Topology of Open Surfaces
around a landmark result of C. P. Ramanujam

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1 Introduction

In 1971, in a landmark paper in the theory of algebraic surfaces [R], C. P. Ramanujam proved the following result:

Theorem 1 *Let X be a non singular complex algebraic surface which is contractible AND simply connected at infinity. Then X is isomorphic to the affine two-space as an algebraic variety.*

Recall the uniformization theorem for Riemann surfaces viz., *any simply connected Riemann surface is bi-holomorphic to either the unit disc* $\mathbf{E}$ *in* $\mathbb{C}$, *the complex plane* $\mathbb{C}$ *or the extended complex plane* $\hat{\mathbb{C}}$. Ramanujam's theorem is a 2-dimensional analogue of this classical 1-dimensional result. That should be enough to convince anyone of its fundamental nature and importance.

The statement of theorem 1 is reproduced from [R]. Observe the capitals 'AND' in the statement of the theorem, which emphasize the hypothesis that follows. This is substantiated by including in the paper, a wonderful example of a contractible smooth algebraic surface which is not simply connected at infinity (and hence is not isomorphic to $\mathbb{C}^2$). This example, the first one of its kind, was the only one known for several years.

In 1961, a powerful topological technique was developed by Mumford [Mu], in the study of isolated singularities of a surface. Subsequently, this was extensively used by several authors for the same purpose. Ramanujam showed how this tool can be effectively used elsewhere, by successfully employing it in the study of open algebraic surfaces. Later, this has been employed by other authors in the study of fibrations of algebraic surfaces as well, see e.g., [Ne]. This tool has come to be known as the Mumford-Ramanujam method.

It is the aim of this article to present some of the salient features of this paper of Ramanujam [R], keeping in mind the general mathematical reader in this area. In §2, we recall a bare minimum of basic concepts from algebraic surfaces. The reader is advised to refer to the excellent book [B-P-V] for more details. We recall other necessary concepts and results as and when needed. In §3, we describe the Mumford-Ramanujam method and the counter example of Ramanujam. In §4, we give a summary of some later developments, directly related to the M-R method. No effort has been made to be exhaustive.

2 Preliminaries

By a complex affine algebraic variety X in $\mathbb{C}^n$, we mean the subspace of common zeros of a finite set of polynomials in n variables with coefficients in $\mathbb{C}$. The set of all $\mathbb{C}$-valued maps on X which can be represented by polynomials is denoted by $k[X]$ and is called the coordinate ring of X. One may say that the geometry of X is completely encoded in the algebraic structure of the ring $k[X]$. Indeed, the coordinate rings $k(X_1)$ and $k(X_2)$ are isomorphic iff the affine varieties X_1, X_2 are. The coordinate ring of $\mathbb{C}^n$ is isomorphic to the ring $\mathbb{C}[T_1, \ldots, T_n]$ of polynomials in n-variables with coefficients in $\mathbb{C}$. In general, the set of all polynomials which vanish on X form an ideal I in this ring and $k[X]$ is isomorphic to the quotient ring $\mathbb{C}[T_1, \ldots, T_n]/I$. Geometric concepts about X may be translated into algebraic statements about $k[X]$ and also quite often, vice versa.

The n-dimensional complex projective space $\mathbb{CP}^n$ is defined to be the space of all 1-dimensional linear subspaces of the complex vector space $\mathbb{C}^{n+1}$. Alternately, this space can be viewed as the quotient space of $\mathbb{C}^{n+1}\setminus\{0\}$ under the following equivalence relation: $\mathbf{v} \sim \mathbf{u}$ iff $\mathbf{v}, \mathbf{u}$ are linearly dependent.

Observe that it makes sense to say whether a given homogeneous polynomial in $n+1$ variables vanishes at a point in the space $\mathbb{CP}^n$. Such a point may be called a zero of the given homogeneous polynomial. A projective algebraic variety in $\mathbb{CP}^n$ is the set of common zeros of a set of homogeneous polynomials in $n+1$ variables. By an algebraic variety we mean either an affine variety or a projective variety. Amongst the three spaces that occur in the statement of the 1-dimensional uniformization theorem, the first one viz. the unit disc $\mathbf{E}$, is not an algebraic variety of any sort. Of course $\mathbb{C}$ is an affine variety. The extended complex plane can be easily identified with $\mathbb{CP}^1$.

An algebraic variety inherits the usual topology from the ambient space $\mathbb{C}^n$ or $\mathbb{CP}^n$. This is called the C^∞-topology. However, it is customary to take another topology on an algebraic variety. This one is known as the *Zariski topology* which is defined by taking as closed sets, all algebraic subvarieties. Observe that a non empty Zariski-open set is open and dense in the C^∞-topology.

A variety is called irreducible if it cannot be written as a union of two proper Zariski closed subsets. An affine variety X is irreducible iff the coordinate ring $k[X]$ is an integral domain. For example, the union of two axes in $\mathbb{C}^2$ is not irreducible. Any variety is the finite union of its irreducible components. Often, we shall refer to them merely as 'components', and this should not be confused with connected components.

If an algebraic variety X happens to be a complex manifold also, with respect to the C^∞-topology, then we say that it is a smooth (non singular) complex algebraic variety. Thus for instance, $\mathbb{C}^n$ is a smooth complex algebraic n-dimensional variety. To emphasize the 'algebraic structure', it is customary to call it *affine n-space over* $\mathbb{C}$ and denote it by $\mathbb{A}^n_{\mathbb{C}}$ or simply by $\mathbb{A}^n$. The set of smooth points of a variety forms a manifold, the dimension of which is also the dimension of the variety. A 1-dimensional variety is called a *curve* and a 2-dimensional variety is called a *surface*.

[1] The set of all polynomial maps which vanish at a given point p of an affine variety X is a maximal ideal m in $k[X]$. Once again, geometry of X around this point is encoded in the ring $k[X]_m$, the localization of $k[X]$ at the maximal ideal m. For instance, the point is a smooth point iff the local ring is regular. It is called a *'normal'* point if the ring is an integrally closed domain. A smooth point is of course always normal but the converse is far from being true. A variety is normal if all points in it are normal. In a normal variety, the set of nonsmooth points form a codimension 2, Zariski-closed subset.

3 The Mumford-Ramanujam (M-R) Method

Consider a surface X over $\mathbb{C}$ and a point $p \in X$. Assume that p is at worst, a normal singularity. Topologically, this means that the singularity p is isolated and the surface has a single branch at the point. Our story begins with the following question raised by Abhyankar around 1954:

Abhyankar's Question: *Suppose p has a fundamental system of neighborhoods $\{U_j\}$ in X such that $U_j \setminus \{p\}$ are simply connected. Then, is p a simple point?*

To say that p is a simple point means that, topologically the surface is a manifold at the point p. In that case, it has a fundamental system of neighborhoods U_j such that each U_j is homeomorphic to $\mathbb{R}^4$. In particular, each $U_j \setminus \{p\}$ is simply connected. So, the condition in the above question is necessary. In 1961, in the paper that we have cited above [Mu], Mumford answered this question in the affirmative. We first describe some ideas from this paper.

Mumford's Theorem Given a normal singularity on a surface, one first *resolves* the

[1] You may ignore this paragraph and read on.

singularity to get a surface X' with an exceptional divisor D. Resolution of a singularity of a surface consists of a finite sequence of normalizations and quadratic transformations, (or *blows-ups*) centered at the given point and its transforms. This process does not affect any part of the surface other than the point p. The end result gives a surface $\overline{X}$ and a surjective morphism $\phi : \overline{X} \longrightarrow X$. The curve $D = \phi^{-1}(p)$ so obtained is called the *exceptional curve.* The map $\phi : \overline{X} \setminus \{P\} \longrightarrow X \setminus \{p\}$ is an isomorphism and for a small neighborhood U of p in X, $\phi^{-1}(U)$ does not contain any singularity of $\overline{X}$. Using the fact that p is a normal singularity one deduces that the curve D is connected. In general, D need not be irreducible. However, each component of D is a complete curve. By taking further quadratic transforms, one can assume that irreducible components of D are smooth, and whenever they intersect each other the point of intersections are *simple normal crossings*. For instance, this means that no three of the components meet in a single point and if two of them meet at a point then they are transversal to each other at that point. In the figure below only the third one is an example of a simple normal crossing.

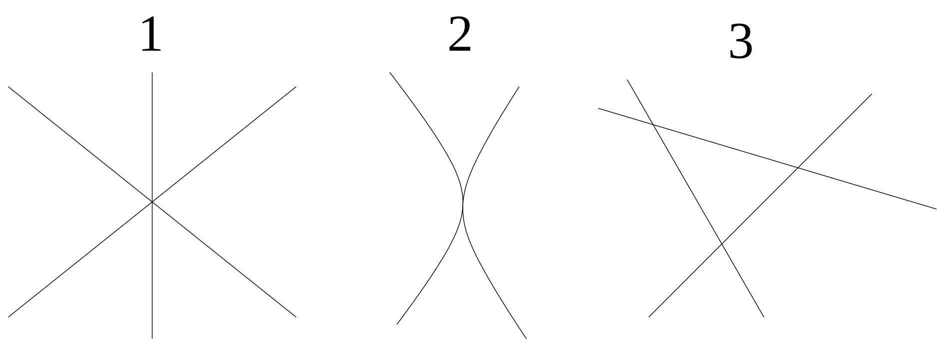

Fig. 1

Mumford opened up the gateway to study the topology of $U \setminus \{P\}$ through the information about the curve D, such as the pattern in which various components intersect with each other, the self-intersection numbers of various components and the topology of each individual component.

For instance, associated to such a curve is a weighted graph Γ defined as follows: The vertex set of Γ is indexed by the set of irreducible components of D. Two vertices are joined in Γ with an edge for each point of intersection of the corresponding two components. Further a number (weight) is assigned to each vertex viz., the self-intersection number of the corresponding component.

The graph Γ can actually be drawn on D itself as follows: For each vertex, choose a point on the corresponding component other than any point of intersection with other components. Now draw the edges joining the respective vertices, so that they pass through the point of intersection of the two components to which it corresponds. It can be seen without much difficulty that the graph so drawn is actually a retract of D. In particular, the graph Γ is simply connected. It is a standard result in topology that any simply connected graph is a tree, i.e., it has no loops.

Now, one can show that each component of D is also a retract of D and hence, simply connected. Thus, each component of D is a simply connected compact Riemann surface and hence, by the 1-dimensional uniformization theorem, is isomorphic to $\mathbb{C}P^1$, the 1-dimensional complex projective space.

Mumford gave a method to write down a presentation of the fundamental group of $U \setminus \{P\}$, where U is a tubular neighborhood of D. This presentation depends merely upon the curve D —the way the various irreducible components intersect each other and their self-intersection numbers. The former information has been coded in the graph Γ defined above. The latter information can be coded by assigning a weight, viz., $w(v) = (C_v \cdot C_v) =: (C_v^2)$, the self-intersection number of the component of D corresponding to v.

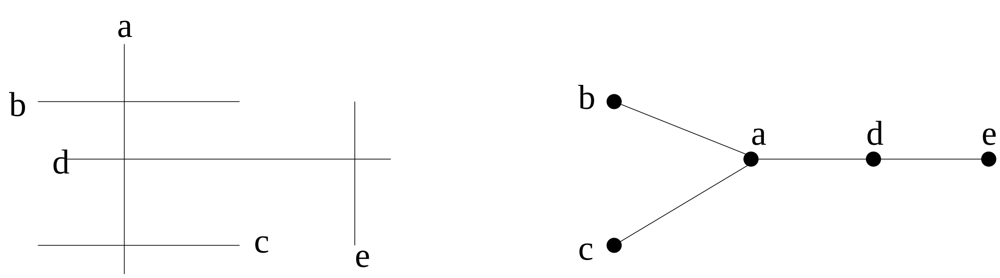

Fig. 2

In the above picture, the letters a, b, c, d, e indicate the self-intersection numbers of the corresponding curves, which are also the weights assigned to the corresponding vertices. Let Γ be any (finite) tree with integer-valued weights assigned to its vertices. Associated with this, we define a group $\pi(\Gamma)$ as follows: generate a free group F on the vertex set $\{v_1, \ldots, v_n\}$ of Γ. Put the commuting relations $[v_i, v_j] = 1$ whenever v_i and v_j are adjacent. Further, for each vertex v_i, let $\{v_{i1}, \ldots, v_{ik}\}$ be all the vertices adjacent to v_i. Put the relations $r_i := v_i^{w(v_i)} v_{i1} v_{i2} \cdots v_{ik} = 1$. That is $\pi(\Gamma)$ is a quotient of F by the normal subgroup generated by all elements $[v_i, v_j]$'s and r_i's.

Using the Seifert Van-Kampen theorem, Mumford shows that the fundamental group of $U \setminus \{P\}$ is the same as $\pi(\Gamma)$ defined above.

Let v be any vertex and let $\Gamma_1, \ldots, \Gamma_k$ be the components of the graph obtained from Γ by removing v and all the edges incident at v. It is easily seen that

Lemma 3.1 *The quotient group $\pi(\Gamma)/ < v >$ is isomorphic to the amalgamated free product $\pi(\Gamma_1) \star \cdots \star \pi(\Gamma_k)/ < v_1 v_2 \cdots v_k >$, the quotient of the free product by the normal subgroup generated by the single element $v_1 \cdots v_k$, where $v_i \in \Gamma_i$ are the vertices adjacent to v in Γ.*

Mumford now makes the following key observation:

Lemma 3.2 [2] *If a group $\pi(\Gamma_1) \star \pi(\Gamma_2) \star \pi(\Gamma_3)/ < v_1 v_2 v_3 >$ is trivial then one of the $\pi(\Gamma_i)$ is trivial.*

Mumford uses this lemma in a very ingenious way to prove that the curve D is not minimal, i.e., there is a component C of D with the self-intersection number -1 and which intersects at most two of the other components of D. Such a component can be blown-down to a smooth point and the resulting curve is still a curve with normal crossing. Repeated application of this argument leads to the conclusion that D could have been chosen to be a single line with self-intersection number -1. This single line therefore can now be blown-down to a smooth point and hence the point p was, after all, a smooth point to begin with.

To a weighted graph Γ as above, one associates a quadratic space $Q(\Gamma)$ as follows: The underlying vector space has a basis consisting of the vertices of Γ. The symmetric bilinear form is given by:

$$\langle v, u\rangle := \begin{cases} 1 & \text{if } v \neq u \quad \text{and } v \text{ is joined to } u \\ 0 & \text{if } v \neq u \quad \text{and } v \text{ is not joined to } u \\ w(v) & \text{if } v = u. \end{cases}$$

The fact that the curve D is the exceptional divisor of a blow-up of a point on a surface implies that the quadratic space $Q(\Gamma)$ is negative definite. This result plays a crucial role in the above proof.

We can now discuss Ramanujam's result.

Ramanujam's Theorem Recall that a topological space V is contractible iff it is homotopy equivalent to a singleton space. Alternatively, V is contractible iff the identity map of V is homotopic to the constant map.

One of the topological concepts that is not commonly found in standard texts but is central in Ramanujam's paper is the notion of 'simple-connectedness-at-infinity'. This can be explained best by a few examples. This notion is defined only for non compact spaces. To begin with let us consider a simpler notion viz., the notion of 'connectedness-at-infinity'. For instance, the real line $\mathbb{R}$ is not connected at infinity. Indeed, it has precisely two connected components at infinity. On the other hand, $\mathbb{R}^n$ for $n \geq 2$, are all connected at infinity. Roughly speaking, a (non compact) topological space X is said to be connected at infinity if for every compact subset K of X there is another compact subset K_1 of X such that $K \subset K_1$ and $X \setminus K_1$ is connected.

For noncompact spaces which are connected at infinity one can talk about whether they are simply connected at infinity or not. The best way to handle connectivity at

[2]The proof given by Mumford in [Mum] of this lemma is incorrect. See [G-S1] or [W] for a correct proof as well as a more general result.

infinity for a 'nice' space X is to take a suitable compactification Y of X and look at the neighborhoods U of $Z := Y \setminus X$ in Y. The topological information at infinity for X is hidden in $U \cap X$ which is called a punctured neighborhood of infinity for X. The space X is said to be simply connected at infinity if Z is connected and there is a fundamental system of neighborhoods $\{U_j\}$ of Z in Y such that $U_j \setminus Z$ are simply connected. With this criterion, one can now easily determine the value of n for which $\mathbb{R}^n$ is simply connected at infinity by taking the one-point compactification of $\mathbb{R}^n$, viz., the n-dimensional sphere $\mathbb{S}^n$. If P denotes the point at infinity we know that it has a fundamental system of neighborhoods $\{U_j\}$ with each U_j homeomorphic to $\mathbb{R}^n$. For $n = 1$, $U_j \setminus \{P\}$ is disconnected. So, $\mathbb{R}$ is not connected at infinity. For $n \geq 2$, $U_j \setminus \{P\}$ is connected. Since each $U_j \setminus \{P\}$ is of the same homotopy type as S^{n-1}, it follows that $\mathbb{R}^2$ is not simply connected at infinity and $\mathbb{R}^n$, $n \geq 3$ are all simply connected at infinity.

It turns out that simple connectivity at infinity is a homeomorphism invariant. (It is not a homotopy invariant). In particular, $\mathbb{C}^2$ being homeomorphic to $\mathbb{R}^4$, is simply connected at infinity. It is of course contractible and smooth. Therefore, all the conditions in Theorem 1 are necessary.

Observe that the one-point compactification of $\mathbb{R}^n$ is again a manifold. The extra point that we added to the space is not a singularity. This is going to play an important role now. The problem with the one-point compactification is that it does not yield algebraic varieties, and often not even a complex manifold. The remedy is to allow other nicer compactifications. Thus for instance, $\mathbb{C}P^n$ is a compactification of $\mathbb{C}^n$, which is a smooth variety. The divisor at infinity in this case is $\mathbb{C}P^{n-1}$ which is as simple as one can expect.

Indeed, even the converse is true for $n = 2$ and Theorem 1 is proved precisely by showing that for the surface X satisfying the conditions of the theorem, a compactification Y can be chosen so that $Z = Y \setminus X$ is isomorphic to $\mathbb{C}P^1$. An isomorphism of Y onto $\mathbb{C}P^2$ can now be obtained from the complete linear system $|Z|$. Hence X is isomorphic to $\mathbb{A}^2$.

The Nagata compactification theorem says that given any affine surface X, we can choose an embedding of X in a normal algebraic surface Y such that the curve $Z := Y \setminus X$ is a hyperplane section. So, we should aim at showing that Y can be chosen such that Z is isomorphic to $\mathbb{C}P^1$.

By resolution of singularities, we can arrange Z to have normal crossings. Now the situation with Z is exactly similar to the situation with D in Mumford's theorem, except that the associated quadratic form is no longer negative definite. However, this is not so bad. Again by the Hodge index theorem, it follows that the quadratic form has exactly one positive eigenvalue, in this situation.

Ramanujam proceeds in a way similar to Mumford's, using combinatorial group theory to show that one can choose Y in such a way that Z consists of a single line L with $(L)^2 = 1$. It turns out that it is enough to show that the associated graph does not have any branch points. Let the number of branch points be k. The idea is to arrive at a contradiction if $k \geq 1$.

However, the non availability of negative definiteness of the quadratic form raises hurdles already for the case $k = 1$ and soon these hurdles seem to become unsurmountable. Ramanujam now pulls out an innocent looking weapon from the surface theory. He proves:

Lemma 3.3 *On a complete non-singular surface V with $H^1(V, \mathcal{O}_V) = 0$, there cannot exist a system of five lines $L_i (1 \leq i \leq 5)$ with normal crossings such that the associated weighted graph looks like:*

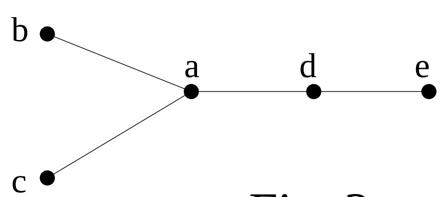

Fig. 3

with $a = -1$ and $e \geq 0$.

With the help of this, the case $k = 1$ is disposed off easily. Once again the case $k = 2$ needs additional treatment, viz., a combination of the unimodularity of the associated quadratic space with the above lemma. The case $k = 3$ and $k \geq 4$ had to be dealt separately using very minute, long drawn combinatorial arguments, which however, do not involve any additional techniques.

Ramanujam's Example The importance of Ramanujam's paper is enhanced by the example that it contains of a contractible smooth surface which is not simply connected at infinity. The construction begins with the projective plane $\mathbb{C}P^2$, a cubic curve C_1 with a cusp on it and a non-degenerate conic C_2 which intersects C_1 in two distinct points P, Q with intersection-multiplicities $5, 1$ respectively. P, Q ought to be chosen different from the cusp and the point of inflexion of the cubic. Now blow-up at the point Q to obtain a surface F. Let C'_i denote the proper transforms of C_i. If we put $X = F \setminus (C'_1 \cup C'_2)$, then X is the required surface.

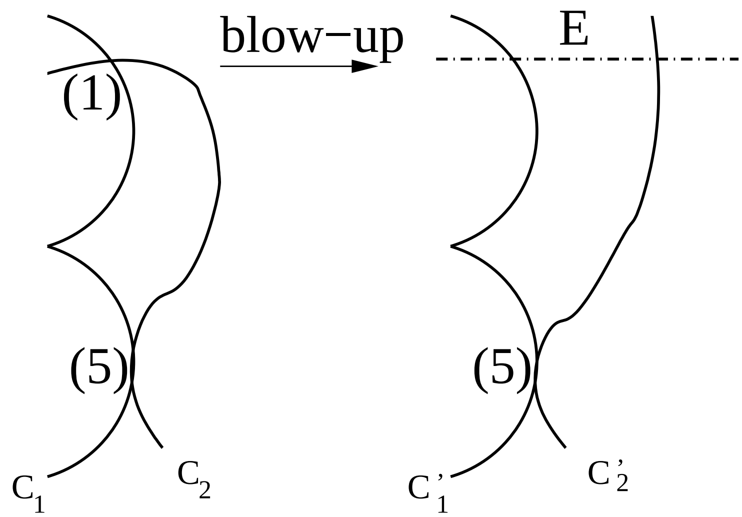

Fig. 4

The existence of such a configuration of curves on $\mathbb{C}P^2$ is justified in the paper as follows: Take C_1 to be any cubic curve with a cusp. Then $C_1 \setminus \{\text{cusp}\}$ is an algebraic group isomorphic to the additive group of $\mathbb{C}$. Such a curve also has a unique point of inflexion which plays the role of 'zero' for the additive group law.

If a conic meets C at points P_i with multiplicity n_i then we must have $\sum_i n_i P_i = 0$ in this additive group and conversely. So, choose P to be any point of the curve other than the cusp and the point of inflexion and $Q = -5P$. There is a unique conic C_2 passing through P and Q and having intersection multiplicities with C_1 as $5, 1$ respectively at these points. [3]

It is somewhat of a mystery as to how Ramanujam arrived at this example.

The proof that X is homologically trivial is not difficult. If one shows that the fundamental group of X is abelian, then it follows that X is actually simply connected, since the first homology of X is nothing but the abelianization of the fundamental group of X. Then by appealing to a theorem of Whitehead, it follows that X is contractible. Ramanujam's proof of the fact that the fundamental group of X is abelian is somewhat involved, and we shall skip it.

There is a readily available compactification of X viz., F as above. Then the divisor at infinity is $Z := F \setminus X = C_1' \cup C_2'$. Observe that after resolving the singularity and the nonnormal intersections of $C_1' \cup C_2'$, the associated weighted graph of the divisor at infinity looks like:

[3] You may still want to see a very specific example. Well! Here is one: Take C_1 to be the curve given by $YZ^2 = X^3$. Then the point $[0,0,1]$ is the point of inflexion and $[0,1,0]$ the cusp. Choose $P = [1,1,1]$. Then C_2 is given by $45X^2 + 15XY - Y^2 - 24XZ - 40YZ + 5Z^2 = 0$, and $Q = [-5, -125, 1]$. There are many different ways of working this out. One can teach a lot of geometry of curves using this example. I strongly feel that this example should be included in text books on this subject.

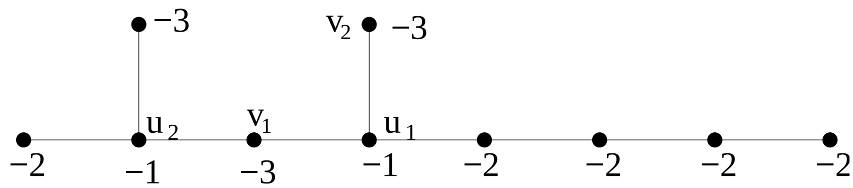

Fig.5

If G denotes the fundamental group at infinity of X, then one can easily read out a presentation for it from the above figure. Assume that G is trivial. Let u_1, u_2 denote the two branch points. Applying Lemma 3.2 at the branch point u_1, we see that the group associated to the branch

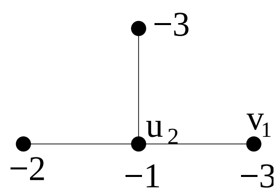

Fig.6

is trivial. Apply Lemma 3.2 again at the branch point u_2 of this graph to arrive at a contradiction.

This completes the proof that X is a contractible smooth affine surface with non trivial fundamental group at infinity. In retrospect, it turns out that the simple-connectedness of X can be derived easily as follows: First of all, by arguments akin to Lefchetz's hyperplane section theorem (see [No]), it can be seen that the fundamental group of X is a quotient of the fundamental group at infinity of X. (Implicitly, this is proved in [R].) Now look at the doubly punctured exceptional line E contained inside X which is obtained by blowing-up the point P (See Fig. 4). This gives an extra relation in $\pi_1(X)$ in addition to those relations which appear in the presentation of the fundamental group at infinity of X, viz., $v_1 v_2 = 1$, where v_1, v_2 are generators corresponding to the proper transforms of C_1', C_2' in D (See Fig.5). From this it is elementary to see that the group $\pi_1(X)$ is trivial.

However, the elegant proof given by Ramanujam has its own educative value.

4 The Impact

It may be noted that Ramanujam arrived at his theorem and the example, in his attempt to solve the following problem.

A cancellation problem Suppose X is an algebraic variety over a field k and $X \times \mathbb{A}^1 \approx \mathbb{A}^3$. Is $X \approx \mathbb{A}^2$?

For some time, Ramanujam's example was thought of as a likely candidate for a counter example, though nobody actually conjectured so. The cancellation problem has since been solved in the affirmative by Miyanishi and others. We discuss more about this below. Observe that if we remove a tubular neighborhood of Z in F occuring in Ramanujam's example, then we get an example of a contractible 4-dimensional topological manifold with boundary which is not a homotopy sphere and whose interior is homeomorphic to Ramanujam's example. Such topological examples had been constructed by several authors, even before Ramanujam's example. Ramanujam's example is special because it is a 2-dimensional complex affine variety.

In 1980, R. V. Gurjar [G1] used the Mumford-Ramanujam method, to study the following question, in his thesis:

Gurjar's Question Let V be a smooth affine surface over $\mathbb{C}$. Suppose there is a proper surjective morphism $f : \mathbb{C}^2 \longrightarrow V$. Is $V \approx \mathbb{C}^2$?

Gurjar got an affirmative answer to this question, except for one case when the fundamental group at infinity of V is the binary icosahedral group.[4]

Around this time, the importance of the algebro-geometric notion of logarithmic Kodaira dimension (introduced by Iitaka) was being realized more and more, in the study of open algebraic surfaces. Let V be embedded in a smooth projective surface $\overline{V}$ and $D = \overline{V} \setminus V$, be the reduced divisor at infinity, with simple normal crossings. Let $K_{\overline{V}}$ be the canonical divisor on $\overline{V}$. For each n, let $\Phi_n : \overline{V} \longrightarrow \mathbb{C}P^{k(n)}$ be the rational map defined by the linear system $|n(K_{\overline{V}} + D)|$ from $\overline{V}$ to some projective space. Then the logarithmic Kodaira dimension of V is defined by

$$\overline{\kappa}(V) := \max \{\dim \Phi_n(\overline{V}) \ : \ n \geq 1\}.$$

For a surface V, $\overline{\kappa}(V)$ can take the values $-\infty, 0, 1$ and 2.

Very neatly fitting into this situation, came another characterization of the affine plane, by Fujita, Miyanishi and Sugie [F-M-S]:

Theorem 2 *Let V be a smooth affine surface over $\mathbb{C}$ such that :*
(i) the co-ordinate ring $\Gamma(V)$ of V is a unique factorization domain,
(ii) The group of units in $\Gamma(V)$ is equal to $\mathbb{C} \setminus \{0\}$.

[4] Incidentally, we have come across the binary icosahedral group several times in our investigations as some kind of a test case.

(iii) The logarithmic Kodaira dimension $\overline{\kappa}(V) = -\infty$.
Then $V \approx \mathbb{C}^2$.

It may be recalled that condition (iii) is the same as saying that $|n(K_{\overline{V}} + D)| = \emptyset$ for all $n \geq 1$. It turns out that the contractibility condition on V occurring in Ramanujam's theorem can be replaced by somewhat weaker conditions (i) and (ii). Thus we see that the role of simple-connectivity at infinity is being played by the condition $\overline{\kappa}(V) = -\infty$. In practice however, the logarithmic Kodaira dimension seems to give better control, as expected. For instance, Gurjar's problem can now be resolved completely in the affirmative. Fujita, Miyanishi and Sugie derived an affirmative answer to the cancellation problem as well from the above characterization of $\mathbb{C}^2$. What is more, using $\overline{\kappa}$, Miyanishi proved the following generalization of Gurjar's result:

Theorem 3 *Let V be a normal affine surface. If there is a proper surjective morphism $f : \mathbb{C}^2 \longrightarrow V$ then V is isomorphic to a quotient of $\mathbb{C}^2$ by a linear action of a 'small' subgroup G of $GL(2, \mathbb{C})$.*

Observe that, in particular, this implies that V has at most one singular point with the local fundamental group isomorphic to G. If we assume that V is actually smooth, it follows that $G = (1)$ and hence V is isomorphic to $\mathbb{C}^2$, thus recovering Gurjar's result.

Gurjar and the author gave a proof of following stronger version of Theorem 3, using the M-R method:

Theorem 4 *Let V be a normal affine surface. If there is a proper surjective morphism $f : \mathbb{C}^2 \longrightarrow V$, then V is isomorphic to a quotient of $\mathbb{C}^2$ by a linear action of a 'small' subgroup G of $GL(2, \mathbb{C})$. If V is smooth, then it is isomorphic to $\mathbb{C}^2$. If the coordinate ring of V is a unique factorization domain, then V is isomorphic to the surface $x^2 + y^3 + z^5 = 0$.*

Later, they proved the following topological version of Miyanishi's result, which is at the same time a generalization of Ramanujam's result:

Theorem 5 *Let V be an affine normal surface which is topologically contractible and has finite fundamental group at infinity. Then V is isomorphic to a quotient of $\mathbb{C}^2$ by a linear action of a 'small' subgroup G of $GL(2, \mathbb{C})$.*

Following the line of arguments used by Ramanujam and using only a little more standard topological tools from 3-dimensional topology, the author gave a complete classification of all connected complete curves with finite local fundamental group, occurring on a projective surface. (See [S1].) The classification is quite analogous to that of 'quotient' singularities of a surface. As an easy corollary of this, one can prove that any affine surface with finite fundamental group at infinity is rational. This was also proved by Gurjar, in a direct way, without appealing to this classification.

All these results suggest a certain relationship between $\overline{\kappa}$ and the fundamental group at infinity. This was discovered by Gurjar and Miyanishi [G-M], using the classification mentioned above:

Theorem 6 *Let V be a smooth*[5] *affine surface with finite fundamental group at infinity. Then $\overline{\kappa}(V) = -\infty$.*

Thus, we see that condition (iii) of Theorem 2 is also somewhat weaker than the corresponding condition of simple connectedness at infinity occuring in Theorem 1. This might suggest that Ramanujam's result (Theorem 1) can be deduced from F-M-S characterization as given in Theorem 2. Alas! This is not quite so. For, the classification theorem of [S] itself is based on the M-R method and already incorporates most of the arguments involved in Ramanujam's proof of Theorem 1.

Very recently, Gurjar was able to give a proof of the Cancellation theorem in [G2], based on the M-R method, fulfilling a long cherished dream. (See [G2]). If $\phi : X \times \mathbb{A}^1 \longrightarrow \mathbb{A}^3$, is an isomorphism, then consider the map $\psi := \pi \circ \phi : \mathbb{A}^3 \longrightarrow X$. If ψ restricted to a generic plane in $\mathbb{A}^3$ is a proper morphism, then we are in the situation of Theorem 4. So, Gurjar has to handle the case when this does not happen. In this case, he shows that X contains 'lots' of $\mathbb{A}^1$, using entirely different techniques. From which it is not difficult to deduce that X is isomorphic to $\mathbb{A}^2$.

In the same paper [G2], there is an appendix due to Pradeep and Gurjar, which offers some simplifications in the arguments that Ramanujam presents in [R], precisely where they become tedious, viz., in handling the case when the branch points in the graph are more than two, by neatly rearranging Ramanujam's arguments. However, the original arguments still have their value. For example, in classifying curves with finite local fundamental group, such rearrangement does not seem to work, whereas Ramanujam's arguments have been successfully adopted (See [S1]).

Compact structures on open surfaces

Let V be a smooth complex n-dimensional manifold. We say V is *compactifiable* if there exists an n-dimensional compact manifold S such that $V \subset S$ as a Zariski open subset. (Observe that S is automatically an algebraic variety.) In this case, we put $C = S \setminus V$, and say (S, C) is a *compact structure* or an *algebraic structure on V*. Observe that the 1-dimensional unit disc **E** does not admit any compact structure. The uniformization Theorem implies that the only compact structure on $\mathbb{C}$ is $(\hat{\mathbb{C}}, \infty)$.

In general, there is a need to introduce an equivalence relation amongst all such pairs (S, C). In the 2-dimensional case, this can be formulated as follows: Two such

[5]It may be possible to relax the condition 'smooth' to 'normal'.

pairs (S, C) and (S', C') are said to be equivalent if there are pairs $(S_i, C_i), 0 \leq i \leq n$, such that $(S, C) = (S_0, C_0), (S', C') = (S_n, C_n)$ and each (S_{i+1}, C_{i+1}) is obtained from (S_i, C_i) either by blowing-up a point in C_i or by its reverse operation.

In [Si], Simha initiated the study of classification of compact structures on $\mathbb{C}^\star \times \mathbb{C}^\star$ which was completed by Susuki [Su] and Ueda [U]. These authors work in the set-up of a holomorphic category and use heavy machinery of cluster sets of holomorphic mappings. In [S2], the author recovers and extends these results by bringing in the M-R method. Here is a brief description of it.

There is a sharper notion of homotopy especially to handle the behavior at infinity of noncompact spaces. Recall that a map $f : X \longrightarrow Y$ of topological spaces is called *proper* if inverse images under f of compact sets are compact. Two maps f_0, f_1 are said to be *proper homotopy equivalent* if there is a homotopy between them which is a proper map. The map f is called a *proper homotopy equivalence* if it has a proper homotopy inverse g. It is not difficult to see that proper homotopy preserves connectivity at infinity as well as the fundamental group at infinity.

Now Ramanujam's Theorem can be restated as follows:

Theorem 7 (Ramanujam) *Let V be a smooth 2-dimensional complex manifold. Then V is biholomorphic to $\mathbb{C}^2$ and carries a unique algebraic structure iff it is compactifiable and is proper homotopy equivalent to $\mathbb{C}^2$.*

In [S2], the author proved the following generalizations of the above result:

Theorem 8 *Let (S, C) be a compact structure on a 2-dimensional complex manifold V.*
(1) If V is proper homotopy equivalent to $\mathbb{C} \times \mathbb{C}^\star$, then (S, C) is equivalent to $(\mathbb{CP}^2, 2L)$ where $2L$ denotes the union of two lines in $\mathbb{CP}^2$.
(2) If V is proper homotopy equivalent to $\mathbb{C}^\star \times \mathbb{C}^\star$, then (S, C) is equivalent to one of the following three:
(a) $(\mathbb{CP}^2, 3L)$, where $3L$ denotes the union of any three lines in general position in $\mathbb{CP}^2$.
(b) (F, E) where F is the total space of a $\mathbb{CP}^1$-bundle over a smooth elliptic curve and E is a section of this bundle with self intersection $(E^2) = 0$.
(c) (H, E) where H is a Hopf surface and E is an elliptic curve on H.

Theorem 9 *Let V be 2-dimensional complex manifold, proper homotopy equivalent to the total space of an affine $\mathbb{C}$-bundle over a Riemann surface R of genus g. Let (S, C) be a compact structure on V. Then (S, C) is equivalent to (X, Δ) where X is the total space of a $\mathbb{CP}^1$-bundle of degree n over a smooth curve of genus g and Δ is a section with $(\Delta^2) = \pm n$, except when $g = 1$. In case $g = 1$, in addition to the above possibility, (S, C) may be equivalent to $(S_{n,\alpha,t}, D_{n,\alpha,t})$, where $S_{n,\alpha,t}$ is an Inoue surface of class VII and $D_{n,\alpha,t}$ is the unique curve with $(D^2_{n,\alpha,t}) = 0$.*

It may be noted that all the possibilities listed in Theorem **??** and **??** actually occur. These results may be viewed as topological characterization of various non compact, compactifiable 2-manifolds. The basic tool involved in the proof is the M-R method, in its full force. The proper homotopy equivalence tells us what the local fundamental group of the divisor C is. Of course one has to use Kodaira's classification of surfaces here. Somewhat surprisingly, the case $g = 0$ is the hardest and one has to use Theorem 1.1 of [Mio] here. Observe the special nature of the case $g = 1$, in Theorem **??**. Here one has to use an important result from [E].

Homology planes A homology plane X is a smooth algebraic surface over $\mathbb{C}$ with all its reduced integral homology groups trivial, i.e., $\tilde{H}_i(X; \mathbb{Z}) = (0)$ for all $i \geq 0$. Note that a contractible smooth surface is a homology plane. Indeed, in the statement of Theorem 1, we can replace 'contractibility' condition on X by saying that X is a homology plane. Thus, the M-R method is indeed available for the study of homology planes.

There was a long standing conjecture of Van de Ven(see [V]) about the rationality of a homology plane. Closely related to this conjecture was another conjecture of Van de Ven about a certain inequality of Chern numbers of non complete algebraic surfaces. This inequality was established by Y. Miyaoka in a landmark paper (see [Mio]). As mentioned before, we already knew that affine surfaces with finite fundamental group at infinity are rational. In [F], Fujita had proved that if $\overline{k}(V) \leq 1$, then V is rational. Thus there was a strong indication of an affirmative answer to the rationality conjecture of Van de Ven.

After an initial success with a few special cases, Gurjar and the author launched a systematic study of this problem around 1985. Throughout, this study was guided by the familiarity in handling a 'unimodular' system of rational curves on an algebraic surface, acquired while working with Ramanujam's paper. The important difference is that there is 'insufficient' information on the fundamental group at infinity. Thus, there was a need to blend Mumford-Ramanujam method with the theory of Zariski-Fujita decomposition, introduced by Zariski for effective divisors and fully developed by Fujita for pseudo-effective divisors.

Rational homology planes

By a *rational homology plane* we mean a surface V with all its reduced homology groups with rational coefficients trivial, i.e., $\tilde{H}_\star(V; \mathbb{Q}) = (0)$. In 1990, R. Kobayashi [K], gave a stronger version of Miyaoka's inequality. In particular, this is applicable to surfaces with quotient singularities. In 1996, employing Kobayashi's inequality in place of Miyaoka's inequality, Pradeep and the author proved that a $\mathbb{Q}$-homology plane with at worst quotient singularities is actually smooth (see [P-S]). Later, together with Gurjar, they proved that there is no $\mathbb{Q}$-homology plane with $\overline{k} = 2$ whose smooth compactification is of general type (see [G-P-S].) Thus it remained to consider only the elliptic case. This work was completed by Gurjar and Pradeep with a lot of effort, using tools from differential topology of 4-manifolds. (See [G-P].) Together with the results in [P-S] and [G-P-S], this establishes the rationality of $\mathbb{Q}$-homology planes with quotient singularities.

Other developments On another front, several examples of homology planes as well as contractible surfaces have been found. Around 1987, Gurjar and Miyanishi carried out a systematic study of homology planes with the logarithmic Kodaira dimension, $\overline{\kappa} \leq 1$ (see [G-M]). Later, tom-Dieck and Petrie found many examples of smooth contractible surfaces of $\overline{\kappa} = 2$, as is the case with Ramanujam's example. In their constructions, the starting point is a 'line configuration' on $\mathbb{C}P^2$, in contrast to the starting point of Ramanujam's example where it is a cubic and a conic. Orevkov has studied homology planes bounded by Seifert spheres (see [O]). Contractible surfaces of general type have been exploited by Zaidenberg [Z] in constructing exotic algebraic structures on affine spaces.

There is a formal aspect to the M-R method which is important for people working in this area. This was implicitly brought out for instance, in [S1]. Russell has presented this aspect neatly in a survey article [Ru].

In conclusion, this area is currently quite alive, and there remains the possibility of going deeper into the M-R method as well as of bringing in new ideas.

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