MA-205 Autumn 2012 Final Examination

QUESTION-CUM-ANSWER SHEET

11-00 am - 1.00 pm   Sept 15, 2012   Total marks - 80

Q.1 Let $f(z) = a\bar{z} + b$ denote the reflection in a line perpendicular to the unit vector $0P$ where the point P is represented by $w \in \mathbb{C}$. Then the value of a is equal to $= -w^2 (= -w/\bar{w} = (\imath w)^2.)$ [3]

Q.2 Define the radius of convergence of a power series. [3]

Answer: $Sup\{\{|z| \ : \ \sum_n a_n z^n$ is convergent$\}$.
OR $Sup\{\{|z| \ : \ \sum_n |a_n z^n|$ is convergent$\}$.
OR $Sup\{r \ : \ \sum_n |a_n| r^n$ is convergent$\}$.

Partial marking:

(i) One mark awarded if the answer is like: 'maximum value of z (instead supremum of —z—) such that $\sum a_n z^n$ is cgt.

(ii) No makrs id fo the answer is like:(a) radius of the largest disc in which the function has no singularity ...
(b) R is the value above which it diverges and below which it converges...

Q.3 Give a formula which relates the radius of convergence R of a power series $\sum_n a_n z^n$ with the coefficients a_n. [3]

Answer: Put $L = \limsup_n |a_n|^{1/n}$. Then $R = 1/L$.

No marks if
(i) modulus sign missing in the formula
(ii) lim instead of limsup
(iii) Ratios instead of n^{th} roots.

Partial marking: Writing limsum=lim will get atmost 1 mark.

Q.4 Define an essential singularity. [3]

Answer: An isolated singularity which is neither a removable singularity nor a pole is called an essantial singularity.

Or Principal part of the Laurent series has infinitely many nonzero terms.

Or there is no n such that $\lim_(z \to a)(z-a)^n f(z) = 0$.

Penalty of one mark for anwers such as
(a) $f(z)$ is infinite at z_0
(b) Limit is infinite from one side and finite from other side.
(c) pole of infinite order.
(d) wrong excample included for illustration.

Q.5 Let R be the radius of convergence of the series $P(z) = \sum_n a_n z^{2n+1}$, where $\lim_{n\to\infty} |a_n|^{1/n} = 5$. Then $R = \frac{1}{\sqrt{5}}$ [4]

Q.6 Show that there is no $z \in \mathbb{C}$ such that $\tan z = \pm\imath$. [4]

Answer: $\tan z = \frac{e^{\imath z} - e^{-\imath z}}{\imath(e^{\imath z} + e^{-\imath z})} = \pm\imath$
iff $e^{\imath z} - e^{-\imath z} = \mp(e^{\imath z} + e^{-\imath z})$
iff $2e^{\pm\imath z} = 0$. which is impossible.

Q.7 Answer the following three questions by first selecting the correct choice (True/False) and justifying it briefly. [4+3+2]

(i) If the map $f(z) = az + b$ represents a rotation then $|a| = 1$ and $a \neq 1$.
(TRUE/) ——1 mark
because $|f(z) - f(0)| = |z - 0|$ for every z implies that $|az| = |z|$ for every z implies that $|a| = 1$ ....(1 mark)
and $a = 1$ implies $f(z) = z + b$ is a translation and not a rotation (2 marks)

OR

f has unique fixed point implies $az + b = z$ has a unique solution implies $(a-1)z = -b$ has a unique solution implies $a \neq 1$. (2 marks)

(ii) Every fractional linear transformation preserves the cross-ratio.

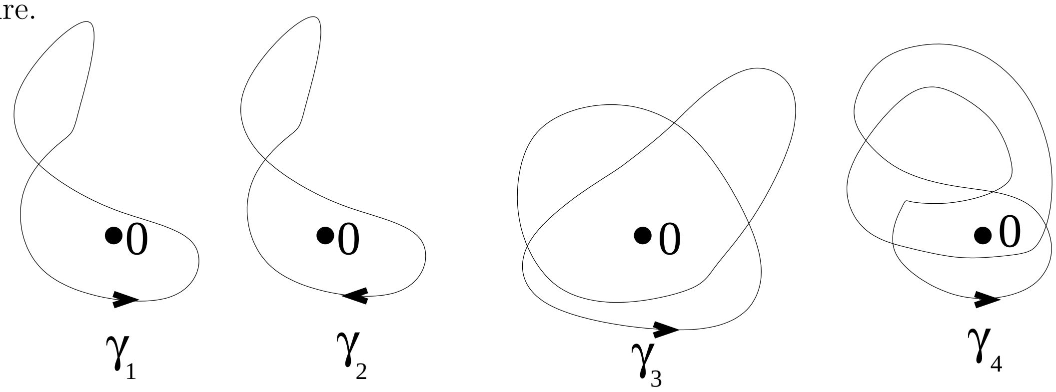
(TRUE) ——-[1]
because every FLT is the composite of some translations, multiplications and the inversion map $z \mapsto 1/z$, and since each of these maps preserve the cross ratio. (2 marks)

(iii) Every entire function with a removable singularity at infinity is a constant.
(TRUE) ——[1 mark]
because such a function is bounded and (by Liouville's theorem,) every bounded entire function is a constant. (1 mark)

Q.8 State Cauchy's integral formula for a holomorphic function f defined on the open disc $B_R(a)$. [3]

Answer: $f(z) = \frac{1}{2\pi\imath}\int_{|z-a|=r} \frac{f(w)}{w-z}dw$, for all $|z-a| < r < R$.
Deduct one mark if the contour of integration is incorrect, or the factor $2\pi\imath$ is missing, or the range of z is incorrect. If all these are incorrect then award 0 mark.

Q.9 Evaluate the integrals $I_j = \int_{\gamma_j} \frac{e^z - e^{-z}}{z^4} dz$ along the four contours drawn in the following picture. [4]



[$I_j = \frac{\eta(\gamma_j,0)2\pi\imath f^{(3)}(0)}{3!}$, where $f(z) = e^z - e^{-z}$. $f^{(3)}(0) = 2$. The values of $\eta(\gamma_j, 0)$ are $1, -1, 2$ and 0 respectively.]

Answer $I_1 = \frac{2\pi}{3}\imath$; $I_2 = -\frac{2\pi}{3}\imath$; $I_3 = \frac{4\pi}{3}\imath$; $I_4 = 0$.

Q. 10 Make a list of all the singularities of $\frac{z}{\sin z}$ and the respective residues at these points. [4]

Answer: $z = 0$ a removable singularity Residue $= 0$ (2 marks)

$z = n\pi$ are all (simple) poles and residue $= (-1)^n n\pi$. (2 marks)

Partial Marking

Deduct one mark if the sign is not correct.

Deduct one mark if the second answer is $2n\pi$.

Award one mark if ∞ is mentioned as an essential singularity with residue equal to 0.

Q.11 Show that $f(z) = \sin z$ defines a conformal mapping on $U = \{x + \imath y \ : \ 0 < x < \pi/2\}$ and describe its image $f(U)$. [6]

Answer:Put $z = x + \imath y, f(z) = u + \imath v$

Then $u = \sin x \cosh y$; $v = \cos x \sinh y$. Dedice from this or otherwise that CR equations are satisfied. (1 mark)

$\sin z_1 = \sin z_2$ iff $z_1 = z_2 + 2n\pi$ or $(2n + 1)\pi - z_2$.

Therefore in U, $f(z) = \sin z$ is one-one. (2 marks)

(Ignore the CR eqaution part if this step is correct.)

(Award one mark if instead $f'(z) \neq 0$ is claimed.)

The line $x = 0$ is mapped onto the line $u = 0$. (1 mark)

$x > 0$ implies $u > 0$. Therefore $f(U)$ is contained in the half plane $u > 0$. (1 mark)

The line $x = \pi_2$ is mapped onto $u = \cosh y \geq 1$ and $v = 0$. Therefore $f(U)$ does not intersect this line which means

$f(U) = \{u + \imath v \ : \ u > 0\} \setminus \{u + \imath v \ : \ u \geq 1, v = 0\}$. (1 marks)

Q.12 Complete the following sentence: A subset A of a domain Ω in $\mathbb{C}$ is called an **isolated set** iff

for every $z \in \Omega$, there is $\delta > 0$ such that $B_\delta(z) \cap A \subset \{z\}$. [2]

Q.13 State Cauchy's estimate for the n^{th} derivative of a holomorphic function defined on an open disc $B_r(a)$. [3]

Answer: Let $M_r = Sup\{|f(z)| \ : \ |z-a| = r\}$
OR let $M_r \geq |f(z)|$ on $B_r(a)$. (1 mark)
Then $|f^{(n)}(a)| \leq \frac{n! M_r}{r^n}$. (2 marks)

No mark for writing Cauchy's integral formula for the n^{th} derivative is written.

Q.14 Suppose f is a holomorphic function on $\mathbb{C}$ such that its 2012^{th} derivative $f^{(2012)}(z) = 0$ on the portion of the y-axis lying between $\imath$ and $-\imath$ and such that $f(1) = 10, f(2) = 15$. Show that for any $w \in \mathbb{C}$ there is $z \in \mathbb{C}$ such that $f(z) = w$. [4]

Answer: The first condition implies that f is a polynomial. (2 marks)
The second condition tells you that f is non constant. (1 mark)
By Fundamental Theorme of Algebra, f takes all the complex values. (1 mark)

Q.15 State the following theorems: [3+3]
(i) **Identity theorem:** Let f and g be holomorphic functions on a region Ω. Suppose K is such that for every $z \in K$, $f(z) = g(z)$ and K has a limit point in Ω. (Alt: K is **not** an isolated subset of Ω.) Then $f \equiv g$ on Ω.

Full marks for taking $g = 0$ in the above answer.

Only one mark if K is assumed to be a non empty open set.

No mark for writing something about flts, even if the statement is correct.

(ii) **Gauss Mean Value theorem:** If f is a holomorphic (or a harmonic) function on $B_R(z_0)$ then for every $0 < r < R$, —-[1 mark]
we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r^{\imath\theta}) d\theta.$$

——[2 marks]

If the factor $\frac{1}{2\pi\imath}$ appears instead of $\frac{1}{2\pi}$ deduct 1 mark.

Q.16 Let $f(z) = \dfrac{z+b}{cz+d}$ be such that $f(5) = -1/3; \ f(-5) = -3$ and $f(100) = 9/11$. Then [3]
$b = -10; c = 1; d = 10$.

Q.17 Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$, [8]

Answer: Put $\cos\theta = \frac{z+z^{-1}}{2}$. (2 marks)

Then the given integral is equal to

$I = \dfrac{1}{2\imath}\displaystyle\int_{|z|=1} \frac{dz}{(z+1/2)(z+2)} = \frac{1}{2\imath}\int_{|z|=1} f(z)dz.$ [2]

The only singularity inside $|z| < 1$ is at $z_0 = -1/2$. (1 mark)

The residue at this point is $\frac{2}{3}$. [1]

Therefore $I = \frac{1}{2\imath} 2\pi\imath\frac{2}{3} = \frac{2\pi}{3}$. (2 marks)

Using directly substitution $t = \theta$ (as in Pre-Jee) and obtaining correct answer carries full mark No parila markings in this method.

Using the class work for the integral $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} d\theta$ and deriving the correct answer carries full mark. No partial marking in this method.

Q.18 Prove or disprove the following statement. [8]

Let C_R be the semicircle lying in the upper half plane with center 0 and radius R. Let $f(z) = \dfrac{ze^{\imath z}}{z^2+z+1}$. Then

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Answer: The statment is true ...... (2 marks)

Put $g(z) = \frac{z}{z^2+z+1} = \frac{z}{(z-a)(z-b)}$

Then for $|z| > R >> 0$ we have $|g(z)| \le \frac{R}{(R-|a|)(R-|b|)} = M_R$. (2 marks)

By Jordan's inequality $\displaystyle\int_0^{\pi} e^{-R\sin\theta} d\theta < \pi/R$. (2marks)

Therefore $|\int_{C_R} f(z)dz| \le M_R \left|\displaystyle\int_0^{\pi} e^{\imath Re^{\imath\theta}} \imath Re^{\imath\theta} d\theta\right|$

$\le M_R R \int_0^{\pi} e^{-R\sin\theta} d\theta < M_R\pi \to 0$

as $R \to \infty$. (2 marks)