

Poincaré Conjecture *

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Abstract:

The century old, so called Poincaré conjecture has been solved at last using geometric techniques. In this talk we chalk-out the development in Topology around this conjecture and surmise a little bit about the future course Topology may take.

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1 Introduction

More than 100 years ago Poincaré (1854-1912), the famous French Mathematician raised, amongst many others the following question:

Question 1: If a closed 3-manifold has trivial fundamental group, must it be homeomorphic to the 3-sphere S^3 ?

A similar question for all dimensions reads as follows:

Question 2: For any positive integer n , if a closed n -manifold is the homotopy type of the n -sphere S^n , must it be homeomorphic to S^n ?

Later these questions came to be known as **Poincaré Conjecture**. The seemingly milder hypothesis in Q.1 is not milder at all because of Poincaré duality relations in the homology groups of a manifold.

The answer turns out to be in the affirmative for all n . For $n = 1, 2$ this is rather trivial and classical (known before Poincaré except that it was not in this modern language). The case for $n \geq 5$ was solved by S. Smale in 1960. The case $n = 4$ was solved by Freedman in 1982. The case $n = 3$ has been in the news for the past three years or so. Last year, Gregory Perelman was awarded coveted Field's medal for his three papers in which he 'proves' Thurston's Geometrization Conjecture which implies, in particular, Poincaré conjecture also.

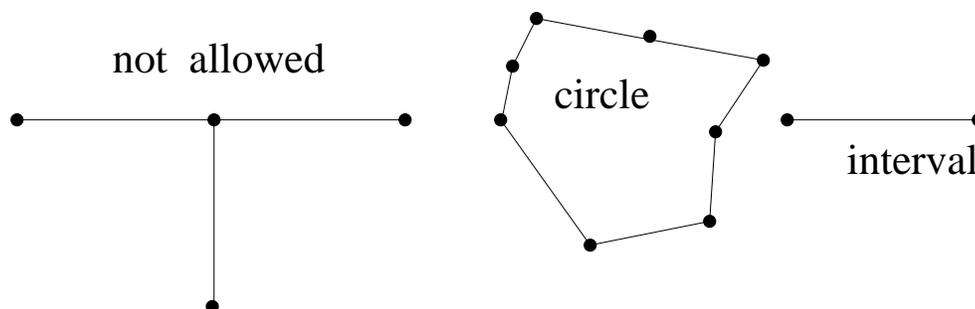
Initially manifolds were introduced by Riemann and they were all smooth to begin with. The concept of triangulation of a manifold is due to Poincaré. He implicitly assumed that all manifolds are triangulable. This was a canonical choice that he made in order to give some foundations to various intuitive ideas of Riemann, Betti and the results of Euler.

As a consequence of these investigations several important topological concepts were born such as the fundamental group, homology, duality, etc. to name a few. No wonder why Poincaré is called **‘father of topology’**. It is another matter that it took more than 50 years of meticulous work by umpteen great mathematician to bring them to the present day level of clarity and sophistication.

We shall only consider manifolds which are compact and with a triangulation. Since every connected component is again a compact manifold, and every compact manifold has finitely many components, it is enough to understand connected compact manifolds.

2 Classification of 1-dim. Manifolds

Consider the case $n = 1$. Let then X be a compact, connected, triangulated 1-manifold. Then X can be thought of as the underlying space of a finite connected graph in which each vertex is of valency at most 1. It follows that either there are precisely two or no vertices of valency 1. Accordingly, it easily follows that X is homeomorphic to a closed interval or a circle.



In other words,

Every compact 1-dimensional manifold X is obtained as a quotient of an interval.

Usually, a first course in Differential topology includes a proof of the classification of 1-dimensional differentiable manifolds. This can be done via Morse theory (see [GP]) or via arc-length parameterization see [M1]). Even the topological case the proof is not so difficult. We get the same result in the three categories.

Before proceeding with other cases, we want to point out to the fact that the above statement in the box itself can be generalized in all dimensions:

Theorem 2.1 *Every (triangulated) compact connected n -manifold is the quotient of a triangulated convex polyhedron in \mathbb{R}^n by some face identifications via affine transformations.*

This result was the backbone of Poincaré's investigations into manifolds, the proof of which is not at all difficult. Observe that **not** all such quotients described as in theorem 2.1 are manifolds. One obvious condition is that a face can be identified with only one other face.

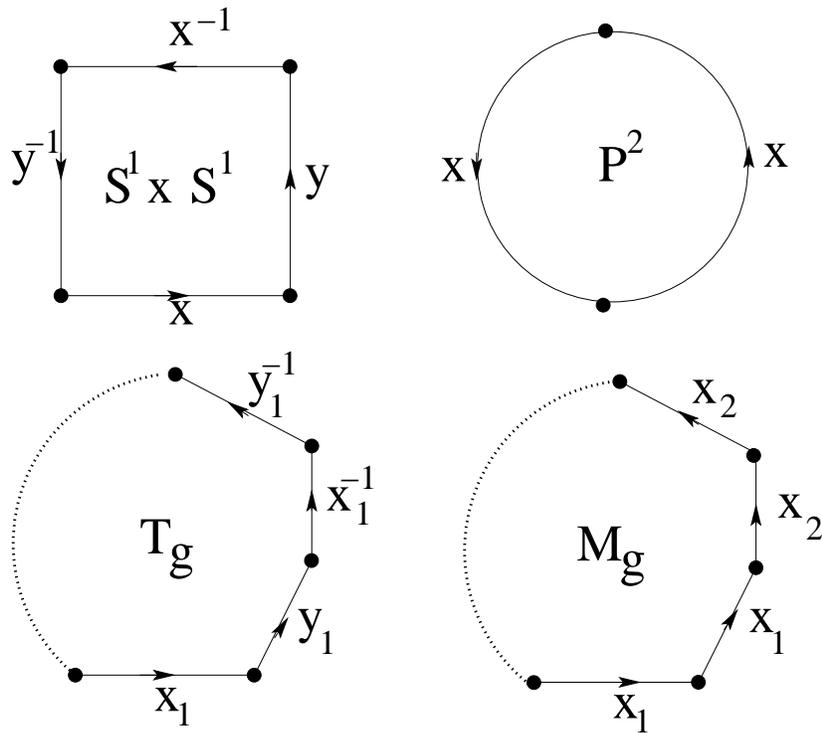
3 The 2-dimensional case

For $n = 2$, this condition itself is enough to ensure that the quotient space is a manifold. Moreover, faces which are not identified with any other face form the boundary of the manifold. Once we understand all closed manifolds (compact manifolds without boundary) it is not so much difficult to understand those with boundary. Therefore, we may restrict our attention to the case when every face is identified with precisely one other face.

Thus, in order to classify all connected compact 2-manifolds, we need to take a (regular) convex polygon, identify its sides pairwise in some pattern and see which patterns give distinct 2-manifolds up to homeomorphism. This leads to the so called canonical polygons. See [Maun-der] or [S] for example.

I will briefly go through the basic steps involved in the classification of surfaces since this is crucial to the understanding of 3-dimensional problem.

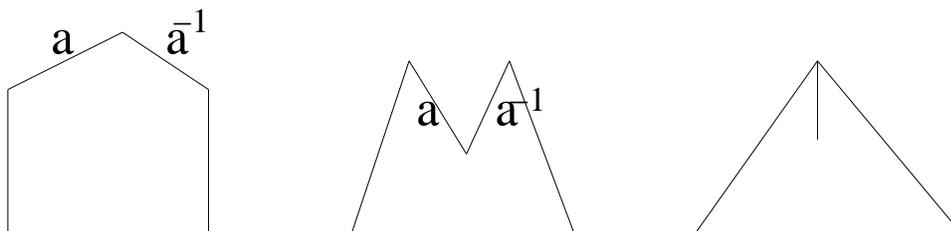
We begin with a representation of a (closed) surface by a convex polygon with even number of sides, with its edges oriented and labeled by symbols $a_j^{\pm 1}$ to indicate which edges are being identified. Thus a letter occurs exactly twice in the list with the signs being arbitrary. The corresponding edges will have to be identified according to the sign though.



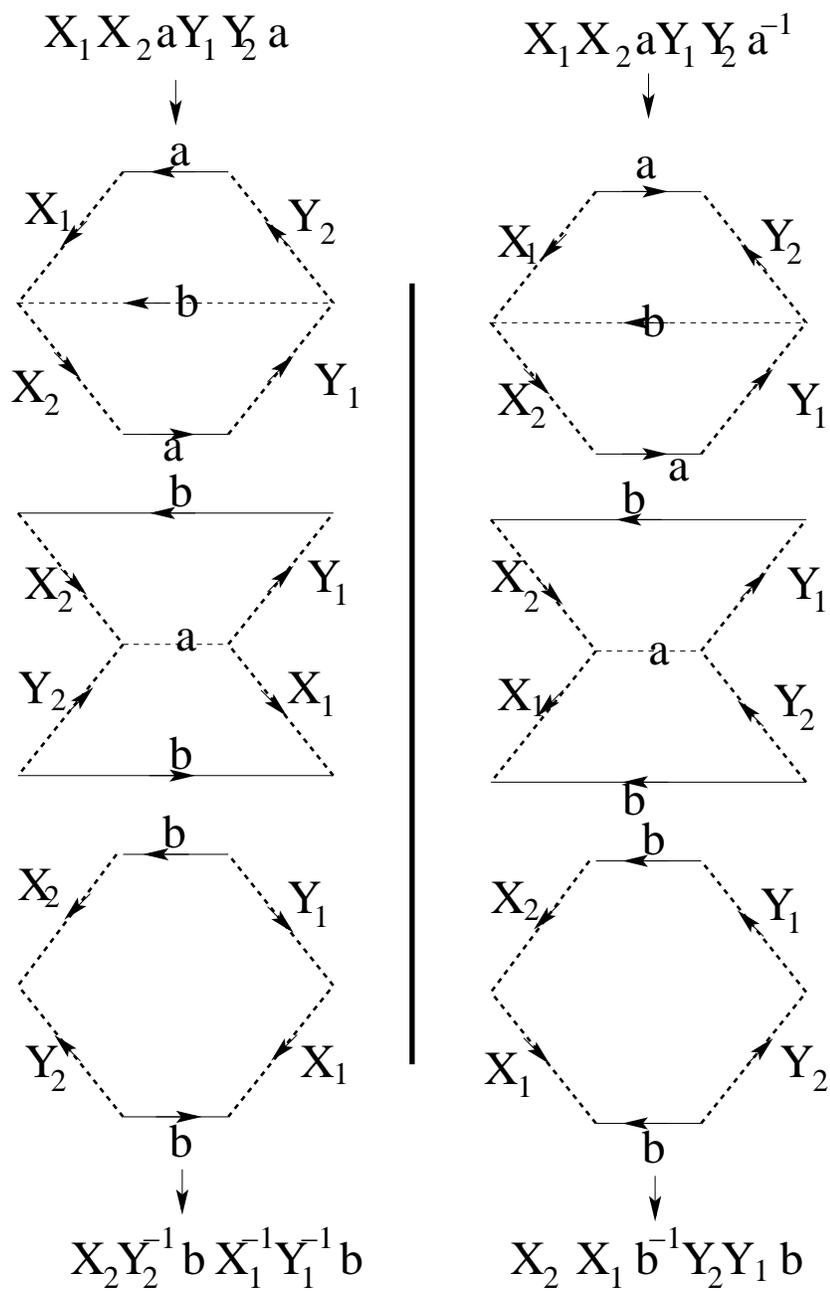
We can cyclically permute the labeling at our will without changing the surface that we get. Therefore we shall consider this labeling sequence only upto cyclic permutation.

We shall now introduce some operations on these data which will not change the homomorphism type of the quotient surface:

I. Elimination of edges of the I-kind If symbols a, a^{-1} occur in succession in the labeling sequence then we call them edges of I-kind. If there are more than four edges in all, then we perform the operation of deleting these to edges from this sequence. The justification is shown in the diagram below. In other words, a sequence of the form Xaa^{-1} is reduced to X .



II Cut and paste:



The final result is that:

Theorem 3.1 *Every connected closed surface is represented by precisely one of the following canonical polygons:*

(i) aa^{-1} ;

(ii) $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}; g \geq 1$.

(iii) $a_1a_1a_2a_2 \cdots a_ga_g, g \geq 1$.

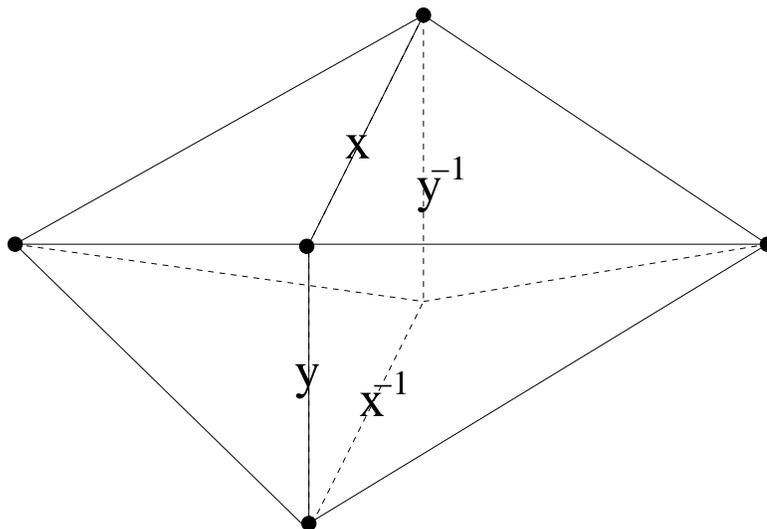
(i) represents the 2-sphere; (ii) gives all orientable surfaces of genus $g \geq 1$ (connected sum of g copies of the torus) and (iii) gives all non orientable surfaces of genus $g \geq 1$ (connected sum of g copies of the projective space P^2).

As a consequence we obtain:

Theorem 3.2 *A connected closed 2-manifold is S^2 iff its Euler characteristic is 2.*

4 The 3-dimensional case

In dimension 3, the first difficulty is in that mere face identification of a polyhedron need not produce a manifold. An easy counter example is obtained by taking the suspension of the Canonical polygon which represents the torus.



Poincaré realized this difficulty and came up with a solution as well.

In modern terminology, in order that a triangulated space be a 3-manifold, it is necessary that the ‘link’ of a vertex should be a triangulated 2-sphere. Having understood the classification of 2-manifolds, in terms of Euler characteristic this condition is easily translated into a combinatorial condition on the identification data as follows:

Consider an equivalence class of a vertex. Suppose it has f vertices. Let the total number of equivalence classes of faces incident at these vertices be e and let the total number of equivalence classes of edges incident at these vertices be v . The condition is that

$$\boxed{v - e + f = 2.} \tag{1}$$

This should be true for each equivalence class of a vertex in order that the quotient is a 3-manifold. Let us refer to this condition by the name ‘**link-condition**’. Thus the recipe for classification for 3-manifold gets ready:

Begin with a triangulated 3-dim. convex polyhedron P (with even number of triangles on the boundary). By an identification data \mathcal{I} on P we mean

- (i) an involution τ on the set of faces;
- (ii) for each face F of P , an affine linear isomorphism $\phi_F : F \rightarrow \tau(F)$ such that $\phi_{\tau(F)} = \phi_F^{-1}$.

With respect to this data, we perform the face identifications on P to obtain the quotient space \hat{P} . Observe that while a face F is getting identified with the face $\tau(F)$ via ϕ_F , its sides and vertices also get identified with the sides and vertices of $\tau(F)$ in some order. It is important to emphasize that no other identification of either edges or vertices by themselves be performed. We shall always assume that the data \mathcal{I} is said to satisfy the link condition viz., that the equivalence classes of vertices, edges and faces with respect to \mathcal{I} satisfy the above link condition.

The big question then is:

Question 3: Given two identification data (P_j, \mathcal{I}_j) , $j = 1, 2$ when are the quotient manifolds homeomorphic.

I call this original programme of Poincaré.

Similar to the 2-dimensional case, it is necessary to find a number of ‘simple operations’ which convert one data into another without changing the homeomorphism type of the quotient space and ‘enough’ of such operations so that if two sets of identification data give the same quotient space type then one can be converted into another via a finitely many such operations and their inverses. Unfortunately both these tasks have been proven to be very very difficult in the 3-dimensional case.

5 Historical Development

Nevertheless, Poincaré's question (along with many of his ideas) has played an extremely important role in the development of Topology which cannot be over-emphasized. In 1910, Max Dehn came up with the following result:

Theorem 5.1 (*Max Dehn*) *A piecewise linearly embedded circle K in S^3 is unknotted iff the fundamental group $\pi_1(S^3 \setminus K)$ is infinite cyclic.*

In a somewhat different direction, yet very foundational, J. W. Alexander (1888-1971) attacked the subject and proved the following:

Theorem 5.2 Alexander *A piecewise linearly embedded 2-sphere in S^3 separates S^3 into two piecewise linearly embedded 3-cells.*

This result became corner stone for Kneser's work who described a general procedure to cut a 3-manifold along piecewise linearly embedded 2-spheres into simpler pieces, which can no longer be cut into further simpler pieces, viz., those which are 'irreducible'. Thus a 3-manifold is irreducible if every pl embedded S^2 in it bounds a 3-cell. As an example observe that $S^3, P^3, S^1 \times S^1 \times S^1$ are all irreducible.

Nineteen years after Dehn's result was published, Kneser also found out that there is a serious gap in Dehn's arguments viz., in one of the lemma's that he claimed and used in his result.

Herbert Seifert(1907-1966) took yet another approach by considering a special class of 3-manifolds. Notice that an easy way to get a 3-manifold is to take Cartesian product of a 2-manifold with a circle. A little sophisticated way would be to introduce certain twists in the product. Seifert went one step further by considering finitely many ‘singular’ fibres. In a nbd of this fiber the manifold can be described as follows: For some integer $n \geq 2$, let ζ be a primitive n^{th} root of unity. Let D^2 denote the open unit disc in \mathbb{C} . Consider the action of $\mathbb{Z}/n\mathbb{Z}$ on $D^2 \times \mathbb{R}$ given by

$$(z, t) \mapsto \left(\zeta z, t + \frac{1}{n} \right)$$

The quotient space X fibers over D^2 with each fiber being a circle. However, this is not a locally trivial fibration; the fiber over 0 is ‘shorter’ than the other fibers. These manifolds are called Seifert fibered spaces which play a very important role in the classification of 3-manifolds and give very useful information in the study of geometric of algebraic surfaces.

Meanwhile, nobody could fill the gap for 35 years or so until Papakariakopoulus filled this gap, in an ingenious way.

Theorem 5.3 Dehn's Lemma (Papakariakopoulus -1957) *Any piecewise linearly embedded loop which is null homotopic in 3-manifold bounds an embedded disc.*

In this now classical work published in Annals of Mathematics, Papa also proved another very important result in the same paper which goes under the name of Sphere Theorem. Even today, there is only a slightly simpler version of this proof of Dehn's lemma which is not all that easy. (See Hempel's book).

These two results provide some kind of a break-through following which 3-manifold topology saw some quick development through the works of Haken, Waldhausen, Jaco, Shalen, Johansson etc..

On the quiet side, those somewhat unpleasant and annoying questions about a topological 3-manifold admitting a triangulation (Moise 1977) and as that every 3-manifold has essentially unique smooth structure (Hirsch-Smale 1959) were answered in the affirmative. Nowadays, everybody knows that the situation in higher dimension is quite different, n -dimension 4 being the weirdest.

Another set of Mathematicians were pursuing yet another track once again opened by Poincaré himself, viz., obtaining manifolds as quotients of known manifolds via group actions. D. Hilbert had posed (Problem no. 18) the question whether there are only finitely many discrete groups of rigid motions of the Euclidean space with compact fundamental domains. This problem was affirmatively answered by Bieberbach in 1910. His work leads to the complete understanding of flat Riemannian manifolds. This was followed by H. Hopf's work on classification of compact 3-manifolds of constant positive curvature.

The manifolds of constant negative curvature were posing a real problem until the works of Thurston in 1970's. Thurston came up with the so called Geometrization conjecture which if proved took care of Poincaré's conjecture as a special case. Thurston's programme came with a big promise but slowly, as the decks were getting cleared, it was becoming more and more evident that Poincaré's purely topological question has to seek answer in hard core differential geometry.

In 1982, R. Hamilton came up with such a programme which is known as Ricci flows. I will not speak about this at all. Suffices it to say that it required tremendous foresight and meticulous work to complete this programme which Perelman seems to have achieved.

6 Surmise Or a Wishful Thinking

It is desirable to have a proof of Poincaré's conjecture on lines indicated by him viz., though not explicitly mentioned, viz., representation of every compact manifold as a quotient of a single 'canonical polyhedron'. Amongst many other things today we have powerful computers on our side as compared to Poincaré's time. (Remember the 4-colour problem!) For example, it is easy to show that we need to restrict our attention to only a very special type of triangulated polyhedrons whose boundary complex can be obtained by successive starring of the boundary complex of a tetrahedron. In particular, it is **shellable**. I strongly believe that Mathematician of this century won't just rest on this great achievement of having solved Poincaré's Conjecture. There are still many things one want to explore in 3-dimensional topology.

References

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