Lecture Notes on Vector bundles and Characteristic Classes [1]

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Contents

Chapter 1

Vector Bundles

1.1 Basics of vector bundles

Definition 1 Let B be a topological space. By a real vector bundle of rank k over B we mean an ordered pair $\xi = (E, p)$, where E is a topological space $p : E \to B$ is a continuous maps such that for each $b \in B$, the fibre $p^{-1}(b) =: \xi_b$ is a k-dimensional $\mathbb{R}$-vector space satisfying the following local triviality condition:
(LTC) To each point $b \in B$ there is an open nbd U of b and a homeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{R}^k$ such that
(i) $\pi_1 \circ \phi = p$
(ii) $\pi_2 \circ \phi : p^{-1}(b') \to \mathbb{R}^k$ is an isomorphism of vector spaces for all $b' \in U$.

Here $\pi_1 : U \times \mathbb{R}^k \to U$ and $\pi_2 : U \times \mathbb{R}^k \to \mathbb{R}^k$ are projection maps.

E is called the total space of ξ and B is called the base.

If $\xi = (E_i, p_i, B_i),\ i = 1, 2$ are two vector bundles, a morphism $\xi_1 \to \xi_2$ of vector bundles consists of a pair $(f, \bar{f})$ of continuous maps such that the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\bar{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

is commutative and such that $\bar{f}|_{p_1^{-1}(b)}$ is $\mathbb{R}$-linear. If both f and $\bar{f}$ are homeomorphisms also, then we say $(f, \bar{f})$ is a vector bundle isomorphism. In this situation we say that the two bundles are isomorphic.

Often while dealing with vector bundles over a fixed base space B, we require a bundle morphism $(f, \bar{f}) : (E_1, p_1, B) \to (E_2, p_2, B)$ to be such that $f = Id_B$.

Given a subspace $E' \subset E$ where $\xi = (E, p, B)$ is a vector bundle, consider the restriction map $p' = p|_{E'}$. We say $\xi' = (E', p', B)$ is a subbundle of ξ iff
(i) $\xi' = (E', p', B)$ is a vector bundle on its own (in particular, p' is surjective).
(ii) The inclusion map $p'^{-1}(b) \subset p^{-1}(b)$ is a linear map for each b.

Remark 1 It is easy to construct ξ' satisfying (ii) without satisfying (i). Also, if ξ is trivial, it does not mean a subbundle ξ' is also trivial.

Example 1

1. The simplest example of a vector bundle of rank k over B is $B \times \mathbb{R}^k$. These are called trivial vector bundles. In fact any vector bundle isomorphic to a product bundle is called a trivial vector bundle. We shall denote this by $\Theta^k := B \times \mathbb{R}^k$, the base space of the bundle being understood by the context. For this trivial bundle, $B \times \mathbb{R}^k$, a section σ corresponds to a continuous map $\pi_2 \circ \sigma : B \to \mathbb{R}^k$.

2. A simple example of a non trivial vector bundle is the infinite Möbius band M : Consider the quotient space of $\mathbb{R} \times \mathbb{R}$ by the equivalence relation $(t, s) \sim (t+1, -s)$. The first projection gives rise to a map $p : M \to \mathbb{S}^1$ which we claim is a non trivial real vector bundle of rank 1 over $\mathbb{S}^1$. It is easy to see that complement of the 0-section in the total space of this bundle is connected. Therefore, the bundle cannot be the trivial bundle $S^1 \times \mathbb{R}$. Indeed the total space of this bundle is not even homeomorphic to $S^1 \times \mathbb{R}$ but to see that needs a little bit more topological arguments.

3. The tangent bundle $\tau(X) := (TX, p, X)$ of any smooth submanifold $X \in \mathbb{R}^N$ is a typical example of a vector bundle of rank n, where $n = dim\, X$. It satisfies the additional smoothness condition viz., both total and base spaces are smooth manifolds, the projection map p is smooth and the homeomorphisms $\phi : p^{-1}(U) \to U \times \mathbb{R}^n$ are actually diffeomorphisms. For a smooth manifold B, a vector bundle which satisfies this additional smoothness condition will be called a smooth vector bundle. On a manifold X embedded in $\mathbb{R}^N$, we get another vector bundle viz., the normal bundle, $\nu(X)$ which is also a smooth vector bundle.

4. Let $B = \mathbb{P}^n$ be the n-dimensional real projective space. The canonical line bundle $\gamma_n^1 = (E, p, \mathbb{P}^n)$ is defined as follows: Recall that $\mathbb{P}^n$ can be defined as the quotient space of $\mathbb{S}^n$ by the antipodal action.

$$E = \{([x], \mathbf{v}) \in \mathbb{P}^n \times \mathbb{R}^{n+1} \ : \ \mathbf{v} = \lambda x\}$$

That is, over each point $[x] \in \mathbb{P}^n$ we are taking the entire line spanned by the vector $x \in \mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Let $p : E \to \mathbb{P}^n$ be the projection to the first factor. The verification that this data forms a line bundle is easy. The case $n = 1$ is an interesting one. The base space $\mathbb{P}^1$ is then diffeomorphic to S^1. However, the bundle γ_1^1 is the infinite Möbius band we considered above.

We begin with the following fundamental criterion to construct/detect isomorphisms between vector bundles.

Lemma 1 Let $f : \xi \to \zeta$ be a bundle map from one vector bundle over B to another. Then f is an isomorphism of vector bundles iff f restricted to each fiber is an isomorphism of vector spaces.

Proof: The only thing that we have to verify is the continuity of f^{-1}. This then can be done locally and hence the problem reduces to the case when ξ, ζ are trivial. In this case, a bundle map $f : B \times \mathbb{R}^k \to B \times \mathbb{R}^k$ is determined by

$$f(b, \mathbf{v}) = (b, A(b)\mathbf{v})$$

where $b \mapsto A(b)$ is a continuous map $A : B \to M(n; \mathbb{R})$ the space of real $n \times n$ matrices. The hypothesis that f restricts to isomorphism on each fiber is the same as saying that each $A(b)$ is invertible and hence we have a continuous map $A : U \to GL(k, \mathbb{R})$. . This then means that $b \mapsto A^{-1}(b)$ is also continuous. Therefore, correspondingly, the map given by

$$(b, \mathbf{v}) \mapsto (b, A^{-1}(b)(\mathbf{v}))$$

is continuous which is nothing but f^{-1}. ♠

Definition 2 Let $\xi = (E, p, B)$ be a vector bundle. By a section of ξ we mean a continuous (smooth) map $\sigma : B \to E$ such that $p \circ \sigma = Id_B$. A section σ is said to be nowhere zero, if $\sigma(b) \neq 0$ for each $b \in B$.

Remark 2
(i) A simple example of a section is the zero-section which assigns to each $b \in B$ the 0-vector in $p^{-1}(b)$. (Use (LTC) to see that the zero-section is continuous.)
(ii) It is easy to see that the trivial bundle has lots of sections. Indeed if $\sigma : B \to B \times \mathbb{R}^k$ is a section then it is for the form,

$$\sigma(b) = (b, f(b))$$

where $f : B \to \mathbb{R}^k$ is continuous and conversely. Thus the set of sections of Θ^k is equal to $\mathcal{C}(B, \mathbb{R}^k)$.
(iii) More generally, the space of sections $\Gamma(\xi)$ can be given a module structure on the ring of continuous functions $\mathcal{C}(B; \mathbb{R})$ and the study of this module is essentially all about the study of vector bundles over B.

Theorem 1 *A vector bundle ξ of rank k is trivial iff there exist sections $\{\sigma_1, \ldots, \sigma_k\}$ which are linearly independent at every point of B.*

Definition 3 By a continuous/smooth vector field on a smooth manifold X we mean a continuous/smooth section of the tangent bundle. By a parallelizable manifold, we mean a smooth manifold X whose tangent bundle is trivial.

Remark 3 Alternatively, a manifold is parallelizable iff there exists n smooth vector fields $\{\sigma_1, \ldots, \sigma_n\}$ such that for each $p \in X$, we have

$$\{\sigma_1(p), \ldots, \sigma_n(p)\}$$

is linearly independent in $T_p(X)$.

1.2 Operations on Vector bundles

Pull-back bundle Given a triple $\xi = (E, p, B)$ (of topological spaces and continuous map) and a continuous function $f : B' \to B$ the pull-back $f^*\xi = (E', p', B')$ is defined by

$$E' = \{(b', e \in B' \times E \ : \ f(b') = p(e)\}; \quad p'(b', e) = b'.$$

If map p satisfies a certain topological properties often it is the case that the same property is satisfied by the map p'. This if the triple ξ is a vector bundle of rank k it follows that so is the triple $f^*\xi$. Moreover, we have a continuous map $\bar{f} : E' \to E$ such that the diagram is commutative:

$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B \arrow[r, "f"] & B \end{tikzcd}$$

Notice that (LTC) for $f * \xi$ follows from the observation that if ξ is the trivial bundle then so is $f^*\xi$. The pull-back bundle has the following universal property. Given any vector bundle $\xi'' = (E'', p'', B')$ over B' and a continuous map $\bar{g} : E'' \to E$ such that $p \circ \bar{g} = f \circ p''$ there exists a unique bundle map $g' : E'' \to E$ over B' i.e., such that $p' \circ g' = p''$ with the property that $\bar{g} = \bar{f} \circ g'$.

$$\begin{tikzcd}
E'' \arrow[dr, "g'"] \arrow[drr, bend left, "\bar{g}"] \arrow[ddr, bend right, "p''"'] & & \\
& E' \arrow[r, "\bar{f}"] \arrow[d] & E \arrow[d, "p"] \\
& B' \arrow[r, "f"] & B
\end{tikzcd}$$

A special case of the pull-back construction is obtained when B' is a subspace of B and $f = \eta : B' \hookrightarrow B$ is the inclusion. We then denote $\eta^*(\xi)$ by $\xi|_{B'}$.

Now suppose $\xi' = (E', p', B')$ and $\xi = (E, p, B)$ are two vector bundles and $(f, \bar{f}) : \xi' \to \xi$ is a bundle map

$$\begin{tikzcd}
E' \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

We then have a unique bundle map $f' : \xi' \to f^*\xi$. We leave it to you to see that

Theorem 2 *The bundle map $f' : \xi' \to f^*\xi$ is an isomorphism iff $\bar{f}$ restricts to an isomorphism on each fiber.*

Remark 4 Note that f itself is just a continuous map and need not be a homeomorphism. However, if f is a homeomorphism, then f is covered by a homeomorphism $\bar{f}$ iff the two bundles ξ and $f^*(\xi')$ over B are isomorphic. More generally, if f is a homeomorphism, then there is 1-1 correspondence between bundle maps $(f, \bar{f}) : \xi \to \xi'$ and bundle maps $(Id_B, g) : \xi \to f^*(\xi')$. This is the reason why we assume that a bundle map $\xi \to \zeta$ of two vector bundles over the same base space B are of the form (Id_B, g).

Cartesian product Given ξ, ξ' we can take $\xi \times \xi' = (E \times E', p \times p', B \times B')$ in the usual way, as a vector bundle of rank $= rk(\xi) + rk(\xi')$. For this bundle, the fiber over a point (b, b') clearly equals $\xi_b \times \xi'_{b'}$. Of particular interest is the special case when ξ' is of rank 0 i.e., $p : E' \to B'$ is a homeomorphism. We denote the product in this case simply by $\xi \times B'$.

Whitney Sum Let now ξ, ξ' be bundles over the same base B. Consider the diagonal map $\Delta : B \to B \times B$. The Whitney-sum of ξ and ξ' is defined by

$$\xi \oplus \xi' = \Delta^*(\xi \times \xi')$$

the pull back of the Cartesian product via the diagonal map. Denoting $\xi \oplus \xi' = (E(\xi \oplus \xi'), q, B)$, we have a commutative diagram

$$\begin{tikzcd} & E(\xi \oplus \xi') \arrow[dl] \arrow[dd, "q"] \arrow[dr] & \\ E \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Whitney-sum indeed corresponds to taking sums of subbundles in the following sense.

Lemma 2 Let ξ_1, ξ_2 be subbundles of ζ such that for each $b \in B$, ζ_b is equal to the direct sum of the subspaces $(\xi_1)_b$ and $(\xi_2)_b$. Then ζ is isomorphic to $\xi_1 \oplus \xi_2$.

Proof: Define $\phi : E(\xi_1 \oplus \xi_2) \to E(\zeta)$ by $\phi(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$. ♠

Riemannian Metric Structure Let ξ be a real vector bundle of rant k. A Riemannian metric on ξ is a continuous function $\beta : E(\xi \oplus \xi) \to \mathbb{R}$ such that restricted to each fiber β is an inner product.

It is easy to see that on any trivial bundle we can give the standard inner product of $\mathbb{R}^k$ itself on each fiber. More generally, given a continuous map $\hat{\beta} : B \to M(k, \mathbb{R})$ taking values inside non degenerate symmetric matrices, we can associate a Riemannian metric on $\Theta^k = B \times \mathbb{R}^k$ by the rule:

$$\beta(\mathbf{v}_1, \mathbf{v}_2)_b = \mathbf{v}_1 \hat{\beta}(b) \mathbf{v}_2^t.$$

And conversely, every Riemannian metric on Θ^k corresponds to a continuous map from B to the space of symmetric non degenerate real $k \times k$ matrices.

Given two bundles with Riemannian metrics one can seek bundle maps which respect the inner products. We can then talk about 'isometries' of such bundle. The simplest question one can ask is: 'what are all isometrically inequivalent metrics on a trivial bundle?' The answer is:

Theorem 3 *Any two Riemannian metric on Θ_k are isometrically equivalent.*

Proof: Gram-Schmidt process. ♠

Orthogonal Complement

Given a Riemannian bundle ξ and a subbundle ξ', the orthogonal complement $(\xi')^\perp$ of ξ' in ξ is defined by

$$E((\xi')^\perp) = \{\mathbf{v} \in \xi_b \ : \ \mathbf{v} \perp \xi'_b\}$$

together with the projection $p : E((\xi')^{\perp}) \to B$. The non trivial thing to verify is the (LTC) which follows once again, from Gram-Schmidt's process.

Remark 5 It is not true that every vector bundle can be given a Riemannian structure. The following result is the 'most' general in this respect in a certain sense.

Theorem 4 *Let B be a paracompact space. Then every vector bundle over B has a Riemannian structure on it.*

Proof: Partition of unity. ♠

Transition functions

Given an open covering $\{U_i\}$ of B and local trivializations $\phi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^k$, of a bundle $\xi = (E, p, B)$ for each pair (i, j) of indices, consider the isomorphisms of the trivial bundles:

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k.$$

They are of the form

$$(b, \mathbf{v}) \mapsto (b, \lambda_{ij}(b)(\mathbf{v}))$$

for some continuous maps $\lambda_{ij} : U_i \cap U_j \to GL(k; \mathbb{R})$. These are called the **transition functions** of the bundle ξ. They satisfy the following two 'cocycle conditions':
(CI) $\lambda_{ii}(b) = Id$ for all i;
(CII) For $b \in U_i \cap U_j \cap U_t$ we have

$$\lambda_{ti}(b) \circ \lambda_{ij}(b) = \lambda_{tj}(b).$$

We would like to reverse the picture: Starting with an open covering $\{U_i\}$ of B and a family $\lambda = \{\lambda_{ij}\}$ of continuous functions $\lambda_{ij} : U_i \cap U_j :\to GL(k; \mathbb{R})$, we define a vector bundle $\xi_\lambda = (E_\lambda, p_\lambda, B)$ of rank k as follows: On the disjoint union $\tilde{E} = \sqcup_i U_i \times \mathbb{R}^k$ define an equivalence relation by saying that

$$(b, \mathbf{v}) \sim (b, \lambda_{ij}(b)(\mathbf{v}))$$

for each pair (i, j) such that $b \in U_i \cap U_j$ and for all $\mathbf{v} \in \mathbb{R}^k$.

The two cocycle conditions ensure that the identifications are compatible and define an equivalence relation. Denote the quotient space by E_λ.

Observe that the projection maps $\pi_1 : U_i \times \mathbb{R}^k \to U_i$ all patch-up to define a continuous map $p_\lambda : E \to B$. Indeed verify that the inclusion $U_i \times \mathbb{R}^k \to \tilde{E}$ followed by the quotient map $\tilde{E} \to E_\lambda$ is a homeomorphism onto $p_\lambda^{-1}(U_i)$ and so we obtain homeomorphisms $\psi_i : U_i \times \mathbb{R}^k :\to p^{-1}(U_i)$. Since each identification map $\lambda_{ij}(b) : \{b\} \times \mathbb{R}^k \to \{b\} \times \mathbb{R}^l$ is an isomorphism of vector bundles we get a unique vector bundle structure on each fibre $p^{-1}(b)$. Taking $\phi_i = \psi^{-1} : p_\lambda^{-1}(U_i) \to U_i \times \mathbb{R}^k$, we get (LTC) for the bundle ξ_λ. For these local trivializations, one can easily verify that

$$\phi_i \circ \phi_j^{-1}(b, \mathbf{v}) = (b, \lambda_{ij}(b)(\mathbf{v}))$$

getting back where we have started.

It is obvious that the topology of the total space as well as the bundle will heavily depend upon the nature of the transition functions. Indeed, if we start off with a bundle ξ and a local trivialization, the union of all local trivializations defines a map $\Phi : \tilde{E} \to E$ which in turn defines a bundle isomorphism $\xi_\lambda \to \xi$.

The transition function description allows us a sure way of carrying out vector space operations on vector bundles. For example if ξ and η are two bundles over B, get a common open covering on which we have local trivializations for both the bundles. Let $\lambda_\xi, \lambda_\eta$ be the corresponding families of transition functions. Define the family $\lambda_\xi \oplus \lambda_\eta$ by the formula

$$(b, \mathbf{v}, \mathbf{u}) \mapsto (b, (\lambda_\xi)_{ij}(b)(\mathbf{v}), (\lambda_\eta)_{ij}(b)(\mathbf{u})).$$

It is a matter of straight forward verification to see that the resulting vector bundle is isomorphic to the Whitney sum $\xi \oplus \eta$. If you want to construct the bundle $\mathrm{Hom}(\xi, \eta)$ all that you have to do is to consider the transition functions

$$\mathrm{Hom}(\lambda_\xi, \lambda_\eta) : (U_i \cap U_j) \times \mathrm{End}(\mathbb{R}^k, \mathbb{R}^l) \to (U_i \cap U_j) \times \mathrm{End}(\mathbb{R}^k, \mathbb{R}^l)$$

defined by

$$(b, \alpha) \mapsto (b, (\lambda_\eta)_{ij}^{-1} \circ \alpha \circ (\lambda_\xi)_{ij}.$$

Likewise, the exterior powers $\Lambda^i \xi$ are constructed out of the transition functions which are fibrewise i^{th} exterior power of the transition functions of ξ.

Exercise 1 Show that

$$\Lambda^2(\xi \oplus \eta) \cong \Lambda^2(\xi) \oplus \Lambda^2(\eta) \oplus \xi \otimes \eta.$$

Remark 6 A simplistic point of view of the entire theory of vector bundles is that it is nothing but continuous/smooth version of linear and multi-linear algebra. A simple illustration of this occurs in the construction of the normal bundle: local triviality of the normal bundle is a consequence of carrying out Gram-Schmidt process, on a set of continuous/smooth vector valued functions which are independent everywhere. Another simple example is that the polar decomposition is continuous smooth process and hence yields the following: If μ, μ' are two Riemannian metrics on a given vector bundle ξ then there exists a fibre preserving homeomorphism $f : E(\xi) \to E(\xi)$ such that $\mu \circ (f, f) = \mu'$.

Example 2 Recall that a finite dimensional vector space V and its dual V^* are isomorphic to each other. However, given a vector bundle ξ its dual bundle, in general, may not be isomorphic to ξ. The reason is that the isomorphism between V and V^* is **not** canonical. On the other hand, it follows easily that $(\xi)^{**}$ is isomorphic to ξ. However, if ξ carries a Riemannian metric then fixing one such, we get an isomorphism $\xi \cong \xi^*$.

Example 3 Consider the tangent bundle $\tau := \tau(\mathbb{P}^n)$. Using the double covering map $\phi : \mathbb{S}^n \to \mathbb{P}^n$, we can describe the total space of τ by

$$E(\tau) = \{[\pm x, \pm \mathbf{v}] \ : \ x \in \mathbb{S}^n, \mathbf{v} \perp x, \mathbf{v} \in \mathbb{R}^{n+1}\}.$$

Observe that $D\phi : E(\tau(\mathbb{S}^n)) \to E(\tau)$ has the property $D(\phi)(x, \mathbf{v}) = D(\phi)(y, \mathbf{u})$ iff $(y, \mathbf{u}) = \pm(x, \mathbf{v})$. Therefore, $D\phi$ is actually the quotient map.

On the other hand a pair $(x, \mathbf{v}) \in \mathbb{S}^n \times \mathbb{R}^{n+1}$ such that $\mathbf{v} \perp x$ also determines a linear map on the 1-dim. subspace $[x]$ spanned by x to its orthogonal complement. Note that the pair $(-x, -\mathbf{v})$ also determines the same linear map. Therefore, we can identify the quotient space with the space of linear maps from 1-dimensional subspaces to their complements in $\mathbb{R}^{n+1}$. This then also describes the vector bundle $\mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp)$ over $\mathbb{P}^n$. We have established:

Theorem 5 $\mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp) \cong \tau(\mathbb{P}^n)$.

Exercise 2 If ξ_j are all vector bundles over the same base space B, prove that

$$\mathrm{Hom}(\xi_1, \xi_2 \oplus \xi_3) \cong \mathrm{Hom}(\xi_1, \xi_2) \oplus \mathrm{Hom}(\xi_1, \xi_3).$$

Exercise 3 If η is a line bundle show that $\mathrm{Hom}(\eta, \eta) \cong \Theta^1$, the trivial line bundle.

Theorem 6 *Let τ denote the tangent bundle of $\mathbb{P}^n$. Then*

$$\tau \oplus \Theta^1 \cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1$$

the n-fold Whitney sum of the canonical line bundle.

Proof:

$$\begin{aligned}
\tau \oplus \Theta^1 &\cong \operatorname{Hom}(\gamma_n^1, \gamma_n^\perp) \oplus \operatorname{Hom}(\gamma_n^1, \gamma_n^1) \\
&\cong \operatorname{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \\
&\cong \operatorname{Hom}(\gamma_n^1, \Theta^{n+1}) \\
&\cong \operatorname{Hom}(\gamma_n^1, \Theta^1) \oplus \cdots \oplus \operatorname{Hom}(\gamma_n^1, \Theta^1).
\end{aligned}$$

Since $\mathbb{P}^n$ is compact, every vector bundle over it admits a Riemannian metric. Therefore, every vector bundle is isomorphic to its dual over $\mathbb{P}^n$. The theorem follows. ♠

1.3 Homotopical Aspect

Lemma 3 Let $B = X \times [a,c], a < b < c$. Suppose ξ is a vector bundle over $B \times [a,c]$ such that $\xi|_{B\times[a,b]}$ and $\xi|_{B\times[b,c]}$ are trivial bundles. Then ξ itself is trivial.

Proof: Let $\phi_1 : \xi|_{X\times[a,b]} :\to (X \times [a,b]) \times \mathbb{R}^k$ and $\phi_2 : \phi_1 : \xi|_{X\times[b,c]} :\to (X \times [b,c]) \times \mathbb{R}^k$ be some trivializations. Consider the isomorphism $\phi_1 \circ \phi_2^{-1} : X \times \{b\} \times \mathbb{R}^k \to X \times \{b\} \times \mathbb{R}^k$ which can be written in the form

$$(x, b, \mathbf{v}) \mapsto (x, b, \lambda_x(\mathbf{v})).$$

It follows that if $\lambda(x,t,\mathbf{v}) = (x, t, \lambda_x(\mathbf{v}))$ then λ is an automorphism of the trivial bundle $B \times [b,c] \times \mathbb{R}^k$. Now define $\phi : E(\xi) \to X \times [a,c] \times \mathbb{R}^k$ by

$$\phi(e) = \begin{cases} \phi_1(e) & \text{if } \pi(e) \in B \times [a,b] \\ \lambda \circ \phi_2(e) & \text{if } \pi(e) \in B \times [b,c]. \end{cases}$$

Verify ϕ defines a trivialization of ξ. ♠

Lemma 4 Let ξ be a vector bundle over $X \times [a,b]$. Then there is an open covering U_i of X such that $\xi|_{U_i\times[a,b]}$ is trivial for each i.

Proof: Easy.

Theorem 7 *Let ξ be a vector bundle over $X \times [0,1]$ where X is paracompact. Then $\xi, (\xi|_{X\times 1}) \times I$ and $(\xi|_{X\times 0}) \times I$ are all isomorphic to each other.*

Corollary 1 Let $f, g : X \to Y$ be two homotopic maps. Then for any vector bundle ξ' over Y, we have $f^*\xi' \cong g^*\xi'$.

Proof of the Corollary If $H : X \times I \to Y$ is a homotopy from f to g consider the bundle $\xi = H^*(\xi')$ over $X \times I$. By the above theorem, $\xi|_{X\times 0}$ and $\xi|_{X\times 1}$ are isomorphic. But they are respectively equal to $f^*(\xi')$ and $g^*(\xi')$.

The proof of the theorem itself is obtained easily via the following proposition.

Proposition 1 Let X be a paracompact and Hausdorff space and ξ be a vector bundle over $X \times I$. Then there is a bundle map $(r, \bar{r}) : \xi \to \xi$, where $r(x,t) = (x,1)$ and $\bar{r}$ is an isomorphism on each fibers.

We shall prove this proposition for the case when X is compact and Hausdorff. The general case does not involve any deeper ideas but only technically more difficult.

Since a compact Hausdorff space is normal, we can get a finite open covering $\{U_1, \ldots, U_n\}$ of X such that

(i) $\xi|_{U_i\times[0,1]}$ is trivial for each i;

(ii) there is a continuous map $\alpha_i : X \to [0,1]$ such that $\overline{\alpha_i^{-1}(0,1]} \subset U_i$, for each i; and

(iii) for every $x \in X$, $max\{\alpha_1(x), \ldots, \alpha_n(x)\} = 1$.

For each i, choose trivializations $h_i : U_i \times [0,1] \times \mathbb{R}^k \to p^{-1}(U_i \times I)$ over $U_i \times [0,1]$ and define bundle maps $(r_i, \bar{r}_i) : \xi \to \xi$ as follows: $r_i(x,t) = (x, max\{\alpha_i(x), t\})$; whereas,

$$\bar{r}_i(e) = \begin{cases} h_i(x, max\{\alpha_i(x), t\}, \mathbf{v}), & \text{if} \quad e = h_i^{-1}(x,t,\mathbf{v}) \in p^{-1}(U_i \times I) \\ e, & \text{if} \quad e \notin p^{-1}(U_i \times I). \end{cases}$$

Then clearly r_i is continuous. Since $\bar{r}_i$ is identity outside the support of α_i and is continuous over $U_i \times I$, it is continuous all over. Moreover, restricted to each fiber, it is a linear isomorphism also. Now consider the composition

$$(r, \bar{r} = (r_1, \bar{r}_1) \circ \cdots (r_n, \bar{r}_n).$$

All that you have to do is to check that $r(b,t) = (b,1)$. ♠

1.4 The Grassmannian Manifolds and the Gauss Map

Fix integers $1 \leq k \leq n$. Let $G_{n,k}$ denote the set of all k-dimensional subspaces of $\mathbb{R}^n$. Let $V_{k,n}$ denote the subspace of $\mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$ (k factors) consisting of ordered k-tuples $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$. There is a surjective map $\eta : V_{n,k} \to G_{n,k}$ and we declare this as a quotient map so as to topologise $G_{n,k}$. This is called the Grassmannian manifold of type (n,k).

Exercise 4 Show that there is a diffeomorphism of the homogeneous space

$$O(n)/O(k,) \times O(,n-k) \to G_{n,k}$$

Consider the triple $\gamma_n^k = (E, \pi, B)$ where, $B = G_{n,k}$,

$$E = E(\gamma_n^k) = \{(V, \mathbf{v}) \in G_{n,k} \times \mathbb{R}^n \ : \ \mathbf{v} \in V\}$$

and $\pi = \pi_1$ the restriction of the projection to the first factor. One can show as in the case $k = 1$ that this defines a k-plane bundle over $G_{n,k}$.

Now consider $\mathbb{R}^n$ as the subspace $\mathbb{R}^n \times 0$ of $\mathbb{R}^{n+1}$. This then induces an inclusion of $G_{n,k} \xrightarrow{\iota} G_{n+1,k}$. Moreover there is a bundle inclusion:

$$\begin{tikzcd} E(\gamma_n^k) \arrow[r] \arrow[d] & E(\gamma_{n+1}^k) \arrow[d] \\ G_{n,k} \arrow[r, "\iota"] & G_{n+1,k} \end{tikzcd}$$

Now consider the spaces

$$G_k = \cup_{n \geq k} G_{n,k}, \quad E(\gamma^k) = \cup_{n \geq k} E(\gamma_n^k)$$

with the weak topology, i.e, $F \subset G_k$ (resp. $E(\gamma^k)$) is closed iff $F \cap G_{n,k}$ (resp. $F \cap E(\gamma_n^k)$) is closed in $G_{n,k}$ (resp.in $E(\gamma_n^k)$). It is not difficult to see that the corresponding projection maps patch up to define a projection map $\pi : E(\gamma^k) \to G_k$ giving a vector bundle γ^k of rank k over G_k.

G_k is called the infinite Grassmannian. Indeed, this is nothing but the space of all k dimensional subspaces of the infinite direct sum $\mathbb{R}^\infty = \mathbb{R} \oplus \mathbb{R} \oplus \cdots$. Also, γ^k is the tautological (canonical) vector bundle over G_k.

Definition 4 Let ξ be a k-plane bundle over B. A map $g : E(\xi) \to \mathbb{R}^n, \quad k \leq n \leq \infty$ is called a Gauss map on ξ, if $g|_{\xi_b}$ is a linear monomorphism for all $b \in B$.

Example 4 The second projection $\pi_2 : G_{n,k} \times \mathbb{R}^n \to \mathbb{R}^n$ restricted to $E(\gamma_n^k)$ is a Gauss map on γ_n^k for all $k \leq n \leq \infty$. Indeed, these Gauss maps give rise to all other Gauss maps as elaborated in the following lemma.

Lemma 5 Let $(f, \bar{f}) : \xi \to \gamma_n^k$ be a bundle map which is an isomorphism on each fiber. Then $\pi_2 \circ \bar{f}$ is a Gauss map for ξ. Conversely, given a Gauss map $g : E(\xi) \to \mathbb{R}^n$ there exists a bundle map $(f, \bar{f}) : \xi \to \gamma_n^k$ which is an isomorphism on each fibre such that $\pi_2 \circ \bar{f} = g$.

Proof: The first part is clear. To prove the converse, we define $f(b) = g(\xi_b) \in G_{n,k}$ and $\bar{f}(e) = (f(p(e)), g(e))$. Use (LTC) to see that f is continuous and therefore $\bar{f}$ is continuous. Other requirements are verified straight forward. ♠

Proposition 2 Any k-plane bundle ξ over a paracompact Hausdorff space admits a Gauss map into $\mathbb{R}^\infty$.

Proof: Since B is paracompact, there exists a countable open covering U_i of B, a partition of unity α_i subordinate to the cover $\{U_i\}$ and trivializations $h_i : U_i \times \mathbb{R}^k \to p^{-1}(U_i)$. Define $g(e) = \sum_i g_i(e)$ where $g_i : E(\xi) \to \mathbb{R}^k$ is zero outside $p^{-1}(U_i)$ and on $p^{-1}(U_i)$, we have

$$g_i(e) = \alpha_i(p(e)\pi_2(h_i^{-1}(e)).$$

Theorem 8 *Let B be a paracompact space. Given a k dimensional vector bundle ξ over B there exists a continuous map $f : B \to \mathbb{G}_k$ such that $\xi \cong f^*(\gamma^k)$. Moreover, if $f' : B \to G_k$ is another such continuous map then f is homotopic to f'.*

Proof: Let $g : \xi \to \mathbb{R}^\infty$ be a Gauss map as in the previous proposition. Then by the above lemma, we get a bundle map $(f, \bar{f}) : \xi \to \gamma^k$ such that $\pi_2 \circ \bar{f} = g$ and $\bar{f}$ is an isomorphism of fibres. This in turn induces a bundle map $(Id, \eta) : \xi \to f^*\gamma_k$ which is again an isomorphism on fibres and hence is bundle isomorphism. This proves the first part.

To prove the second part, we note that an isomorphism $\xi \cong f'^*(\gamma^k)$ induces a bundle map $(f', \bar{f}') : \xi \to \gamma^k$ which in turn corresponds to a Gauss map $g' : E(\xi) \to \mathbb{R}^\infty$. Likewise to get a homotopy between f and f' it is enough to produce a homotopy $g_t : E(\xi) \to \mathbb{R}^\infty$ of g and g' through Gauss maps.

Let $\mathbb{R}^{ev}, \mathbb{R}^{odd}$ be subspaces of $\mathbb{R}^\infty$ consisting of elements whose odd-place coordinates (resp. even-place coordinates) are zero. Let $ev : \mathbb{R}^\infty \to \mathbb{R}^{ev}$ and $od : \mathbb{R}^\infty \to \mathbb{R}^{odd}$ be the maps defined by

$$ev(x_1, \ldots, x_n, 0, \ldots,) \mapsto (0, x_1, 0, x_2, \ldots);\ od(x_1, x_2, \ldots, x_n, 0, \ldots) \mapsto (x_1, 0, x_2, 0, \ldots).$$

Then ev and od are monomorphisms and are homotopic through monomorphisms to the identity map:

$$tx + (1-t)ev(x); \quad tx + (1-t)od(x).$$

Therefore it follows that g is homotopic to $ev \circ g$ and g' is homotopic to $odd \circ g'$. Now consider the homotopy

$$g_t(e) = (1-t)(ev \circ g)(e) + t(od \circ g')(e)$$

between $ev \circ g$ and $od \circ g'$. Injectivity of g_t follows from the fact that the line joining $ev \circ g(e)$ and $od \circ g'(e)$ does not pass through the origin in the vector space $\mathbb{R}^\infty$ since $ev \circ g(e)$ and $od \circ g'(e)$ are linearly independent for all e. ♠

Remark 7 We have come to a junction in the study of isomorphism class of vector bundles over a fixed base space B. We can proceed now in different directions. In the next chapter we take up the study of characteristic classes.

Chapter 2

Characteristic Classes

We shall now need a little generalization of the notion of vector bundles namely, **fiber bundles,** where the fibres are not necessarily vector spaces, but homeomorphic (diffeomorphic) to a fixed topological space (to a manifold). The (LTC) is formulated similarly, except that the transition functions are no longer linear isomorphisms but take values in some specific subgroup of the group of all homeomorphisms $Homeo(F)$ (diffeomorphisms $(Diff(F))$ of F. One such important construction is the projective bundle $P(\xi)$ associated to a vector bundle ξ in which each fiber ξ_b is replaced by the projective space $P(\xi_b)$. However, note that this is not a vector bundle. Yet another example is the sphere bundle associated to a k-plane bundle with a Riemannian metric.

2.1 Orientation and Euler Class

(Recall how one defines an orientation on a smooth manifold.)

Definition 5 Let V be real vector space of dimension $k > 0$. By an orientation on V one means an equivalence class of an ordered basis; two bases being equivalent if the transformation matrix taking one to the other is of positive determinant.

Let now ξ be a vector bundle over B. Then by a pre-orientation on ξ we mean a choice of orientation on each fibre ξ_b. A pre-orientation is called an orientation, if it satisfies the following local constancy condition: There exists an open covering U_i of B on which we have trivializations $h_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^k$ such that the restriction map $h_i : \xi_b \to b \times \mathbb{R}^k$ preserves orientations, where we orient $\mathbb{R}^k$ with the standard orientation.

Remark 8
(i) Thus on a vector space, there are precisely two orientations.

(ii) Consider two oriented vector spaces. Then we give the orientation on $V \times W$ by first taking the basis for V and then following it up with a basis for W. Thus it is easily seen that $W \times V$ will receive the orientation equal to $(-1)^{kl}$ times that of $V \times W$ where $dim(V) = k, dim(W) = l$.
(iii) Let V_0 denote the space $V \setminus \{0\}$. Then one knows that $H_k(V, V_0; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$; so is $H^k(V, V_0)$. Consider the standard k-simplex and its embedding in $\mathbb{R}^k$ given by

$$x \mapsto x - \beta_k$$

where, $\beta_k = e_1 + e_2 + \cdots + e_k/k$ is the barycenter of Δ_k. It is not hard to verify that this embedding defines a singular simplex generating $H_k(\mathbb{R}^k, \mathbb{R}^k_0; \mathbb{Z})$. Choosing the standard orientation on Δ_k, we get one generator and by changing the orientation on Δ_k we get another. Therefore, it follows that choosing an orientation on V is equivalent to choosing a generator for $H_k(V, V_0; \mathbb{Z})$.

Similar statement can be made by using cohomology groups as well. Moreover, note that there exist a unique class $\mu \in H^k(U \times \mathbb{R}^k, U \times \mathbb{R}^k_0; \mathbb{Z})$ such that for each $b \in B$, if $\iota_b : \mathbb{R}^k \to b \times \mathbb{R}^k \subset U \times \mathbb{R}^k$ is the inclusion map, then $\iota^*(\mu)$ is the orientation class corresponding to the standard orientation.

Definition 6 Let $\xi = (E, p, B)$ be a k-plane bundle. By a pre-orientation on ξ we mean a choice of an orientation on each fibre. [This is equivalent to fixing a generator for the group $H^k(\xi_b, (\xi_b)_0; \mathbb{Z}) \approx \mathbb{Z}$.] We say a pre-orientation is an orientation if it satisfies following local constancy condition, viz., for each $b \in B$, there exists a neighbourhood V of b in B and an element $\mu_V \in H^k(p^{-1}(V); p^{-1}(V) \cap E_0; \mathbb{Z})$ which, on each fibre over $b' \in V$, restricts to the generator given by the pre-orientation.

Theorem 9 *A vector bundle ξ is orientable iff there is an element $\mu \in H^n(E, E_0; \mathbb{Z})$ whose restriction to any fibre is a generator of $H^n(\xi_b, (\xi_b)_o; \mathbb{Z})$.*

Proof: The 'if' part is obvious. We need to prove the 'only if' part here.

For the simplicity of the exposition, we shall prove this for the case when ξ is of finite type, viz., when there is a finite cover of B which trivializes p. (See [Sp] page 262 OR [M-S] chapter 10 for a complete proof.)

The statement is trivially verified in the case when the entire bundle is trivial. Now it suffices to prove the theorem for the case when B is covered by two open sets V_1, V_2 over which the bundle is trivial. Put

$$E_i = p^{-1}(V_i), i = 1, 2, E' = p^{-1}(V_1 \cap V_2), (E_i)_0 = E_i \cap E_0, i = 1, 2, E'_0 = E' \cap E_0.$$

Let $\mu_i \in H^k(E_i;(E_i)_0;\mathbb{Z}), i = 1, 2$ be such that restricted to each fibre they give the pre-orientation class. Now in the exact Mayer-Vietoris sequence

$$H^{k-1}(E', E'_0) \to H^k(E, E_0) \to H^k(E_1,(E_1)_0) \oplus H^k(E_2,(E_2)_0) \to H^k(E', E'_0) \quad (2.1)$$

since the bundle over $V_1 \cap V_2$ is trivial, it is easily verified that the element $\mu_1 - \mu_2$ is mapped to zero. Therefore there exists $\mu \in H^k(E, E_0)$ which maps onto (μ_1, μ_2). By the canonical property of cohomology elements, it follows that μ restricts to the generator of $H^k(\xi_b,(\xi_b)_0)$ for each $b \in B$. This completes the proof of the theorem. ♠

This theorem allows us to define orientability over any associative ring with a unit.

Definition 7 We say ξ is (cohomologically) orientable over a ring R if there exists $\mu \in H^k(E, E_0; R)$ such that $\iota_b^*(\mu)$ is a generator of $H^k(\xi_b,(\xi_b)_0; R)$.

Indeed, analogous to the above theorem, we have

Theorem 10 *A vector bundle ξ is cohomologically orientable over R iff there is there is an open covering $\{V_i\}$ of the base B and a compatible family $\{\mu_j\}$ where $\mu_j \in H^n(p^{-1}(V_j), E_0 \cap p^{-1}(V_j); R) \approx R$ is a generator.*

Here 'compatible' means whenever $V_i \cap V_j \neq \emptyset$, we have, $\mu_i|_{V_i \cap V_J} = \mu_j|_{V_i \cap V_j}$.

Remark 9
(i) Every vector bundle is orientable over $\mathbb{Z}_2$.
(ii) A bundle ξ is orientable iff it is (cohomologically) orientable over $\mathbb{Z}$.
(iii) A manifold is orientable iff its tangent bundle is orientable.
(iv) One can identify the base space with the image of the zero section of a bundle. It then follows that the projection map $p : E \to B$ is a strong deformation retraction of $E(\xi)$ onto the base. In particular, p induces isomorphisms between homology (and cohomology) groups of the total space with those of the base space.

Exercise 5 (a) Show that a bundle is orientable iff there exist local trivializations so that the corresponding transition functions have positive determinant.
(b) Show that if ξ is orientable then all the exterior powers $\Lambda^i(\xi)$ are orientable.
(c) Show that a k-plane bundle ξ is orientable iff $\Lambda^k(\xi)$ is orientable iff $\Lambda^k(\xi)$ is trivial. ($\Lambda^k(\xi)$ is called the determinant bundle of ξ.)
(d) Let ξ be a line bundle over B. Then ξ is orientable iff it is trivial.
(e) For every complex bundle ξ, the underlying real bundle $\xi_{\mathbb{R}}$ is orientable.

Definition 8 Let $\xi = (E, p, B)$ be an oriented real k-plane bundle with an orientation class $\mu \in H^n(E, E_0; \mathbb{Z})$. Then

$$e(\xi) := (p^*)^{-1} i^*(\mu) \in H^n(B; \mathbb{Z})$$

is called the Euler class of class of ξ.

If ξ is a complex k-plane bundle, then $\xi_{\mathbb{R}}$ has a canonical orientation with respect to which we take the Euler class, i.e., $(\xi) = e(\xi_{\mathbb{R}})$.

Remark 10
(i) Let ξ and ξ' be oriented k-plane bundles and $(f, \bar{f}) : \xi \to \xi'$ is a map which is an isomorphism on each fibres and preserves orientations. Then by naturality property of cohomology classes, it follows that $e(\xi) = f^*(e(\xi'))$.
(ii) If we change the orientation on ξ, then it follows that $e(\xi)$ also changes its sign. Thus if we agree to denote the bundle with opposite orientation by $-\xi$ then we have $e(-\xi) = -e(\xi)$.
(iii) Combining (i) and (ii), it follows that if k is odd, then $2e(\xi) = 0$. For consider the automorphism $(Id, \eta) : \xi \to -\xi$ given by $v \mapsto -v$. This is orientation reversing if k is odd. Therefore from (i), it follows that $e(\xi) = Id^*(e(-\xi)) = e(-\xi) = -e(\xi)$.

Example 5 If ξ is a trivial bundle then $e(\xi) = 0$. To see this first of all note that $B \times \mathbb{R}^k$ is orientable and an orientation class looks like $\mu = \pi_2^*(\nu)$ where $\nu \in H^2(\mathbb{R}^2; \mathbb{R}^2_0; \mathbb{Z})$ is a generator. Therefore, it follows that $i^*(\mu) = 0$ in $H^2(B \times \mathbb{R}^2; \mathbb{Z})$.

Example 6 Now consider the canonical complex line bundles $\gamma^1_{\mathbb{C}}$ over the complex projective space $\mathbb{P}^1_{\mathbb{C}} = \mathbb{S}^2$. From the definition of the projective space, it is clear that the associated sphere-bundle viz., the bundle restricted to the subspace of norm one vectors is the Hopf-bundle $p : \mathbb{S}^3 \to \mathbb{S}^2$ with fibres $\mathbb{S}^1$. Clearly the bundle restricted to $V_i = \mathbb{S}^2 \setminus \{P_i\}$ is trivial where P_i denote the north and south pole and we can orient each piece with the orientation coming from the complex structure on the fibre $\mathbb{R}^2 = \mathbb{C}$. The compatibility of this choice over $V_1 \cap V_2$ follows since the transition function $([z_2, z_2], z) \mapsto ([z_1, z_2], z \cdot z_1/z_2)$ is holomorphic and hence orientation preserving. Therefore, the bundle is orientable. (This follows from exercise 5 (e) above also.) Indeed, since $\mathbb{S}^3 \subset E_0$ is a deformation retract, we have $H^1(E_0) = 0 = H^2(E_0)$ and hence

$$H^2(E, E_0) \approx H^2(E) \approx H^2(\mathbb{S}^2) \approx \mathbb{Z}.$$

Therefore, the orientation class μ has to be a generator of $H^2(E, E_0)$ which is mapped onto a generator of $H^2(\mathbb{S}^2)$. This just means $e(\gamma^1_{\mathbb{C}}) = (p^*)^{-1}(i^*(\mu))$ is a generator of $H^2(\mathbb{S}^2)$.

Since $\mathbb{S}^2 = \mathbb{P}^1_{\mathbb{C}}$ is a complex manifold, it has a canonical orientation which fixes a generator $z \in H^2(\mathbb{P}^1_{\mathbb{C}}; \mathbb{Z})$ and hence $e(\gamma^1_{\mathbb{C}}) = \pm z$. We need determine the sign.

This is best understood if we work inside $\mathbb{P}^2_{\mathbb{C}}$ in which there is a natural embedding of the total space of γ^1 as an open subspace:

$$E(\gamma^1) = \{[z_0, z_1, z_2] \; : \; z_0 \neq 0\} \subset \mathbb{P}^2_{\mathbb{C}}.$$

Note that the zero section of the bundle is identified with the line at infinity in $\mathbb{P}^2_{\mathbb{C}}$.

There is an orientation on the total space $E(\gamma^1)$ which is obtained by taking the orientation on the base followed by the orientation on the fibre. With respect to this orientation, it is clear that each fibre has intersection number with the zero section equal to 1. This implies that $p^*(z) \cup \mu$ is the generator of $H^4(\mathbb{P}^2_{\mathbb{C}}; \mathbb{Z})$. On the other hand we also have $p^*(z) \cup p^*(z)$ is also this generator. Therefore, it follows that $p^*(z) = \mu$. Inside E this just implies that

$$e(\gamma^1_{\mathbb{C}}) = z. \tag{2.2}$$

Example 7 Orientation double cover Given any k-bundle $\xi = (E, p, B)$, we shall construct an orientable $k-$bundle $\tilde{\xi} = (\tilde{E}, \tilde{p}, \tilde{B})$ and a bundle map $(q, \bar{q}) : \tilde{\xi} \to \xi$ as follows: Choose any atlas $\{U_i, h_i\}$ of trivializations of ξ. Fix an orientation on each of $U_i \times \mathbb{R}^k$ take orient each $p^{-1}(U_i)$ so that h_i preserve orientations. Now for each i take two copies of $V_i = p^{-1}(U_i)$ and label them by $V_i^{\pm}$. Let X be the disjoint union of $\{V_i^{\pm}\}$. For each un-ordered pair of indices $\{i, j\}$, let $W_{ij} = U_i \cap U_j$. On X we define an equivalence relation by the following rule: Given $(b, v) \in W_{ij}^{\pm} \times \mathbb{R}^k$ identify it with $h_i \circ h_j^{-1}(b, v) \in W_{ij}^{\pm} \times \mathbb{R}^k$ if $h_i \circ h_j^{-1} : \{b\} \times \mathbb{R}^k \to \{b\} \times \mathbb{R}^k$ is orientation preserving; otherwise identify it with $h_i \circ h_j^{-1}(b, v) \in W_{ij}^{\mp} \times \mathbb{R}^k$. Let $\tilde{E}$ denote the quotient space of X. Then the projection maps p factor through a map $\tilde{p} : \tilde{E} \to \tilde{B}$ defining a k-plane bundle $\tilde{\xi}$. Moreover there is an obvious quotient map which defines a bundle map $(q, \bar{q}) : \tilde{\xi} \to \xi$. Observe that

(i) $\tilde{\xi}$ is always orientable.

(ii) $(q, \bar{q})$ is a double covering.

(iii) $\tilde{\xi}$ is a disjoint union of two copies of ξ iff ξ is orientable.

Theorem 11 *The Euler class of the Cartesian product is the cross product of the Euler classes; also the Euler class of a Whitney sum is the cup product of the Euler classes.*

$$e(\xi_1 \times \xi_2) = e(\xi_1) \times e(\xi_2); \quad e(\xi_1 \oplus \xi_2) = e(\xi_1) \cup e(\xi_2).$$

Proof: Note that for any two oriented vector spaces V, W, $V \times W$ is oriented by first taking the basis of V followed by the basis of W. It follows easily from this that $\mu(\xi \times \xi) = \mu(\xi) \times \mu(\xi')$ from which the first property follows. But then $\xi_1 \oplus \xi_2 = \Delta^*(\xi_1 \times \xi_2)$ and the second property follows from the above remark (i). ♠

Corollary 2 Let M be a smooth oriented manifold such that $e(M) \neq 0$. Then the tangent bundle $\tau(M)$ does not admit any subbundle of odd rank.

Proof: Suppose ξ is an oriented subbundle of odd rank. Fixing a Riemannian metric on τ, we can take the orthogonal complement $\xi^\perp$ and orient it in such a way so that the direct sum orientation coincides with that of τ. Therefore,

$$2e(M) = 2e(\tau(M)) = 2e(\xi) \cup e(\xi^\perp) = 0 \in H^n(M, \mathbb{Z}) \approx \mathbb{Z}$$

which contradicts the hypothesis $e(M) \neq 0$. ♠

Now, in the situation of the above corollary, it may happen that ξ is not orientable. From the earlier example, there is a double cover $\phi : \tilde{M} \to M$ such that $\phi^*(\xi)$ is orientable. On the other hand, it is a subbundle of $\phi^*(\tau(M)) = \tau(\tilde{M})$. Now we are in the orientable case. ♠

Exercise 6 Extend the notion of orientability to the sphere bundles $\dot{\eta} = (\dot{E}, p, B)$. Show that a sphere bundle is orientable iff there is $\mu \in H^{k-1}(\dot{E}; \mathbb{Z})$ whose restriction to each fiber is a generator of $H^{k-1}(\dot{\eta}_b; \mathbb{Z})$.

Exercise 7 Fix a Riemannian metric on a $k-$plane bundle ξ. Let $\dot{E}$ denote the subspace of unit vectors in E. Then $p : \dot{E} \to B$ defines a (locally trivial) $\mathbb{S}^{k-1}$-bundle over B.

Theorem 12 *Let ξ be a k-plane bundle with a metric and $\dot{\xi}$ be the sphere bundle. Let η be the tautological line bundle over the total space of $\dot{\xi}$. Of the three bundles $\xi, \dot{\xi}, \eta$, if two of them are orientable then the third is also orientable.*

Theorem 13 *For any oriented vector bundle ξ, $e(\xi) = 0$ if ξ admits a nowhere vanishing section.*

Proof: Suppose ξ admits a nowhere vanishing section. Then we can write $\xi \cong \xi' \oplus \Theta^1$. It follows that $e(\xi) = e(\xi') \cup e(\Theta^1) = 0$. ♠

Lemma 6 Let M^m be closed submanifold of a manifold N^{m+k}. Then for any coefficient ring R, there is a canonical isomorphism

$$H^*(E, E_0; R) \to H^*(N, N \setminus M; R)$$

where E is the total space of the normal bundle of M in N and E_0 is the subspace of E consisting of non zero normal vectors.

Proof: Fixing a Riemannian metric on N recall that for some suitable $\epsilon > 0$ the exponential map $\exp : E(\tau(N)) \to N$ restricts to a diffeomorphism

$$E(\epsilon) \to N(\epsilon)$$

of the space of all vectors in $\nu(M)$ of norm less than ϵ to a tubular neighbourhood $E(\epsilon)$ of M in N. This diffeomorphism is identity on M. On the other hand we have the excision map

$$(N(\epsilon), N \setminus M) \hookrightarrow (N, N \setminus M)$$

inducing an isomorphism in cohomology. Combining this with $(Exp)^*$ gives the required isomorphism. ♠

Remark 11 The isomorphism does not depend upon the choice of $\epsilon > 0$, nor on the choice of the Riemannian metric. This is so because the homotopy type of the tubular neighbourhood is independent of such choices. Now suppose that the normal bundle of M is oriented. Then the image of the fundamental cohomology class μ in $H^k(N, N \setminus M)$ under the above isomorphism will be denoted by μ'. Since the diffeomorphism Exp is Identity on M it follows that under the inclusion induced maps $M \to N$ followed by $N \to (N, N \setminus M)$ the element μ' is mapped onto $e(\nu(M))$. As an immediate consequence we have:

Theorem 14 *Let M^m be a closed manifold embedded in $\mathbb{R}^{m+k}$ so that the normal bundle $\nu(M)$ is orientable. Then $e(\nu(M)) = 0$.*

Proof: You have to only notice that $H^k(\mathbb{R}^{m+k}) = (0)$. ♠

Finally, we come to the result that justifies the name of this characteristic class:

Theorem 15 *Let M be a closed oriented k-dimensional manifold. Then $e(M) = \chi(M)[\bar{M}]$ where $[\bar{M}] \in H^k(M;\mathbb{Z})$ is the dual to the fundamental class $[M] \in H_m(M,\mathbb{Z})$.*

Proof: Consider the diagonal embedding $\Delta : M \to M \times M$. We shall identify M with $\Delta(M)$.

First note that the tangent bundle of $\tau(M)$ is canonically isomorphic to the normal bundle $\nu(M)$ in $M \times M$ via

$$(\mathbf{v}, \mathbf{v}) \mapsto (\mathbf{v}, -\mathbf{v}).$$

Therefore the normal bundle is also oriented. Let the fundamental orientation class μ of $\nu(M)$ correspond to a class $\mu' \in H^m(M \times M, M \times M \setminus \Delta(M);\mathbb{Z})$. Let μ'' denote its image in $H^m(M \times M;\mathbb{Z})$ under the inclusion induced map.

Then $e(M) = e(\tau(M)) = e(\nu(M)) = \Delta^*(\mu'')$.

We now prefer to work over a field $\mathbb{K}$ containing $\mathbb{Z}$ say $\mathbb{K} = \mathbb{Q}$ or $= \mathbb{R}$. We need :

Theorem 16 Poincaré Duality Theorem: *Let M be a closed oriented manifold with the fundamental class $[M] \in H_m(M;\mathbb{Z})$. Then for each $0 \le r \le m$, there exist basis $\{b_{rj}\}$ of $H^r(M)$ such that for each r*

$$(b_{r,i} \cup b_{m-r,j}) \cap [M] = \delta_{ij}.$$

and the following lemma:

Lemma 7 $\Delta^*(\mu'') = \sum_r (-1)^r \sum_j b_{rj} \cup b^*_{m-r,j}$.

Proof: This follows from the fundamental property of the fundamental class $[M] \in H_m(M;\mathbb{Z})$ viz., for each $x \in M$ there is a preferred generator $M_x \in H_m(M, M \setminus \{x\})$ such that $[M]$ is mapped onto M_x under the inclusion induced map $M \to (M, M \setminus \{x\})$. The rest of the argument involves standard properties of cup, cap, cross and slant products. ♠

Continuing with the proof of theorem 15, it follows that if $e(\tau(M)) = \lambda[\bar{M}]$ then

$$\begin{aligned}
\lambda &= e(\tau(M)) \cap [M] = \Delta^*(\mu'') \cap [M] \\
&= \textstyle\sum_r (-1)^r (b_{r,j} \cup b^*_{m-r,j}) \cap [M] = \sum_r (-1)^r \mathrm{rank}\,(H^r(M)) \\
&= \chi(M).
\end{aligned}$$

Alternative proof of theorem 15.

We shall prove two lemmas, from which the theorem would follow immediately.

Lemma 8 Let M is a smooth oriented closed n-manifold. If $s : M \to \tau(M)$ is a smooth section of the tangent bundle of a manifold with finitely many zeros, then the index $\iota(s)$ of s is equal to $e(M) \cap [M]$ where $[M] \in H_n(M; \mathbb{Z})$ denotes the fundamental class of M.

Lemma 9 Let K be a triangulation of a closed oriented manifold M. Then there exists a smooth vector field s on M, with finitely many zeros such that $\iota(s) = \chi(K)$.

For other avatars of Euler characteristic of a smooth closed manifold see [Sh2].

Proof of Lemma 8: Choose disjoint disc neighbourhoods $B_1, \ldots, B_k$ around the zeros $z_1, \ldots, z_k$, respectively. Put $P = \cup_i B_i$ and $Q = M \setminus intB$, $E' = p^{-1}(Q)$, where $p : E \to M$ is the projection of the tangent bundle. Put $E_0 = E \setminus s_0(M)$ where $s_0 : M \to E$ is the zero section, and $E'_0 = E' \cap E_0$. Then it follows that $s : Q \to E'$ factors through $s' : Q \to E_0 \to E'$ and therefore, $s^*(\mu)|_Q = 0$. Therefore the computation of $e(M) \cap [M] = \langle s^*(\mu), [M] \rangle$ can be effectively carried out by restricting attention on P. In other words, $s^*(\mu)$ can be thought of as a relative n-cochain on $(P, \partial P)$ and

$$e(M) \cap [M] = \sum_i s^*(\mu) \cap [B_i, \partial B_i]. \tag{2.3}$$

Now on each $B_i = B$ the bundle is trivial and fixing some trivialization on each of them we can write, $s(x) = (x, \sigma(x)$, where $\sigma : B \to \mathbb{R}^n$ is a smooth map such that $\sigma^{-1}(0) = \{z_i\}$. Also over B we can represent μ as $1 \times v$ where $v \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ and $1 \in H^0(B)$. Therefore $s^*[M] \cap [B, \partial B] = \sigma^*[v] \cap [B, \partial B]$. This is nothing but the winding number of $\sigma|_{\partial B}$ around the point 0 which is also equal to the degree d_i of the map $\sigma / \|\sigma\| : \partial B \to \mathbb{S}^{n-1}$. The index of s at the point $z = z_i$ is equal to d_i. Now (2.3) implies $\iota(s) = \sum_i d_i = e(M) \cap [M]$. ♠

Proof of lemma 9 Let K', K'' denote respectively the first and second barycentric subdivision of K. Then each vertex v of K'' as an element of $|K|$ belongs to the interior of a unique simplex $s \in K$. Treating s as a vertex of K', this assignment gives a vertex map $\varphi : V(K'') \to V(K')$. Check that this actually defines a simplicial map $\varphi : K'' \to K'$. It is easily checked that on the vertices of K', φ is identity. In fact, we claim that the fixed points of $|\varphi|$ are precisely vertices of K'. To prove this, let $x \in \langle \tau \rangle$ for $\tau = \{v_0, \ldots, v_k\} \in K''$. Then $x = \sum_i t_i v_i$, with $t_i \neq 0$. Therefore, $|\varphi|(x) = x$ implies $\sum_i t_i v_i = \sum_i t_i \varphi(v_i)$. The two convex combinations on either side are taken inside a single simplex of K and hence it follows that $\{v_0, \ldots, v_p\} = \{\varphi(v_0), \ldots, \varphi(v_k)\}$. This is possible iff $k = 0$ and v_0 is a vertex of K'.

[With respect to the CW-structure of K' $|\varphi|$ is a cellular map such that under the canonical orientation of the simplices τ of K', $|\varphi| : |\tau| \to |\tau|$ defines a degree 1 map.

Therefore it follows that the Lefschetz number of $|\varphi|$ is equal $\chi(K') = \chi(M)$. This fact also follows from the general theory since φ is a simplicial approximation to Id_M.]

We shall now construct a smooth vector field s on M such that the zero set of the section s is precisely $V(K')$, the vertex set of K', so that its index at $\beta(\tau)$ is equal to $(-1)^{\dim \tau}$. Taking the sum of all these indices it would follow that $\iota(s) = \chi(K) = \chi(M)$.

For points $x \in U = |K| \setminus V(K')$ the line segment $[x, \varphi(x)]$ in $|K|$ defines a smooth parameterised curve. Let $\alpha(x)$ denote the unit tangent vector to this curve at x. Then α is a continuous vector field. In fact, it is smooth wherever $|\varphi|$ is smooth. Consider the star neighbourhoods

$$St(v, K'') = \{\lambda \in |K''| \ : \ \lambda(v) \neq 0\}$$

of $v \in K'$. Then

$$|K| = \cup\{\overline{St}(v, K'') \ : \ v \in K'\}$$

form a cell $n-$ decomposition of $|K|$ and α is smooth restricted to each $\overline{St}(v, K'') \setminus \{v\}$. Let $\eta : [0,1] \to [0,1]$ be a smooth 1-1 mapping such that $\eta(0) = \eta'(0) = 0 = \eta'(1)$ and $\eta(1) = 1$. Using polar coordinates for points for $\overline{St}(v, K'')$ we now define $s(t,z) = \eta(t)\alpha(z)$ where $z \in Lk(v, K'') = \partial\overline{St}(v, K'')$ and $0 \leq t \leq 1$. It is easily checked that s is the required vector field. It remains to compute the index of s at each of the vertices $v \in K'$. Suppose $v = \beta(\tau)$ for some $\tau \in K$ of dimension d. The vector field s is pointing towards v at each point of $|\tau| \setminus \{v\}$. On the other hand, if τ' is the dual $(n-d)$-cell in K'' then s is pointing away from v at all the points $|\tau'| \setminus \{v\}$. This means $\iota_v(s) = (-1)^d$ which is what we wanted. ♠

Corollary 3 Let M be a smooth closed manifold. Then M has a smooth nowhere vanishing section iff $e(M) = 0$.

Proof: We have already see the only if part. Suppose now that $e(M) = 0$. We may assume that M is connected. We can take any vector field s with finitely many zeros and then $\iota(s) = 0$. We can assume that all the zeros of s are contained in the interior of a single disc in M (by an ambient isotopy). I then follows by arguments similar to that used in Hopf index theorem, that we can modify s on this disc so as not have any zeros. (For more details of this arguments, see section 7.7 in [Sh2].) ♠

2.2 Construction of Steifel Whitney Classes and Chern Classes

Convention:

c	$K = K_c$	$F = F_c$
1	$\mathbb{Z}_2$	$\mathbb{R}$
2	$\mathbb{Z}$	$\mathbb{C}$

We shall consider a F-vector bundle $\xi = (E, p, B)$ of rank n. Let E_0 denote the open space consisting of non zero vectors in E and let $p_0 = p|_{E_0}$. Let $P(\xi) = (P(E), q, B)$ denote the projectivized bundle. We have the natural projection $E_0 \to P(E)$ and q is the factorization of p_0 through this map. Note that elements of $P(E)$ are lines in $p^{-1}(b), b \in B$.

$$\begin{tikzcd}
E(\gamma^1) \arrow[d] & E(\lambda_\xi) \arrow[l] \arrow[r, "\iota"] \arrow[d] & q^*(\xi) \arrow[dl] \arrow[dr] & \\
F\mathbb{P}^\infty & P(E) \arrow[l, "f_\xi"'] \arrow[dr, "q"'] & E_0 \arrow[l] \arrow[r, "\iota"] \arrow[d, "p_0"] & E(\xi) \arrow[dl, "p"] \\
 & & B &
\end{tikzcd}$$

Consider the pullback bundle $q^*(\xi)$ over $P(E)$:

$$q^*(\xi) = \{(L, v) \in E' \times E \ : \ q(L) = p(v)\}$$

This bundle has a natural line subbundle given by

$$\lambda_\xi = \{(L, v) \ : \ v \in L\}$$

Also observe that, for each $b \in B$, the fiber $q^{-1}(b)$ can be identified with the projective space $\mathbb{P}_F^{n-1}$ and if $j_b : q^{-1}(b) \to P(E)$ is the inclusion map, then $j_b^*(\lambda_\xi)$ is the canonical line bundle on the projective space $\mathbb{P}_F^{n-1}$.

[It can be seen that if ξ is a numerable bundle then so is λ_ξ]

Therefore, we have a unique homotopy class $f_\xi : E(P(\xi)) = P(E) \to \mathbb{P}_F^\infty = G_1(F^\infty)$ such that $\lambda_\xi = f_\xi^*(\gamma_F^1)$ where γ_F^1 is the universal line bundle over $\mathbb{P}_F^\infty$.

Recall:

Theorem 17 *The cohomology ring $H^*(\mathbb{P}^\infty_{F_c}, K_c)$ is the polynomial algebra $K_c[z]$, where* $\deg z = c, c = 1, 2$. *The rings $H^*(\mathbb{P}^n_{F_c}; K_c)$ are got by putting one relation $z^{n+1} = 0$ under the map induced by the inclusion $\mathbb{P}^n_{F_c} \subset \mathbb{P}^\infty_{F_c}$.*

Remark 12 In case of real projective spaces, ($c = 1$), there is no ambiguity in the choice a generator $z \in H^2(\mathbb{P}^n_{\mathbb{R}}; \mathbb{Z}_2) \approx \mathbb{Z}_2$. In case of complex projective spaces, first of all note that there is a canonical choice for the orientation class of $\mathbb{P}^1_{\mathbb{C}}$ being a 1-dimensional compact complex manifold, which is taken as the generator z for $H^2(\mathbb{P}^1_{\mathbb{C}}; \mathbb{Z})$. Under inclusion induced map this defines the choice of the generator for all $H^2(\mathbb{P}^n_{\mathbb{C}}; \mathbb{Z})$ ($c = 2$).

Let $f_\xi : P(E(\xi)) \to \mathbb{P}^\infty_F$ be the classifying map for the line bundle λ_ξ as constructed in the above diagram. We denote

$$a_\xi = -f_\xi^*(z) \in H^c(P(\xi)) \tag{2.4}$$

which depends on the homotopy class of f_ξ and not on the specific choice of f_ξ.

Theorem 18 *The mapping $p^* : H^*(B) \to H^*(P(\xi))$ is a monomorphism and the elements $\{1, a_\xi, \ldots, a_\xi^{n-1}\}$ form a $H^*(B)$ base for $H^*(P(\xi))$.*

[See Spanier]

In particular consider the universal n-plane bundle $\xi = \gamma^n$ over $G_n(F^\infty)$. The element $a_\xi^n \in H^{nc}(P(\xi))$ can be expressed in a unique way

$$a_\xi^n = \sum_{i=1}^{n} (-1)^{i-1} x_i(\gamma^n)) a_\xi^{n-i}, \tag{2.5}$$

where $x_i(\gamma^n) \in H^{ci}(G_n; K_c)$. These are called universal characteristic classes for n-plane bundles.

For any n-plane bundle $\xi = g^*(\gamma^n)$ over B, where $g : B \to G_k(F^\infty)$, we define

$$x_i(\xi) = g^*(x_i(\gamma^n)), \quad i \leq n, \text{ and } x_i(\xi) = 0, \quad i > n. \tag{2.6}$$

We put

$$x(\xi) = 1 + x_1(\xi) + \cdots + x_n(\xi) + 0 + \cdot \tag{2.7}$$

For $F = \mathbb{R}$ (resp. $\mathbb{C}$), the element $x_i(\xi) \in H^i(B, \mathbb{Z}_2)$ resp. $\in H^{2i}(B; \mathbb{Z})$ is called the i^{th} Steifel-Whitney class (Chern class) of ξ denoted by $w_i(\xi)$ (resp. $c_i(\xi)$. Also, $x(\xi)$ is called the total Steifel-Whitney class denoted by $w(\xi)$ (total Chern class denoted by $c(\xi)$.)

2.3 Fundamental Properties

The following four properties of characteristic classes are so fundamental that they have been upgraded to the status of being called axioms.
(A1) If ξ and η are isomorphic bundles over B then $x(\xi) = x(\eta)$.
(A2) If $g : B' \to B$ is a continuous map, then $x(g^*(\xi)) = g^*(x(\xi))$.
(A3) For any two vector bundles ξ and η over B, we have,

$$x(\xi \oplus \eta) = x(\xi) \cup x(\eta).$$

(A4) $x(\gamma^1) = 1 + z$, where $\gamma^1 := \gamma^1_F$ is the universal line bundle over $\mathbb{P}^\infty_F$.

The properties (A1), (A2) are verified easily. To verify property (A4), we note that when $\xi = \gamma^1$ then $P(\xi) = G_1(F^\infty) = \mathbb{P}^\infty_F$ and q is the identity map. Therefore $\lambda_\xi = \gamma^1$ and $f = Id$. Therefore $a_\xi = z$ is the generator of $H^c(\mathbb{P}^\infty_F)$. On the other hand the identity (2.5) reduces to the identity $a_\xi = x_1(\gamma^1)$. This proves (A4). Property (A3) is the one which will take some effort to verify. We post-pone this for a while. First, let us derive some easy and beautiful consequences of these axioms.

Corollary 4
(1) $x(\Theta^k) = 1$.
(2) If η and ξ are stably equivalent, then $x(\eta) = x(\xi)$.
(3) If B is stably parallelizable manifold, then $w(B) := w(\tau(B)) = 1$.
In particular $w(\tau(\mathbb{S}^n)) = 1$.
(4) $x(\mathbb{P}^n_F) := x(\tau(\mathbb{P}^n_F)) = (1 + z)^{n+1}$.

Proof: (1) The trivial bundle Θ^k is induced by the constant map $f : B \to G_k(F^\infty)$.
(2) We have by property (A3) $x(\eta) = x(\eta \oplus \theta^k) = x(\xi \oplus \theta^k) = x(\xi)$.
(3) follows from (2).
(4) If γ^1 denotes the canonical line bundle over $\mathbb{P}^n_F$ then we have seen that the tangent bundle $\tau(\mathbb{P}^n_F)$ is stably equivalent to $(n + 1)\gamma^1$. Now use (A3) and (A4). ♠

Corollary 5 (Stiefel) The class $w(\mathbb{P}^n) := w(\tau(\mathbb{P}^n))$ is equal to 1 iff $n + 1$ is a power of 2. Thus the only projective spaces which can be parallelizable are $\mathbb{P}^{2^k-1}, k \geq 1$.

It is known that $\mathbb{P}^n$ is parallelizable iff $1, 3, 7$. This requires digging deeper into the properties of characteristic classes which we shall not deal with here. (See [M] for further references.)

Some applications
(a) **Division Algebras**

Theorem 19 *Suppose there is a bilinear map $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which has no zero divisors. Then $\mathbb{P}^{n-1}$ is parallelizable. In particular, n is a power of 2.*

Proof: Let $\{\mathbf{e}_j\}_{1\leq j\leq}$ denote the standard basis for $\mathbb{R}^n$. That there are no divisors for β means that

$$\alpha_j : x \mapsto \beta(x, \mathbf{e}_j)$$

define isomorphisms or each j. Also note that for any fixed $x \neq 0$, the set $\{\alpha_j(x)\}$ is linearly independent. [For

$$0 = \sum_j r_j\alpha_j(x) = \beta(x, \sum_j r_j\mathbf{e}_j)$$

implies that $\sum_j r_j\mathbf{e}_j = 0$ which means each $r_j = 0, j = 1, 2, \ldots, n$.]

Putting $v_j = \alpha_1^{-1} \circ \alpha_j$, we get, for each non zero $x \in \mathbb{R}^n$, a linearly independent set $\{v_1(x) = x, v_2(x), \ldots, v_n(x)\}$. If p_x denotes the projection on the plane $x^\perp$, orthogonal to x, then it follows that $\{p_xv_2(x), \ldots, p_xv_n(x)\}$ forms a basis of $x^\perp$. This then defines a trivialization of the bundle $Hom(\gamma^1_{n-1}, {\gamma^1_{n-1}}^\perp) \cong \tau(\mathbb{P}^{n-1})$.

Remark 13 Note that by applying Gram-Schmidt process to $\{v_1(x) = x, v_2(x), \ldots, v_n(x)\}$ we obtain trivialization of $\tau(\mathbb{S}^{n-1})$ also.

Remark 14 Of course, it is known that only $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ admit bilinear forms without any zero divisors but we cannot prove this with the techniques developed so far.

(b) **Steifel-Whitney numbers and Un-oriented Cobordism**

Given any closed (i.e, compact and without boundary) n-dimensional manifold M, one knows that $H_n(M; \mathbb{Z}_2) = \mathbb{Z}_2$. A generator of this group is called a fundamental class μ_M (or orientation class) of M. We shall write $w_i(M)$ for the i^{th} Steifel-Whitney class of the tangent bundle of M.

Now consider a sequence of variables $T_1, \ldots, T_k$ where we give (weighted) degree $deg(T_i) = i$. Then each monomial $m(T) = T_1^{r_1} \cdots T_k^{r_k}$ is of total degree

$$deg(m(T)) = \sum_i r_i i.$$

For each $m(T)$ of total degree n we get a number

$$SW(T) = [w_1(M)^{r_1} \cdots w_n(M)^{r_n}] \cap \mu_M \in \mathbb{Z}_2.$$

The collection $\{SW(T)\}$ where T varies over all the monomials of total degree n is referred to as the collection of Steifel-Whitney numbers of M.

Example 8 Let us prove that all S-W numbers of $\mathbb{P}^{2n-1}$ vanish. We know $w(\mathbb{P}^m) = (1+a)^{m+1}$. Putting $m = 2n-1$, we see that $w(\mathbb{P}^{2n-1}) = (1+a^2)^n$. In particular, $w_{2i}(\mathbb{P}^{2n-1}) = 0$ for all i. Now any monomial of total degree odd will have at least one of the variables of odd degree, the conclusion follows.

On the other hand, for $m = 2n$, $w_{2n}(\mathbb{P}^{2n}) = (2n+1)a^{2n} = a^{2n} \neq 0$. Similarly, $w_1(\mathbb{P}^{2n}) = (2n+1)a = a$ and hence $w_1^{2n} = a^{2n} \neq 0$. So, there are at least two of them which are non zero.

In the special case when $m = 2^n$, we have $w(\mathbb{P}^m) = 1 + a + a^m$ and so there are no other non zero S-W numbers.

This computation may not be so impressive. However, the following two results due to two great topologists take the cake.

Theorem 20 Pontrjagin *If M is the total boundary of a compact manifold W, then the Steifel-Whitney numbers of M are all zero.*

Proof: Let $\partial : H_{i+1}(W, M) \to H_i(M)$ and $\delta : H^i(M) \to H^{i+1}(W, M)$ denote the canonical connecting homomorphism in the respective long homology (cohomology) exact sequence of the pair (W, M). The relative fundamental class $\mu_W \in H_{n+1}(W, M; \mathbb{Z}_2)$ has the property that $\partial(\mu_W) = \mu_M$. By the projection formula for the cap product we have, for any $u \in H^n(M)$

$$v \cap (\partial \mu_W) = (\delta v) \cap \mu_M$$

We know that the tangent bundle $\tau(W)$ restricted to $\partial W = M$ has the tangent bundle $\tau(M)$ of M as a subbundle. Moreover, the normal bundle of M in W is a trivial 1-dimensional bundle with, for example a strictly outward normal drawn at each point of M. Thus we have

$$\tau(W)|_M \cong \tau(M) \oplus \Theta^1.$$

This then means that $\iota^* w(W) = w(M)$. Therefore, each class $w_1(M)^{r_1} \cdots w_n(M)^{r_n} \in H^n(M)$, $(\sum r_i = n)$, is in the image of $\iota^* : H^n(W) \to H^n(M)$. By the long exact sequence

$$H^n(W) \xrightarrow{\iota^*} H^n(M) \xrightarrow{\delta} H^{n+1}(W, M)$$

it follows that $\delta(w_1(M)^{r_1} \cdots w_n(M)^{r_n}) = 0$. ♠

The converse of this theorem is a very deep result due to Thom:

Theorem 21 René Thom *If all the Steifel-Whitney numbers of a closed manifold vanish then M is the total boundary of some manifold.*

(See [T].) For strengthening these results, another type of characteristic classes called *Pontrjagin Classes* were invented. We shell study them in the next section.

(c) **Immersions and embeddings** For any immersed manifold M in $\mathbb{R}^N$ we can talk about the normal bundle $\nu(M)$ and then we have

$$\tau(M) \oplus \nu(M) \cong \theta^N.$$

It follows that

$$w(M)w(\nu(M)) = 1.$$

That is $w(\nu(M))$ is the multiplicative inverse of $w(M)$ in the cohomology algebra $H^*(M;\mathbb{Z}_2)$. In particular, the total Steifel-Whitney class of the normal bundle $\nu(M)$ is independent of the dimension of the immersion. On the other hand, since $\nu(M)$ is a $(N-n)$-plane bundle, we know that $w_i(\nu(M)) = 0$ for $i > N - m$. This then puts an obvious lower bound for the immersion dimension, provided we can compute the inverse of $w(M)$.

This is where we use the formal graded-algebra approach. Consider

$$H^{\Pi}(M) = K + H^1(M;K) + \cdots + H^n(M,K) + \cdots$$

be the direct product of $H^i(M), i \geq 0$ for any connected topological space. An element of this direct product is a finite or infinite sum

$$a_0 + a_1 + \cdots$$

where $a_i \in H^i(M,K)$. One defines componentwise addition and Cauchy product as multiplication to make it into a K-algebra. It follows easily that $H^{\Pi}(M)$ is a graded-commutative algebra in which an element of the form above is invertible iff $a_0 \neq 0$. In particular, let us compute the inverse of $w(\mathbb{P}^9)$. Indeed

$$w(\mathbb{P}^9) = (1+a)^{10} = 1 + a^2 + a^8$$

and hence

$$w(\mathbb{P}^9)^{-1} = 1 + a^2 + a^4 + a^6.$$

Therefore we cannot immerse $\mathbb{P}^9$ in $\mathbb{R}^N$ for $N < 9 + 6 = 15$. For $n = 2^r$, we actually get a sharp result. Here

$$w(\mathbb{P}^n) = (1+a)^{n+1} = 1 + a + a^n$$

and therefore

$$w(\nu) = w(\mathbb{P}^n)^{-1} = 1 + a + \cdots + a^{n-1}.$$

This then implies that we cannot immerse $\mathbb{P}^n$ in $\mathbb{R}^N$ for $N < 2n-1$. On the other hand, by a celebrated theorem of Whitney, any manifold can be immersed in $\mathbb{R}^{2n-1}$.

2.4 Splitting Principle and Uniqueness

Let $\xi = (E, p, B)$ be any vector bundle. Call a map $f : B_1 \to B$ a splitting map for ξ if $f^* : H^*(B, K) \to H^*(B_1, K)$ is injective and $f^*(\xi)$ is a direct sum of line bundles. Clearly

(1) If $f : B_1 \to B$ is a splitting map for ξ and $g : B_2 \to B_1$ is such that $g* : H^*(B_1, K) \to H^*(B_2, K)$ is injective then $f \circ g$ is also a splitting map for ξ.

(2) Any map $f : B_1 \to B$ such that $f^* : H^*(B; K) \to H^*(B_1; K)$ is injective is a splitting map for all line bundles over B.

(3) Given any ξ over B let $q : P(\xi) \to B$ be the associated projective bundle. Then $H^*(B, K) \to H^*(P(\xi); K)$ is injective. Moreover, $q^*(\xi) = \lambda_\xi \oplus \sigma_\xi$, where λ_ξ is the canonical line subbundle and σ is some complementary subbundle.

(4) Thus by induction on the rank of ξ, it follows that there is a splitting map for each ξ.

(5) Indeed, by repeated application of (1) it also follows that for any finitely many bundles ξ_i over B, there is a common splitting map.

Theorem 22 Uniqueness of Characteristic Classes *If $x(\xi), y(\xi)$ satisfy the properties (A1)–(A4), then $x = y$.*

Proof: To begin with for any line bundle λ, we have $x(\lambda) = 1 + x_1(\lambda) = 1 + y_1(\lambda) = y(\lambda)$ (Use (A4) and (A2).) Given ξ let $f : B_1 \to B$ be a splitting map, to show that $x(\xi) = y(\xi)$ it is enough to show that $f^*(x(\xi)) = f^*(y(\xi))$. But we have

$$\begin{aligned} f^*(x(\xi)) = x(f^*(\xi)) &= x(\lambda_1 \oplus \cdots \oplus \lambda_n) = x(\lambda_1)x(\lambda_2)\cdots x(\lambda_n) \\ &= y(\lambda_1)y(\lambda_2)\cdots y(\lambda_n) \\ &= y(\lambda_1 \oplus \cdots \oplus \lambda_n) \\ &= y(f^*(\xi) = f^*(y(\xi)). \end{aligned}$$

Verification of Property (A3)

Lemma 10 Let $\xi = \lambda_1 \oplus \cdots \oplus \lambda_k$, a direct sum of line bundles. Then

$$x(\xi) = x(\lambda_1)\cdots x(\xi_k) = (1 + x_1(\lambda_1))\cdots(1 + x_1(\lambda_k)). \tag{2.8}$$

Proof: Let $q : P(\xi) \to B$ be the projective bundle. Consider the line subbundle λ_ξ of $q^*(\xi) = \oplus_i q^*(\lambda_i)$. Proving (2.8) is the same as proving that the product

$$(a_\xi + x_1(\lambda_1))\cdots(a_\xi + x_1(\lambda_k)) = 0. \tag{2.9}$$

Upon tensoring with λ_ξ^*, the line subbundle λ_ξ, yields a trivial subbundle of $\oplus_{i=1}^n[\lambda_\xi^* \otimes q^*(\lambda_i)]$. This is the same as having a nowhere vanishing section s of the direct sum. When projected to any of the summands, this yields a cross section s_i of the line bundle $\lambda_\xi^* \otimes q^*(\lambda_i)$. Let $V_i \subset P(\xi)$ on which s_i does not vanish. This means that restricted to V_i, $\lambda_\xi^* \otimes q^*(\lambda_i)$ is trivial. Therefore $a_\xi + q^* x_1(\lambda_i) = x_1(\lambda_\xi) + x_1(q^*(\lambda_i)) = 0$ on V_i. Since $\cup_i V_i = P(\xi)$, from a general observation on cup products, (2.9) follows. ♠

Given ξ, η on B, let $f : B_1 \to B$ be a common splitting. Let

$$\oplus_{i=1}^k \lambda_i = f^*(\xi); \qquad \oplus_{j=1}^l \lambda_{n+j} = f^*(\eta).$$

Then

$$\begin{aligned} f^* x(\xi \oplus \eta) &= x(f^*(\xi \oplus \eta)) = x(\lambda_1 \oplus \cdots \oplus \lambda_{m+n}) \\ &= [x(\lambda_1) \cdots x(\lambda_n)][x(\lambda_{n+1} \cdots x(\lambda_{n+m}) \\ &= x(f^*(\xi)) x(f^*(\eta)) = f^*(x(\xi) x(\eta)). \end{aligned}$$

Since f^* is injective, we are through. ♠

Theorem 23 *For any complex k-plane bundle η over a paracompact space, we have $e(\eta) = c_k(\eta)$.*

Proof: By the splitting principle, and the product property of Euler class and Chern class for Whitney sums, it is enough to prove this for line bundles. By the classification of line bundles, it is enough to prove this for the universal line bundle γ^1 over $\mathbb{P}_{\mathbb{C}}^\infty$. Since the inclusion $\mathbb{P}_{\mathbb{C}}^1 \hookrightarrow \mathbb{P}_{\mathbb{C}}^\infty$ induces an isomorphism in second cohomology, it is enough to prove this for $\gamma_{\mathbb{C}}^1$. For this case, in example **??**, we have verified that $e(\gamma^1) = z$ the canonical generator and by definition (A4) $c_1(\gamma^1) = z$. ♠

Exercise 8 Show that for any complex line bundle η over a paracompact space, $w_2(\eta)$ is equal to the mod 2 reduction of $c_1(\eta)$. [Hint: First show that the restriction of canonical line bundle over $\mathbb{P}_{\mathbb{C}}^{n-1}$ to $\mathbb{P}_{\mathbb{R}}^{2n-1}$ is the complexification of the canonical line bundle over $\mathbb{P}_{\mathbb{R}}^{2n-1}$.]

2.5 Complex bundles and Pontrjagin Classes

Definition 9 Let V be a $\mathbb{R}$-vector space of even dimension. By a complex structure on V we mean a $\mathbb{R}$-linear isomorphism $J : V \to V$ such that $J \circ J = -Id$.

Given a complex vector space F, we shall denote by $F_{\mathbb{R}}$ the underlying real vector space of dimension $2(dim_{\mathbb{C}} F)$.

Remark 15 This is an example of a *forgetful functor*. Here it forgets the complex structure retaining only the real vector space structure and the orientation. Clearly the map $J(\mathbf{v}) = \imath\mathbf{v}$ gives a complex structure on $F_{\mathbb{R}}$ which is $\mathbb{C}$-isomorphic to F.

Definition 10 By the complexification of a real vector space V we mean taking $V \otimes_{\mathbb{R}} \mathbb{C}$. The complex structure this is defined by $J(\mathbf{v} \otimes 1) = \mathbf{v} \otimes \imath)$. Under the identification $V \otimes \mathbb{C} \to V \oplus V$ given by

$$(\mathbf{u} \otimes 1 \mapsto (\mathbf{u}, 0); \quad \mathbf{u} \otimes \imath \mapsto (0, \mathbf{u})$$

the complex structure takes the form:

$$J(\mathbf{u}, \mathbf{v}) = (-\mathbf{v}, \mathbf{u}).$$

Definition 11 Given a complex vector space F by the conjugate complex vector space $\bar{F}$ we mean the underlying real vector space together with the complex structure $J(\mathbf{v}) = -\imath\mathbf{v}$.

Lemma 11 (i) We have for any real vector space V, $V \otimes \mathbb{C}$ is canonically isomorphic to the conjugate $\mathbb{C}-$ vector space $\overline{V \otimes \mathbb{C}}$.
(ii) Given a complex vector space F we have a canonical isomorphism $F_{\mathbb{R}} \otimes \mathbb{C} \to F \oplus \bar{F}$.
(iii) Both (i) and (ii) hold for vector bundles as well.

Proof: (i) Define $\Theta(\mathbf{u} + \imath\mathbf{v}) = \mathbf{u} - \imath\mathbf{v}$ and verify that Θ is as required.
(ii) Consider the following two maps $f, g : F \to F_{\mathbb{R}} \otimes \mathbb{C}$ given by

$$g(\mathbf{u}) = (\mathbf{u}, -\imath\mathbf{u}); \quad h(\mathbf{u}) = (\mathbf{u}, \imath\mathbf{u}).$$

Verify that g is a complex linear and h is conjugate linear and both are injective. Moreover, images of the two maps span the entire $F_{\mathbb{R}} \otimes \mathbb{C}$. Therefore, we can identify $F_{\mathbb{R}} \otimes \mathbb{C}$ with $F \oplus \bar{F}$.
(iii) Since this isomorphism is canonical, we get the same statement for vector bundles as well. ♠

Lemma 12 $c_1(\overline{\gamma^1}) = -c_1(\gamma_1)$ where γ^1 is the canonical line bundle over $\mathbb{CP}^\infty$.

Proof: Enough to prove this for the canonical line bundle over $\mathbb{CP}^1$. Consider the map $j : \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by $z \mapsto \bar{z}$. Then $j*(\gamma_1) \cong \overline{\gamma^1}$. Therefore $c(\overline{\gamma_1}) = j^*(1+z) = 1-z$. ♠

Theorem 24 $c_i(\bar{\xi}) = (-1)^i c_i(\xi)$.

Proof: Use splitting principle.

Thus for a real bundle ξ, if the total Chern class is

$$c(\xi \times \mathbb{C}) = 1 + c_1 + \cdots + c_k$$

then

$$c(\overline{\xi \otimes \mathbb{C}}) = 1 - c_1 + c_2 - + \cdots + (-1)^k c_k$$

from lemma 11, it follows that $2c_{2i-1}(\xi \otimes \mathbb{C}) = 0$. So, we concentrate our attention on the even degree terms.

Definition 12 The i^{th} Pontrjagin class of a real vector bundle ξ is defined as

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B; \mathbb{Z})$$

and the total Pontrjagin class

$$p(\xi) = 1 + p_1(\xi) \cdots + p_{[n/2]}(\xi).$$

Remark 16
(a) The sign is introduced so that elsewhere some formula becomes nicer. (See corollary 7.) You may ignore the sign (like some authors), which is OK provided you are consistent.
(b) All the four fundamental properties of the Chern classes hold for Pontrjagin classes as well except that the product formula is valid only up to order 2 terms, i.e.,

$2[p(\xi \oplus \eta) - p(\xi)p(\eta)] = 0$. However, we have the stronger result $p(\xi \oplus \theta^1) = p(\xi)$ which follows directly from the corresponding result for Chern classes. In particular, $p(\tau(S^n)) = 1$.

Theorem 25 *For any complex k-plane bundle ω we have*

$$1 - p_1 + p_2 - + \cdots + (-1)^k p_k = (1 - c_1 + c_2 - + \cdots + (-1)^k c_k)(1 + c_1 + \cdots + c_k).$$

In particular,

$$p_j(\omega) = c_j^2 - 2c_{j-1}c_{j+1} + - \cdots + (-1)^k 2c_{2k}.$$

Corollary 6 $p(\mathbb{CP}^k) = (1 + a^2)^{k+1}$.

Lemma 13 Given any $\mathbb{R}$-bundle of rank k, there is a $\mathbb{R}$-isomorphism $\xi \oplus \xi \to (\xi \otimes \mathbb{C})_{\mathbb{R}}$ which is orientation preserving iff $k(k-1)/2$ is even.

Corollary 7 If ξ is an oriented real bundle of rank $2k$, then $p_k(\xi)$ is equal to the square of the Euler class $e(\xi)$.

Proof: We have $p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C}) = (-1)^k e(\xi \otimes \mathbb{C})$. On the other hand, $e(\xi \otimes \mathbb{C}) = (-1)^{2k(2k-1)/2} e(\xi \oplus \xi) = (-1)^{k(2k-1)} e(\xi)^2 = (-1)^k e(\xi)^2$. ♠

Theorem 26 *The cohomology ring of the oriented infinite real Grassmannian manifold $\tilde{G}_{2n+1}$ (respectively, $\tilde{G}_{2n}$) is, up to 2-torsion, isomorphic to the polynomial ring generated by the Pontrjagin classes $p_1, \ldots, p_n$ (respectively, $p_1, \ldots, p_n$, and $e(\gamma^{2n})$) of the canonical oriented $2n+1$ (respectively $(2n)$)-plane bundle over $\tilde{G}_{2n+1}$ (resp. $\tilde{G}_{2n}$).*

Definition 13 Let $M_i, i = 1, 2$ be any two closed manifolds. We say M_1 is cobordant to M_2 if there exists a compact manifold W with $\partial W = M_1 \dot{\cup} M_2$.

Remark 17 It can be shown that being cobordant is an equivalence relation on the diffeomorphism class of all closed n-dimensional manifolds. Under disjoint union, these classes form an abelian group. Under the Cartesian product, this abelian group becomes a graded commutative ring. We state without proof:

Theorem 27 Thom *The oriented cobordism group Ω_n is finite for $n \not\equiv 0 \mod 4$ and is a finitely generated group of rank equal to the number of partitions of r for $n = 4r$.*

Corollary 8 Let M be a smooth closed oriented manifold. Then some positive multiple of M is oriented null cobordant iff all the Pontrjagin numbers of M vanish.

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