussed later on.

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- The hypothesis that $h'(z_0) \neq 0$ is indeed unnecessary in the above lemma. This stronger version of the above lemma will be perhaps taken up later in the course.

The real case

In contrast, in the real case, consider the function $x \mapsto x^3$ which defines a continuous bijection of the real line onto itself. Its inverse is also continuous but not differentiable at 0 as can be seen easily in different ways.

Derivative of Ln

Example

Let us find out the derivative of a branch $\eta(z)$ of the logarithm. We shall show that $\frac{d}{dz}(\eta z) = \frac{1}{z}$. Since, $\exp \circ \eta = Id$, *it follows from the chain rule that* $(\exp)'(\eta(z))\eta'(z) = 1$. *Therefore, we have,* $z\eta'(z) = 1$ *and hence,* $\eta'(z) = 1/z$, *as claimed.*

Discontinuity of Ln

The principle branch logarithm Ln does not have additive property in the full: For instance $Ln(-1) = \pi$ whereas $0 = Ln(1) = Ln[(-1)(-1)] \neq Ln(-1) + Ln(-1) = 2\pi\imath$.

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