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Some linear Algebra

Writing $z = x + \imath y$ as a coulumn $\binom{x}{y}$ so that $\mathbb{C}$ is identified with $\mathbb{R}^2$.

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In particular, the complex numbers 1 and ι are represented by $\binom{1}{0}$ are $\binom{0}{1}$ respectively.
Exercise: Show that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is multiplication by a complex number iff $a = d$ and $c = -b$.

Some linear Algebra

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Thereofore put $w = a + \imath c$ and check that $wz = (ax - cy) + \imath(ay + cx)$ is the same as

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - cy \\ ay + cx \end{pmatrix}$$

Recall Calculus of 2-variables

- Let U be an open subset of $\mathbb{C}$, $z_0 = (x_0, y_0) \in U$ and $f : U \to \mathbb{C}, f(x,y) = u(x,y) + \imath v(x,y)$ be a given function where u and v are real valued functions of two real variables x, y.

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- Saying that f has continuous partial derivatives is the same saying that u, v have continuous partial derivatives and in that case,

$$f_x = u_x + \imath v_x, \quad f_y = u_y + \imath v_y.$$

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$$f_x = u_x + \imath v_x, \quad f_y = u_y + \imath v_y.$$

- We also write $\nabla f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

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Taking the limit along the lines parellel to the x-axis or y-axis, i.e., putting $h = t$, OR $h = \imath t$, respectively, $t \in \mathbb{R}$ under the limit, we get the two partial derivative of f at z_0.

Cauchy-Riemann Equation

Therefore,

$$f'(z_0) = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} = f_x(x_0, y_0) = f_x(z_0) \, (2$$

Cauchy-Riemann Equation

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Similarly,

$$f'(z_0) = \lim_{t\to 0} \frac{f(x_0, y_0+t) - f(x_0, y_0)}{\imath t}$$

$$= \frac{1}{\imath} \lim_{t\to 0} \frac{f(x_0, y_0+t) - f(x_0, y_0)}{t} = \frac{f_y(z_0)}{\imath}. \quad (3)$$

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Equating the real and imaginary parts, we get **Cauchy Riemann Equations:**

$$\boxed{u_x = v_y ; u_y = -v_x} \tag{5}$$

Cauchy-Riemann Equation

- Moreover we have

$$\boxed{\boxed{|f'(z_0)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2 = u_x v_y - u_y v_x.}} \tag{6}$$

Cauchy-Riemann Equation

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$$\boxed{\begin{array}{l}|f'(z_0)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = u_x^2 + u_y^2 = \\ v_x^2 + v_y^2 = u_x v_y - u_y v_x.\end{array}} \tag{6}$$

- The last expression above, which is the determinant of the matrix

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \tag{7}$$

is called the **jacobian** of the mapping $f = (u, v)$, with respect to the variables (x, y) and is denoted by

$$\boxed{J_{(x,y)}(u, v) := u_x v_y - u_y v_x.} \tag{8}$$

An Application

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The function $z \mapsto |z|^2$ is not complex differentiable for any point except at $z = 0$. It satisfies the CR-equations at 0. That of course does not mean that it is $\mathbb{C}$-differentiable at 0. You have to prove the differentiability directly.

An Example

Example

If $f(z) = \frac{\bar{z}^2}{z}, z \neq 0$ *and* $f(0) = 0$, *show that Cauchy-Riemann equations are satisfied at* $z = 0$, *but* $f'(0)$ *does not exist.*

Sol: Put $f = u + \imath v$. Then
$u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}$; $v(x,y) = \frac{-x^2y+y^3}{x^2+y^2}$. Direct computation shows that

$u_x(0,0) = 1; u_y(0,0) = 0; v_x(0,0) = 0, v_y(0,0) = 1.$

Hence CR equations are satisfied.

Solution continued

However, for $z = re^{i\theta}$, we have

$$\lim_{r\to 0} \frac{f(z) - f(0)}{z} = \lim_{r\to 0} \frac{\bar{z}^2}{z^2} = e^{-4i\theta}.$$

This means that the limit taken along different lines is different. Hence $f'(0) = \lim_{z\to 0} \frac{\bar{z}^2}{z^2}$ does not exist.

Holomorphic functions

Definition

Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ be a function such that $f = u + \imath v$. Suppose u, v have continuous partial derivatives throughout U and u, v satisfy CR equations. Then we say f is a homolorphic function. (Cauchy called them 'Synectic functions')

Holomorphic functions

Theorem

Let f be a complex valued function of a complex variable defined on an open subset U. Then f is complex differentiable in U iff f is holomorphic on U.

Holomorphic functions

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- If f is complex differentiable in U, we have seen that it has partial derivatives in U which satify CR.
- The fact that these partial derivatives are continuous is **not easy** to derive and will not be done in this course.
- The converse part is not difficult. Indeed using MVT of 1-variable calculus, one can first show that the real total derivative Df of f exists. Since the CR equations are satisfied, this means that Df is multiplication by a complex number. That is enough to conclude that the complex derivative f' exists and is equal to Df. Again, we shall skip the details of the proof of this.

Cauchy-Riemann equations under orthogonal transformations

Theorem

If $f(z) = u + \imath v$ is differentiable at a point $z_0 = z_0 + \imath y_0$ in an open set G, then

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}, \quad \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s} \quad (*)$$

at (x_0, y_0) where $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial n}$ denote directional differentiation in two orthogonal directions s and n at (x_0, y_0), such that n is obtained from s by making a counterclockwise rotation.

- **Solution:** Put $\mathbf{s} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \mathbf{n} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

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- Then

$$u_{\mathbf{s}} = \frac{\partial u}{\partial \mathbf{s}} = \nabla u \cdot \mathbf{s} = u_x \cos\theta + u_y \sin\theta;$$

$$v_{\mathbf{s}} = \frac{\partial v}{\partial \mathbf{s}} = \nabla v \cdot \mathbf{s} = v_x \cos\theta + v_y \sin\theta.$$

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- Similarly,

$$u_{\mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = -u_x \sin\theta + u_y \cos\theta;$$

$$v_{\mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = \nabla v \cdot \mathbf{n} = -v_x \sin\theta + v_y \cos\theta.$$

These quantities can be expressed in matrix form as:

$$\begin{pmatrix} u_{\mathbf{s}} & u_{\mathbf{n}} \\ v_{\mathbf{s}} & v_{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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where R_θ is rotation through an angle θ which is a multiplication by a non zero complex number. Therefore the first matrix is a multiplication by a complex number iff the second one is. From the little bit of linear algebra that we saw, in the begining, this establishes (*).

CR equations in polar coordinates

Take a point other than the origin. (At the origin polar coordinate is singular.) Say, $z_0 = (x_0, y_0) \neq (0, 0)$ and let $f = u + \imath v$. Then the equations

$$\boxed{\boxed{ru_r = v_\theta; \quad rv_r = -u_\theta.}} \tag{9}$$

are equivalent to CR-equations and obtain the formula:

$$\boxed{\boxed{f'(z_0) = e^{-i\theta_0}(u_r + iv_r) = -\frac{i}{z_0}(u_\theta + iv_\theta)}} \tag{10}$$

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- $u_r = u_x\cos\theta + u_y\sin\theta; v_r = v_x\cos\theta + v_y\sin\theta;$ $u_\theta = -u_x r\sin\theta + u_y r\cos\theta; v_\theta = -v_x r\sin\theta + v_y r\cos\theta.$

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  $u_\theta = -u_x r\sin\theta + u_y r\cos\theta; v_\theta = -v_x r\sin\theta + v_y r\cos\theta.$
- This can be expressed in the matrix form as:

$$r\begin{pmatrix} \cos\theta & \sin\theta \\ sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_x & v_y \\ u_y & -v_x \end{pmatrix} = \begin{pmatrix} ru_r & v_\theta \\ u_\theta & -rv_r \end{pmatrix}.$$

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- $x_r = \cos\theta; x_\theta = -r\sin\theta; y_r = \sin\theta; y_\theta = r\cos\theta$. Therefore, by the chain rule:
- $u_r = u_x\cos\theta + u_y\sin\theta; v_r = v_x\cos\theta + v_y\sin\theta$; $u_\theta = -u_x r\sin\theta + u_y r\cos\theta; v_\theta = -v_x r\sin\theta + v_y r\cos\theta$.
- This can be expressed in the matrix form as:

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- Note that $r \neq 0$, and the left most matrix is invertible. Therefore the two columns vectors of the second matrix are equal (CR equations in cartesian cooridnates) iff the two columns of the matrix on the RHS are equal (CR equations in polar coordiantes)

CR equations in polar coordinates

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Rewrite the above matrix equation in the form

$$\begin{pmatrix} u_x & v_y \\ u_y & -v_x \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r & v_\theta/r \\ u_\theta/r & -v_r \end{pmatrix}$$

and substituting in $f'(z) = u_x - \imath u_y$, gives

$$\begin{aligned} f'(z) &= u_r \cos\theta - \frac{u_\theta}{r}\sin\theta - \imath\left(u_r \sin\theta + \frac{u_\theta}{r}\cos\theta\right) \\ &= u_r\cos\theta + v_r\sin\theta - \imath u_r \sin\theta + \imath v_r\cos\theta \\ &= e^{-\imath\theta}(u_r + \imath v_r) \\ &= \frac{e^{-\imath\theta}}{r}(v_\theta - \imath u_\theta) = \frac{-\imath}{z}(u_\theta + \imath v_\theta). \end{aligned}$$

Harmonic functions

Let $u : U \to \mathbb{R}$ be a twice continuously differentiable function on an open subset U of $\mathbb{R}^2$. We say u is **Harmonic** if it satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Theorem

If $f = u + \imath v$ is a holomorphic function then u and v are harmonic.

In this situation, we say v is a harmonic conjugate of u. For instance $x^3 - 3xy^2$ is a Harmonic function. Its harmonic conjugates are $3x^2y - y^3) + c$ where c is any constant.

Harmonic functions

Theorem

Every harmonic funtion u on an open disc in $\mathbb{R}^2$ is the real part of a holomorphic function.

Proof: We want to find a function v such that $f = u + \imath v$ is holomorphic.

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It such a v exists then it follows that $v_y = u_x$ and $v_x = -u_y$. So we set up $v = \int u_x dy$. Then $v_y = u_x$ alright.

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It such a v exists then it follows that $v_y = u_x$ and $v_x = -u_y$. So we set up $v = \int u_x dy$. Then $v_y = u_x$ alright.

Moreover, $v_x = \int u_{xx} dy = -\int u_{yy} dy = -u_y$. Therefore, u, v satisfy CR equations. Since u is twice continuously differentiable, so is v. Therefore u_x, u_y, v_x, v_y are continuous also. This implies f is complex differentiable. $\spadesuit$

Formal derivatives

We introduce the following differential operators:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right).$$

They can be operated on any function f of two variables which has partial derivatives.

Formal derivatives

Note that

$$\frac{\partial}{\partial z} z = 1, \frac{\partial}{\partial \bar{z}} z = 0; \frac{\partial}{\partial z}(\bar{z}) = 0, \frac{\partial}{\partial \bar{z}}(\bar{z}) = 1.$$

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For composite functions, we also have formal chain rule: Suppose g is a function of $w = f(z)$ then

$$\frac{\partial}{\partial z}(g \circ f(z_0) = \frac{\partial g}{\partial w}(f(z_0))\frac{\partial}{\partial z} f(z_0) + \frac{\partial g}{\partial \bar{w}}(f(z_0))\frac{\partial}{\partial z}\bar{f}(z_0).$$

etc.

Formal derivatives

Theorem

Suppose f is a holomorphic function. Then $\frac{\partial}{\partial z} f = f'(z)$ and $\frac{\partial}{\partial \bar{z}} f = 0$.

Remark: Converse is is also true provided we assume f has continuous partial derivatives in an open set.

Definition

We say f is anti-holomorphic if $\frac{\partial}{\partial z}(f) = 0$.

Theorem

f is holomorphic iff $\bar{f}$ is antiholomorphic.

Formal derivatives

The formal derivative approach allows to interpret a holomorphic function as something which independent of $\bar{z}$. Such an interpretation can be fully justified later as we advance in our study of holomorphic functions. However at this stage, let us make use of this interpretation.

Finding Harmonic conjugates without integration

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Let consider a harmonic function u defined on a disc around the origin and such that $u(0,0) = 0$.
Suppose f is such that $f = u + \imath v$ is holomorphic. Put $g = \bar{f}$. Then g is an anti-holomorphic function. Using $x = z + \bar{z}/2$ and $y = z - \bar{z}/2\imath$ we consider u as a function of z and $\bar{z}$.

Finding Harmonic conjugates without integration

Since $u = \frac{f+g}{2}$, we have

$$u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \frac{f(z)+g(z)}{2}.$$

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Since g is anti-holomorphic it is independent of z or equivalently a function of $\bar{z}$ alone and $g(0) = 0$. Therefore, upon putting $\bar{z} = 0$ we obtain

$$u\left(\frac{z}{2}, \frac{z}{2\imath}\right) = \frac{1}{2}f(z).$$

We have the magic formula:

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right).$$

Magic formula

Example

(1) Take $u = x^2 - y^2$. *Then* $u(z/2, z/2i) = (z/2)^2 - (z/2i)^2 = z^2/2$. *Therefore* $f(z) = z^2$.

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(1) Take $u = x^2 - y^2$. *Then* $u(z/2, z/2i) = (z/2)^2 - (z/2i)^2 = z^2/2$. *Therefore* $f(z) = z^2$.

(2) Take $u = x^3 - 3xy^2$. *Then* $2u(z/2, z/2i) = 2(z/2)^3 - 3(z/2)(z/2i)^2 = z^3/4 + 3z^3/4 = z^3$.