INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

Department of Mathematics

Seminar Lectures on Foliation Theory [1]

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Lecture 1

Basic requirements for this Seminar Series: Familiarity with the notion of differential manifold, submersion, vector bundles.

1 Some Examples

Let us begin with some examples:
(1) Write $\mathbb{R}^m = \mathbb{R}^d \times \mathbb{R}^{m-d}$. As we know this is one of the several cartesian product decomposition of $\mathbb{R}^m$. Via the second projection, this can also be thought of as a 'trivial vector bundle' of rank d over $\mathbb{R}^{m-d}$. This also gives the trivial example of a codim. d-foliation of $\mathbb{R}^n$, as a decomposition into d-dimensional leaves $\mathbb{R}^d \times \{y\}$ as y varies over $\mathbb{R}^{m-d}$.
(2) A little more generally, we may consider any two manifolds M, N and a submersion $f : M \to N$. Here M can be written as a disjoint union of fibres of f each one is a submanifold of dimension equal to $dim\ M - dim\ N = d$. We say f is a submersion of M of codimension d. The manifold structure for the fibres comes from an atlas for M via the surjective form of implicit function theorem since $df_p : T_pM \to T_{f(p)}N$ is surjective at every point of M. We would like to consider this description also as a codim d foliation. However, this is also too simple minded one and hence we would call them simple foliations. If the fibres of the submersion are connected as well, then we call it strictly simple.
(3) **Kronecker Foliation of a Torus** Let us now consider something non trivial. Fix an irrational number a. To each $\lambda \in \mathbb{S}^1$ consider the map $f_\lambda : \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$ given by

$$t \mapsto (e^{2\pi\imath t}, \lambda e^{2\pi\imath a t}).$$

Clearly, f_λ is an injective immersion. Call he image $f_\lambda(\mathbb{R})$ a leaf. Each leaf is dense in $\mathbb{S}^1 \times \mathbb{S}^1$ (exercise). As λ varies over $\mathbb{S}^1$ these curves cover $\mathbb{S}^1 \times \mathbb{S}^1$. For $\lambda_1 \neq \lambda_2$, either $f_{\lambda_1}(\mathbb{R}) \cap f_{\lambda_2}(\mathbb{R}) = \emptyset$ or they are equal. Indeed $f_{\lambda_1}(\mathbb{R}) = f_{\lambda_2}(\mathbb{R})$ iff $\lambda_1\lambda_2^{-1} \in < e^{2\pi\imath a} >$. Thus the leaves are parameterised by the quotient group $\mathbb{S}^1 / < e^{2\pi\imath a} >$. This group has very poor topological structure: the closure of the identity element is the whole group.

In fact this foliation is induced via the covering projection $p : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ from the foliation of $\mathbb{R}^2$ given by the lines of irrational slope a. On $\mathbb{R}^2$ this is a simple foliation. The induced one on $\mathbb{S}^1 \times \mathbb{S}^1$ is not a simple one.

[1] These lectures were part of the Year Long Programme in Geometry and Topology 2008

Later we shall see that if $p : M \to N$ is a covering projection then giving a foliation on M is equivalent to giving a foliation on N.

(4) **Möbius Band** Think of Möbius band $\mathcal{M}$ as a quotient of $\mathbb{R}^2$ by the relation $(x, y) \sim (x', y')$ iff $x - x'$ is an integer and $y' = (-1)^{x-x'}y$. Then the product foliation $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ induces a foliation on $\mathcal{M}$. All lines $\mathbb{R} \times \{y\}$ are mapped to circles, the ones with $y = n + \frac{1}{2}$ being mapped onto the central circle. Note that this is a non simple foliation, in the sense, it does not correspond to any submersion. If you interchange the (x, y) coordinates, then you get another foliation which makes $\mathcal{M}$, the twisted line bundle over $\mathbb{S}^1$.

(5) **The Reeb Foliation** of $\mathbb{S}^3$. Think of $\mathbb{S}^3$ as a union of two solid tori $\mathbb{S}^1 \times D^2$ and $D^2 \times \mathbb{S}^1$ glued along their boundary via the identity map.

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \ : \ |z_1|^2 + |z_2|^2 = 1\}.$$

$$M_1 = \{(z_1, z_2) \in \mathbb{S}^3 \ : \ |z_1|^2 \leq 1/2\}; \quad M_2 = \{(z_1, z_2) \in \mathbb{S}^3 \ : \ |z_2|^2 \leq 1/2\}$$

$$\mathbb{S}^3 = M_1 \cup M_2; \quad M_1 \cap M_2 = \{(z_1, z_2) \ : \ |z_1|^2 = \frac{1}{2} = |z_2|^2\}.$$

We shall construct a foliation of $\mathbb{S}^3$ by constructing one on each of the solid tori M_j in which the boundary surface occurs as a leaf and glue them together to get a foliation on $\mathbb{S}^3$.

Consider $\phi : int\ D^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x, y, t) = e^{\frac{1}{1-x^2-y^2}} - t.$$

This map is clearly a submersion and hence defines a simple foliation on $int\ D^2 \times \mathbb{R}$. For each $s \in \mathbb{R}$, we get a leaf which is the the graph of the function $(x, y) \mapsto e^{\frac{1}{1-x^2-y^2}} - s$. Each leaf is diffeomorphic to $int\ D^2$ (diffeomorphic to $\mathbb{R}^2$). Every point on the boundary of $D^2 \times \mathbb{R}$ is the limit point of the leaves and each leaf is 'tangential at infinity' to the boundary. By adding the boundary also as a leaf, we get a foliation of $D^2 \times \mathbb{R}$. The covering map $D^2 \times \mathbb{R} \to D^2 \times \mathbb{S}^1$ then induces a foliation on $D^2 \times \mathbb{S}^1$. Note that except the boundary piece, every other leaf is diffeomorphic to $\mathbb{R}^2$. As subspaces of $D^2 \times \mathbb{S}^1$, they are submanifolds but **not** closed.

Two copies of these are then put together to get a foliation of $\mathbb{S}^3$.

We can consider the space of leaves as a quotient space of $\mathbb{S}^3$. Here too, this space has a single point whose closure is the whole space.

So far, we have **not** defined a number of terms here such as foliation, leaves etc.. The idea was to motivate appropriate definitions by looking at these examples.

2 Basic Definitions

Throughout these notes, M will denote a m-dimensional smooth manifold.

Definition 2.1 Let $1 \leq d \leq m-1$. By a *foliation-atlas* $\mathcal{U} = \{(U_i, \phi_i)\}$ *on* M *of codimension* $m-d$ (or equivalently, ... *of dimension* d) we mean an open covering $\{U_i\}$ of M and diffeomorphisms $\phi_i : U_i \to \mathbb{R}^d \times \mathbb{R}^{m-d}$ satisfying the compatibility condition (FA) *for every pair* (i, j) *the change of charts*

$$\phi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is of the form

$$\phi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)), \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^{m-d}.$$

Two foliation-atlases $\mathcal{U}_1, \mathcal{U}_2$ are said to be compatible with each other, if the change of chart from any member of $\mathcal{U}_1$ to any member of $\mathcal{U}_2$ satisfies (FA). In that case, it follows that $\mathcal{U}_1 \cup \mathcal{U}_2$ itself is a foliation-atlas. Therefore every foliation-atlas is contained in a unique maximal one.

Definition 2.2 A maximal foliation-atlas $\mathcal{F}$ on M is called a foliation on M. Members (U_i, ϕ_i) of a foliation $\mathcal{F}$ will be called local product structures for M.

Example 2.1 Let us verify that any submersion $f : M \to N$ defines a foliation in a natural way. By surjective form of implicit function theorem, to each point $p \in m$ we have a nbd U_i of p and diffeomorphisms $\psi_i : \mathbb{R}^n \to U_i$ and $\alpha_i : \mathbb{R}^{n-d} \to f(U_i)$ such that $\psi_i(0) = p$ and $f \circ \psi_i(x_1, \ldots, x_n) = \alpha(x_{d+1}, \ldots, x_n)$. Therefore we can take $\phi_i = \psi_i^{-1}$ and then $h_{ij} = \alpha_j^{-1} \circ \alpha_i$ will fit the condition (FA). Thus the collection $\{(U_i, \psi_i^{-1})\}$ forms an atlas of submersions on M. The important point to note here is that the class of $\mathcal{F}$ obtained this way is independent of the choices of U_i and ψ_i.

Exercise 2.1 Verify that each of the examples discussed above fits the above definition of a foliation.

Remark 2.1
(i) We may allow $d = 0$. Thus a differentiable structure on M is nothing but a 0-foliation.
(ii) Given a foliation $\mathcal{F}$, we take any foliation atlas $\mathcal{U}$ belonging to $\mathcal{F}$ to study its geometric aspects. Thus, if $p \in U_i$, put $\phi(p) = (x, y) \in \mathbb{R}^d \times \mathbb{R}^{m-d}$. Then $\phi_i^{-1}(\mathbb{R}^d \times y)$ is a submanifold of U_i diffeomorphic to $\mathbb{R}^d$ and contains the point p. We call this a **slice** of $\mathcal{F}$ through p. Observe that any two slices of $\mathcal{F}$ through p coincide in a nbd of p (i.e., independent of the index i chosen). Each open set U_i is a disjoint union of slices. Whenever two slices belonging to different charts meet, they will meet in an open set in the slice and hence their union is a submanifold of dimension d. This is then true for any finite union of slices. However, when we take infinite union something breaks down. We shall come back to these objected which are going to be called *leaves* of the foliation, a little later.

3 Vector fields

Let now TM denote the tangent bundle of M, $\mathcal{C}(M)$ denote the ring of all smooth functions on M and $\mathfrak{X}(M)$ denote the space of all smooth sections of M i.e., smooth vector fields on M. Observe that $\mathfrak{X}(M)$ is a $\mathcal{C}(M)$-module.

Definition 3.1 A family of vector fields $\{X^j\}$ on M is said to be *locally finite* if for every point $p \in M$, there is a nbd U of p in M such that for all but finitely many j, we have $X^j|U \equiv 0$. A submodule V of $\mathfrak{X}(M)$ is said to be *complete* if for each locally finite family $\{X^j\}$ in V, we have $\sum_j X^j \in V$.

Example 3.1 (i) Let M be a non compact manifold and $\mathfrak{X}_c(M)$ be the submodule of $\mathfrak{X}(M)$ consisting of those fields with compact support. Then $\mathfrak{X}_c(M)$ is **not** a complete submodule. For, there are definitely vector fields X whose support is non compact and we can write $X = \sum_j \eta_j X$ where $\{\eta_j\}$ is a smooth partition of unity associated to a relatively compact, locally finite atlas.
(ii) On the other hand if M is compact, then every submodule V of $\mathfrak{X}(M)$ is complete. For, if $\{X^j\}$ is a locally finite subfamily of V we can find a finite cover $\{U_k\}$ of M such that restricted to each U_k the family is finite, which in turn implies that the family itself is finite.

Given a vector subspace V of $\mathfrak{X}(M)$, as such we are not interested in the rank (dimension) of V as a vector space. But we are interested in another number which we shall call dimension of V.

Definition 3.2 Let V be a vector subspace of $\mathfrak{X}(M)$. Given any point $p \in M$ consider the mapping $V \to T_p(M)$ given by $X \mapsto X(p)$, which is clearly a linear map of vector spaces. Let us denote the image of this map by $E(V)_p$. Suppose V is such that for each p, $dim\ E(V)_p = d$. Then we call this common number d the dimension of V. We also call a rank d-subbundle of tangent bundle TM a *d-field* on M.

We are interested in the question when does the collection $\{E(V)_p\}$ define a subbundle of TM.

Theorem 3.1 *Let E be a d-field on M and $V(E)$ denote the set of all vector fields X such that $X(p) \in E_p$ for all p. Then $V(E)$ is a complete submodule of $\mathfrak{X}(M)$ and is of dimension d. Moreover, the assignment*

$$E \mapsto V(E)$$

defines a one-to-one correspondence between d-fields on M and d-dimensional complete submodules of $\mathfrak{X}(M)$.

Proof: Check that $V(E)$ is a complete submodule. To see that $V(E)$ is of dimension d, it is enough to check that $V(E)_p = E_p$. Again, clearly $V(E)_p \subset E_p$. To see the other way inclusion, given $\mathbf{v} \in E_p$, it is enough to produce a vector field X such that $X_p = \mathbf{v}$ and $X_q \in E_q$ for each $q \in M$. Since E is locally trivial, we can find a vector field Y in a nbd U of p such that $Y_q \in E_q, q \in U$ and $Y_p = \mathbf{v}$. Now choose a bump function η at p on M and consider $X = \eta Y$. Then outside U, $\eta = 0$ and hence it follows that $X_q \in E_q$ for all $q \in M$. Also $\eta(p) = 1$ and hence $X_p = \mathbf{v}$ as required.

Now let V be a d-dim. complete submodule of $\mathfrak{X}(M)$. We shall show that $E(V)$ is a subbundle of TM. Given any point $p \in M$, let $X^1, \ldots, X^d \in V$ be such that $\{X_p^1, \ldots, X_p^d\}$ forms a basis for $E(V)_p$. Fix a product structure to the tangent bundle TM over a nbd U of p and let us write $TU = U \times \mathbb{R}^m$. In this product structure, the vector fields X^j are nothing but smooth maps $X^j : U \to \mathbb{R}^m$ which are independent at p. We may complete $\{X_p^1, \ldots, X_p^d\}$ to a basis $\{X_p^1, \ldots, X_p^d, \mathbf{v}_1, \ldots, \mathbf{v}_{m-d}\}$ of $\mathbb{R}^m$. By continuity, there is a nbd W of p on which $\{X_q^1, \ldots, X_q^d, \mathbf{v}_1, \ldots, \mathbf{v}_{m-d}\}$ will form a basis for $\mathbb{R}^m$. This then gives subbundle structure to $E(V)$ over W. Thus $E(V)$ is a d-field.

It remains to verify that $E(V(E)) = E$ and $V(E(V)) = V$. The first one is verified since the same holds point-wise, i.e, $E(V(E))_p = V(E)_p = E_p$. For the second equality, it is clear that $V \subset V(E(V))$. On the other hand, let $X \in V(E(V))$, i.e., a vector field X such that $X_p \in E(V)_p$ for each p.

Pick up an open covering $\{U_\alpha\}$ for M and for each i, vector fields $X^{\alpha,1}, \ldots, X^{\alpha,d} \in V$ which span $E(V)_q$ for all $q \in U_\alpha$. Write

$$X(q) = \sum_j t_{\alpha,j}(q) X^{\alpha,j}(q), q \in U_\alpha$$

where $t_{\alpha,j}$ are smooth function on U_α. Choose a partition of unity $\{\eta_\alpha\}$ subordinate to $\{U_\alpha\}$ and put

$$Y^\alpha(q) = \eta_\alpha(q) \left(\sum_{j=1}^{d} t_{\alpha,j}(q) X^{\alpha,j}(q) \right), q \in U_\alpha.$$

Then each $Y^\alpha \in V$ and this is a locally finite family. Therefore $X = \sum_\alpha Y^\alpha \in V$, since V is complete. ♠

Corollary 3.1 A d-dimensional submodule V of $\mathfrak{X}(M)$ is complete iff it satisfies the following condition: " $X \in \mathfrak{X}(M)$ is in V iff $X_p \in V_p$ for every $p \in M$.

Lecture 2

Remark 3.1 Given a submersion $\phi : M \to N$, $ker\ (D\phi)$ defines a d-field on M where $d = dim\ M - dim\ N$. However, a d-field need not define a submersion. On the other hand locally, it does precisely the same. This motivates the following definition.

Definition 3.3 By an *atlas $\mathcal{S}$ of submersions* of codim d on a manifold M, we mean a family $f_i : U_i \to \mathbb{R}^{m-d}$ of submersions, where $\{U_i\}$ is an open covering of M, satisfying the following compatibility condition:
(AS) For each point $p \in U_i \cap U_j$, there exists a nbd W of $f_i(p)$ in $\mathbb{R}^{m-d}$ and a diffeomorphism h of W onto a nbd of $f_j(p)$ such that $f_j(q) = h \circ f_i(q), q \in (f_i)^{-1}(W) \cap U_j = V$.

$$\begin{tikzcd} & V \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ W \arrow[rr, "h"] & & h(W) \end{tikzcd}$$

Fig.1

Remark 3.2 Given two atlases of codim d submersions the union will be an atlas, if members of one are compatible with the members of the other. As usual, every atlas of submersions is contained in a unique maximal atlas.

Remark 3.3 Given a foliation-atlas $\mathcal{U} \in \mathcal{F}$, put $f_i = \pi_2 \circ \phi_i$ where $\pi_2 : \mathbb{R}^d \times \mathbb{R}^{m-d} \to \mathbb{R}^{m-d}$ is the second projection.

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\ \mathbb{R}^d \times \mathbb{R}^{m-d} \arrow[rr, "{(g(x,y),h(y))}"] \arrow[d, "\pi_2"'] & & \mathbb{R}^d \times \mathbb{R}^{m-d} \arrow[d, "\pi_2"] \\ \mathbb{R}^{m-d} \arrow[rr, "h"] & & \mathbb{R}^{m-d} \end{tikzcd}$$

Then $\{(U_i, f_i)\}$ becomes an atlas of submersions. We can then take the maximal atlas of submersions containing this and call this and denote it by $\mathcal{F}_s$. Observe that if we had chosen a different atlas belonging to $\mathcal{F}$, we still get the same maximal atlas of submersions.

Conversely given a maximal atlas of submersions $\mathcal{S}$, we can take a refinement $\{(U_i, f_i)\}$ of it for which there exist diffeomorphisms $\phi_i : U_i \to \mathbb{R}^d \times \mathbb{R}^{m-d}$ such that $\pi_2 \circ \phi_i = f_i$. (In the above diagram, given one of the two horizontal maps, the other one can be constructed.) It then follows that $\{(U_i, \phi)\}$ is a foliation-atlas. We can now take the maximal one containing this and denote this by $\mathcal{S}_a$. It is easy to check that that $((\mathcal{F})_s)_a = \mathcal{F}$ and $(\mathcal{S}_a)_s = \mathcal{S}$ once you observe that a refinement of an atlas (in either situation) is compatible with the given atlas. Thus we have an alternative definition of foliation in the form of the following theorem.

Theorem 3.2 *The assignments $\mathcal{F} \rightsquigarrow \mathcal{F}_s$ and $\mathcal{S} \rightsquigarrow \mathcal{S}_a$ are inverses of each other and so define a 1-1 correspondence of foliations on M with the class of maximal atlases of submersions.*

Remark 3.4 Given a d-foliation $\mathcal{F}$, we associate a d-field $E(\mathcal{F})$ as follows. For each $p \in U_\alpha$, take $E(\mathcal{F})_p = Ker\ D(f_\alpha)_p$. Because of the compatibility condition (AS), this is independent of α chosen so that $p \in U_\alpha$. Therefore each $E(\mathcal{F})_p$ is a well defined d-dimensional subspace of T_pM. On the other hand, the subspaces being kernel of local submersions, the local triviality of the subbundle follows. Thus $E(\mathcal{F})$ is a d-field on M. We consider the problem of when and how to construct a foliation out of a given d-field.

The following lemma plays the key role in the next result. It can be proved as a direct application of implicit function theorem.

Lemma 3.1 Let U be an open subset of $\mathbb{R}^m$, $f, g : U \to \mathbb{R}^{m-d}$ be two submersions such that $Ker\ D(f)_q = Ker\ D(g)_q$ for all $q \in U$. Then for every $p \in U$, there is an open nbd W of $f(p)$ and a diffeomorphism $h : W \to h(W)$ onto an open nbd of $g(p)$ such that $h \circ f = g$ in a nbd of p.

As an immediate consequence we have:

Theorem 3.3 *Let E be a d-field on a manifold M, $\{U_\alpha\}$ be an open covering of M and $f_\alpha : U_\alpha \to \mathbb{R}^{m-d}$ such that $Ker\ D(f_\alpha)_q = E_q, q \in U_\alpha$. Then $\{(U_\alpha, f_\alpha)\}$ is an atlas of submersions.*

Theorem 3.4 *The association $\mathcal{F} \mapsto E(\mathcal{F})$ is one-one.*

Proof: Suppose $E(\mathcal{F}_1) = E(\mathcal{F}_2)$. Then the hypothesis in the above theorem is satisfied for the atlas $\mathcal{F}_1 \cup \mathcal{F}_2$. Therefore, $\mathcal{F}_1 \cup \mathcal{F}_2$ is as atlas of submersions. Since both $\mathcal{F}_i$ are maximal, $\mathcal{F}_1 = \mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}_2$. ♠

Definition 3.4 A d-field E on M is called *completely integrable*(CI), if there exists a d-foliation $\mathcal{F}$ such that $E(\mathcal{F}) = E$.

Remark 3.5 Thus d-foliations on M are in 1-1 correspondence with completely integrable d-fields on M. This result however does not really give us a new definition of a foliation since complete integrability is not defined independent of the foliation. This we shall take up soon.

It turns out that not all d-fields are CI. However, it is not so difficult to see this to be true for $d = 1$.

Theorem 3.5 *Every 1-field is completely integrable.*

Proof: By lemma 3.1, the problem becomes a local one. So, consider a point $p \in M$ and a nbd U of p in which the line bundle is trivial. This then gives us a vector field X on U which is non zero on U. We may assume that $p = 0$ and that $X_0 = \mathbf{e}_n$. The Initial value problem

$$f(q, 0) = q, \frac{\partial f}{\partial t}(q, t) = X(f(q, t)) \tag{1}$$

has a unique solution $f : W \times (-\epsilon, \epsilon) \to U$, where W is some nbd of 0 in $\mathbb{R}^n$ and $\epsilon > 0$. Put $V' = W \cap \mathbb{R}^{n-1} \times \{0\} \times (-\epsilon, \epsilon)$ and $h = f|V'$. Since $h(x, 0, 0) = (x, 0)$, it follows that $\frac{\partial h}{\partial x_i}(0) = \mathbf{e}_i, i = 1, 2, \ldots, n-1$. Also $\frac{\partial h}{\partial t}(0) = X_0 = \mathbf{e}_n$. Therefore h is of maximal rank at $(0, 0)$ and hence there exist a nbd V of 0 in V' on which h is a diffeomorphism. Now look at $\eta = \pi \circ h^{-1}$ on $h(V)$, where $\pi : V' \to \mathbb{R}^{n-1}$ is the natural projection. Clearly η is a submersion. From (1) it follows that $D(h)_{(x,0,t)}(\partial_t) = X_{h(x,0,t)}$. Since $D(\pi)(\partial_t) = 0$, it follows that $X_q \in Ker\ D(\eta)_q, q \in h(V)$. Since $Ker\ D(\eta)$ is 1-dimensional, it is spanned by X. ♠

4 Frobenius Theorem

Let us recall some more basic facts about vector fields.

Given a smooth vector field X on a smooth manifold M and a smooth function $f : M \to \mathbb{R}$, let us define the function Xf by the formula

$$(Xf)_p = X_p(f),$$

where the rhs represents the directional derivative of f in the direction of X_p (up to scalar). Indeed, here we are using the definition of a tangent vector X_p at $p \in M$ to be a map defined on the set of all smooth real valued functions in a nbd of p to the real numbers with the properties:
(a) If f and g agree on a nbd of p then $X_p(f) = X_p(g)$
(b) $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$.
(c) $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$.
Verification that $p \mapsto X_p f$ is smooth as a function of p (since X is smooth) is straight forward once you convert the whole thing in local coordinates. And the same argument shows that X also satisfies (b) and (c). Thus

Theorem 4.1 *Every smooth vector field X defines a derivation on the ring $C^\infty(M)$, i.e., a linear map of the vector spaces $\mathcal{C}(M) \to \mathcal{C}(M)$ which satisfies the Leibniz rule: $X(fg) = X(f)g + fX(g)$.*

What we are now interested in is the converse of this.

Theorem 4.2 *Every derivation on $C^\infty(M)$ corresponds to a smooth vector field on M.*

Proof: Given a derivation χ on $\mathcal{C}(M)$, for each $p \in M$, define $X_p(f) = \chi(f)(p)$. Then X_p satisfies (b) and (c). To prove (a) we shall simply show that if f vanishes in a nbd U of p, then $chi_(f)(p) = 0$. By smooth Urysohn's lemma, there exists a smooth map λ on M which is 1 outside U and is $\lambda(p) = 0$. Thus $f = \lambda f$. Now $\chi(f)(p) = \chi(\lambda(p) = \lambda(p)\chi(f)(p) + \chi(\lambda)(p)f(p) = 0$. ♠

Remark 4.1 The partial derivatives ∂_j w.r.t. to the variable x_j on $\mathbb{R}^m$ are simple examples of vector fields on $\mathbb{R}^m$. One of the important properties of these vector fields is that successive operations by several of them can be performed in any order without affecting the end result, i.e., ∂_j denotes the partial derivative w.r.t x_j then we know that $\partial_j\partial_k(f) = \partial_k\partial_j(f), f \in \mathcal{C}(\mathbb{R}^m)$. This is no longer true of arbitrary elements of $\mathfrak{X}(M)$ (even when $M = \mathbb{R}^m$). In order to capture this non commutativeness of vector fields, we consider the operation

$$(X, Y) \mapsto [X, Y] := XY - YX$$

called the Lie-bracket of X with Y. The following properties are easily verified.

Lemma 4.1 The binary operation bracket

$$[\ \ ,\ \ ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfies the following properties.
(i) $[X, Y]$ is bilinear.
(ii) Anti-symmetric:$[Y, X] = -[X, Y]$.
(iii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.
(iv) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

Exercise Verify (iv).

Remark 4.2 Observe that $[\ \ ,\ \ ]$ need not be even associative. Property (iii) is a replacement for non associativity and is called Jacobi Identity. In the abstract set up properties (i),(ii),(iii) make a vector space with an binary operation $[\ \ ,\ \ ]$ into a *Lie Algebra.* Any vector subspace which is closed under $[\ \ ,\ \ ]$ will then be called a Lie subalgebra. The special property (iv) in the case of $\mathfrak{X}(M)$, gives the relation between the $\mathcal{C}(M)$-module structure and the bracket operation, which comes handy when we want to study $\mathcal{C}(M)$-submodules of $\mathfrak{X}(M)$ which are Lie subalgebras. Our study of these objects begins with:

Lemma 4.2 Any 1-dimensional complete submodule of $\mathfrak{X}(M)$ is a subalgebra.

Proof: If V is a 1-dimensional submodule, then V_p is 1-dimensional for every $p \in M$. Let $X, Y \in V$. From cor 3.1, it is enough to prove that $[X, Y]_p \in V_p, \forall\, p \in M$. If $X_p \neq 0$, we can write $Y_q = \lambda X_q$ in a nbd U_p of p. Then $[X, Y]_q = [X, \lambda X]_q = X_q(\lambda)X \in V$. In particular, $[X, Y]_p = X_p(\lambda)X_p \in V_p$. Similarly we can show that if $Y_p \neq 0$, then $[X, Y]_p \in V$. If both $X_p = 0 = Y_p$ then clearly $[X, Y]_p = inV_p$. ♠

Remark 4.3 Observe that a 1-dimensional submodule need not be cyclic.

We are now ready for the celebrated theorem:

Theorem 4.3 Frobenius *A d-field E is completely integrable iff $V(E)$ is a subalgebra of $\mathfrak{X}(M)$.*

Proof: . We have proved that $V(E)$ is a complete submodule. We have also proved that every line field is completely integrable and every 1-dimensional complete submodule is a subalgebra. Combining these together gives the proof of the above theorem for $d = 1$.

Suppose E is CI. Let $X, Y \in V(E)$. Then to each point $p \in M$ there is a nbd U_p and a submersion $f_p : U_p \to \mathbb{R}^{m-d}$ such that $E_q = Ker\ D(f_p)q$ for all $q \in U_p$. Therefore $D(f_p)_q(X) = 0 = D(f_p)_q(Y)$. This implies that $D(f_p)_q([X, Y]) = 0$. This just means that $[X, Y]_p \in E_p$. Therefore, $[X, Y] \in V(E)$.

The converse part is proved by induction on d. Thus we assume $d \geq 2$ and for every $(d-1)$-dimensional field E', such that $V(E')$ is a subalgebra, E' is CI. Let now E be a $d-$ field and $V(E)$ be a subalgebra of $\mathfrak{X}(M)$.

It is enough to find to each $p \in M$ a nbd U of p and a submersion $f : U \to \mathbb{R}^{m-d}$ such that $Ker\ (Df)_q = E_q$ for all $q \in U$. (See theorem 3.3).

The problem is thus completely local in nature and so, we may assume that $p = 0 \in \mathbb{R}^m$ and we have vector fields $\{X^1, \ldots, X^d\}$ which form a basis for E_q at all points q in a nbd W of 0. We may further assume that $X^1 = \partial_1$. By subtracting a suitable multiple of X^1 from $^j, j \geq 0$, we assume that $X^j = \sum_{i=2}^m \alpha_i^j \partial_i$. Put $S = W \cap 0 \times \mathbb{R}^{m-1}$.

Consider the vector fields $Y^j = X^j|S, j = 2, \ldots, d$. Since they are linearly independent, they define a $d-1$ field E' on S. The complete module $V(E')$ on S is generated by $\{Y^2, \ldots, Y^d\}$. Therefore, in order to see that $V(E')$ is a subalgebra, it is enough to know that $[Y^j, Y^k] \in V(E)$. This follows from the fact that $[\partial_j, \partial_k] = 0$ and the property (iv) in lemma 4.1.

By induction hypothesis, cutting down to a nbd of 0 in S we have assume that there is a submersion $f' : S \to \mathbb{R}^{m-d}$ such that $Ker\ D(f')_q = E'_q$ for all $q \in S$.

Let $\pi : U \to S$ be the projection $\pi(x_1, \ldots, x_m) = (x_2, \ldots, x_m)$. We claim $f = f' \circ \pi$ is the required submersion. That f is a submersion is obvious. We have to show that $Ker\ D(f)_{(x,q)} = E_{(x_1,q)}$ for all $q \in S, (x_1, q) \in U$.

This is where we need the hypothesis that $V(E)$ is a subalgebra. And we need the uniqueness of solutions of initial value problems of systems of first order linear differential equations.

By dimensional considerations, it is enough to show that $(Df)X^j = 0, j = 1, \ldots, d$. For $j = 1$ this is obvious. Put $f = (f_1, \ldots, f_{m-d})$. Then $(Df)X^j = (X^j f_1, \ldots, X^j f_{m-d})$. Therefore, we have to show that $X^j f_i = 0$ for $2 \leq j \leq d, 1 \leq i \leq m-d$.

Fix $1 \leq i \leq m-d$. Clearly $X^1 f_i = 0$ (because f_i are independent of x_1). Therefore,

$$X^1 X^j f_i = X^1 X^j f_i - X^j X^1 f_i = [X^1, X^j] f_i = \sum_{k=1}^{d} c_{j,k} X^k f_i = \sum_{k=2}^{d} c_{k,j} X^k f_i.$$

Then the above can be expressed as a solution $\phi_j = X^j f_i$ to the system of linear first order differential equation:

$$\frac{\partial}{\partial x_1} \phi_j = \sum_k c_{j,k} \phi_k.$$

Moreover these solutions satisfy the initial condition

$$\phi_j(0, q) = Y^j f_i = D(f_i)(Y^j) = D(\pi) \circ D(f')(Y^j) = 0.$$

By the uniqueness of the solution, it follows that $\phi_j = 0$ for all j on U as required. ♠

Lecture 3

5 Foliation as a differential ideal of $\Omega(M)$.

Let us now consider another major tool in Differential Geometry viz., the De'Rham complex $\Omega(M) = \oplus_{k=0}^m \Omega^k(M)$ of (smooth) differential forms on M. A 1-form $\omega \in \Omega^1(M)$ on M can be thought of as a linear map $\omega : \mathfrak{X}(M) \to \mathcal{C}(M)$

$$\omega(X)(p) = \omega_p(X_p).$$

Similarly, we can think of k-form ω on Ω as a $\mathcal{C}(M)$-valued alternating k-tensor $\mathfrak{X}(M)$ by the formula

$$\omega(X^1, \ldots, X^k)(p) = \omega_p(X^1_p, \ldots, X^k_p).$$

Proposition 5.1 For any vector $X_p \in T_p(M)$ and a smooth map $f : M \to \mathbb{R}$ we have $df(X_p) = X_p(f)$.

This gives rise a very interesting relation between exterior derivative and the Lie brackets:

Lemma 5.1 Let ω be a $1-$form on M and X and Y be two vector fields on M. Then

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{2}$$

Proof: By the linearity of the equation (2), it suffices to verify this for 1-forms of the form fdg. Now we have

$$\begin{aligned} LHS = (df \wedge dg)(X, Y) &= df(X)dg(Y) - df(Y)dg(X) \\ &= X(f)Y(g) - Y(f)X(g). \end{aligned}$$

and

$$\begin{aligned} RHS &= X(fdg(Y) - Y(fdg(X) - fdg([X, Y]) \\ &= X(fY(g)) - Y(fX(g)) - f[X, Y](g) \\ &= X(f)Y(g) + fXY(g) - Y(f)X(g) - fYX(g) - fXY(g) - fYX(g) \\ &= X(f)Y(g) - Y(f)X(g). \end{aligned}$$

Exercise 5.1 Show that for any p-form ω and any $(p+1)$-tuple $(X^0, X^1, \ldots, X^p)$ of vector fields, we have

$$\begin{aligned} & d\omega(X^1, \ldots, X^p) \\ = & \sum_{0 \le i \le p} (-1)^i X^i(\omega(X^0, \ldots, \widehat{X^i}, \ldots, X^p) \\ + & \textstyle\sum_{0 \le i < j \le p} (-1)^{i+j}(\omega([X^i, X^j], X^0, \ldots, \widehat{X^i}, \ldots, \widehat{X^j}, \ldots, X^p). \end{aligned} \tag{3}$$

Definition 5.1 A graded submodule A of $\Omega(M)$ is called locally trivial of rank q if there is an open covering $\{U_j\}$ of M such that $A|U_j$ is generated by some linearly independent $1-$forms $\{\omega_1, \ldots, \omega_k\}$ on U_j.

Definition 5.2 Given a submodule V of $\mathfrak{X}(M)$ we can consider the submodule $A = ann\ (V)$ of $\Omega(M)$ defined as follows:

$$A^k = \{\omega \in \Omega^k(M)\ :\ \omega(X^1, \ldots, X^k) = 0,\ \forall\ X^j \in V\}.$$

Proposition 5.2 If V is a complete submodule of $\mathfrak{X}(M)$ of dimension d then $ann\ (V)$ is locally trivial of rank $m - d$.

Proof: Recall that V is complete submodule then there is an open covering $\{U_j\}$ and vector fields $\{X^1, \ldots, X^d\}$ which form a basis for V_x at each $x \in U_j$. We can complete this to a basis of T_xM over U_j and take the dual basis of $1-$forms $\{\omega_1, \ldots, \omega_m\}$ for $\Omega^1(U_j)$. Now it is clear that $A^1|_{U_j}$ is spanned by $\{\omega_{d+1}, \ldots, \omega_m\}$. A little bit of usual multi-linear algebra tells you that $A^k|_{U_j}$ is also generated by them. ♠

Definition 5.3 A graded submodule of $\mathcal{A}$ of $\Omega(M)$ is called a (differential) ideal if it is closed under the external derivation, i.e., $d(\mathcal{A}) \subset \mathcal{A}$.

The following theorem, which is an immediate consequence of (2) and (3), gives you one more definition of a foliation.

Theorem 5.1 *A complete submodule V of $\mathfrak{X}(M)$ is a Lie subalgebra iff ann (V) is a locally trivial (differential) ideal of $\Omega(M)$.*

Lecture 5

6 Some Constructions

We shall discuss a few standard constructions now.

Product of foliations Given two foliated manifolds $(M_j, \mathcal{F}_j), j = 1, 2$ there is an obvious way to foliate the product manifold $M_1 \times M_2$ which we shall denote by $\mathcal{F}_1 \times \mathcal{F}_2$: If $\mathcal{F}_i = \{(U_i, \phi_i)\}$ and $\mathcal{F}_2 = \{(V_j, \psi_j)\}$ then take $\mathcal{F}_1 \times \mathcal{F}_2 = \{(U_i \times V_j, T \circ (\phi_i \times \psi_j)\}$ where $T : \mathbb{R}^{d_1} \times \mathbb{R}^{m_1-d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{m_2-d_2} \to \mathbb{R}^{d_1+d_2} \times \mathbb{R}^{m_1+m_2-d_1-d_2}$ is given by

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \mapsto (-1)^{(m_1-d_1)d_2}((\mathbf{x}, \mathbf{z}), \mathbf{y}, \mathbf{w}))$$

Pull-Back Foliation A smooth map $f : N \to (M, \mathcal{F})$ is said to be transversal to $\mathcal{F}$ if f is transversal to every leaf of $\mathcal{F}$ i.e., for every $q \in N$ we have

$$df_q(T_qN) + T_{f(q)}L(q) = T_{f(q)}M$$

where L_q denotes the leaf of $\mathcal{F}$ through $f(q)$. We can then get a foliation $f^*\mathcal{F}$ on N of codimension equal to codimension of $\mathcal{F}$ as follows: Let now the foliation $\mathcal{F}$ be represented by a family $\{(U_i, \phi_i)\}$ of compatible submersions $\phi : U_i \to \mathbb{R}^{m-d}$. Take $V_i = f^{-1}(U_i)$ and $\psi = \phi_i \circ f$. Then $\{(V_i, \psi_i)\}$ will be a compatible family of submersions on N.

Quotient Foliation Let G be a group acting properly discontinuously on a manifold M so that M/G is a Hausdorff manifold. (In other words, the quotient map $q : M \to M/G$ is a covering projection.) Suppose $\mathcal{F}$ is a foliation and the group action maps leaves of $\mathcal{F}$ to leaves. Such an action is said to preserve the foliation structure of M. Start with an open cover $\{U_i\}$ of M/G which is evenly covered by the covering projection, i.e., for each i, $q^{-1}(U_i)$ is a disjoint union of open sets $\{V_{i,g}\}$ of M such that $q : V_{i,g} \to U_i$ is a homeomorphism. Now refine this cover if necessary and assume that $\{(V_{i,g}, \phi_{i,g})\}$ is an atlas of compatible submersions defining $\mathcal{F}$. That the action of G preserves leaves of $\mathcal{F}$ just means that we have commutative diagrams:

$$\begin{tikzcd} V_{i,g}h \arrow[r] \arrow[d, "\phi_{i,g}"'] & V_{i,gh} \arrow[d, "\phi_{i,gh}"] \\ \mathbb{R}^d \arrow[r, "f_h"'] & \mathbb{R}^d \end{tikzcd}$$

where the vertical arrows indicate the submersions and the top horizontal arrow is the action of $h \in G$ and the bottom horizontal on is some diffeomorphism. Now, by taking $\{(U_i, \psi_{i,g})\}$ where $\psi_{i,g} = \phi_{i,g} \circ q^{-1} : U_i \to \mathbb{R}^d$ we get a compatible atlas of submersions on M/G.

Suspension of a Diffeomorphism This is another example of quotient foliation. For any manifold consider the 1-dim. foliation on $\mathbb{R} \times M$ by the leaves $\mathbb{R} \times \{x\}$. Now let $f : M \to M$ be a diffeomorphism. Define an action of $\mathbb{Z}$ on M by the rule:

$$(k, t, x) \mapsto (t + x, f^k(x)).$$

Then $\mathbb{Z}$ acts properly discontinuously and maps leaves onto leaves. This induces a foliation denoted by S_f on the quotient manifold $\mathbb{R} \times_{\mathbb{Z}} M$. This foliation is called the suspension of the diffeomorphism f.

Lie Group Action More generally, suppose a Lie group G acts on a manifold M smoothly. Consider orbit decomposition $M = \coprod Gx$. The isotropy subgroup $G_x = \{g \in G : gx = x\}$ is a closed subgroup of G and we have G/G^x diffeomorphic to G_x and each orbit is a manifold. We say the action of G on M is foliated if all the orbits Gx are of the same dimension. In this case, it follows that (???) the subspace of all vector fields tangent to the orbits forms a Lie subalgebra of $\mathfrak{X}(M)$ and hence we have a foliation structure on M with its leaves being connected components of the orbits of the G-action.

7 Orientability

Definition 7.1 Let (E, p, M) be any rank d vector bundle over M. Recall that we say E is orientable, if the transition functions are orientable, i.e., if $\{U_i\}$ is an open cover for M and $\phi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^d$ are local trivializations, then $\phi_i \circ \phi_j^{-1} : \{x\} \to real^d \to \{x\} \times \mathbb{R}^d$ are orientation preserving for all $x \in U_i \cap U_j$ for all i, j.

Definition 7.2 Let $(M, \mathcal{F})$ be a d-foliated manifold and $E = E(\mathcal{F})$ be the associated subbundle of TM. We say $\mathcal{F}$ is orientable of E is orientable. Similarly, we say $\mathcal{F}$ is transverse orientable, if the quotient bundle TM/E is orientable.

Remark 7.1 (i) If M is orientable, it follows that $\mathcal{F}$ is orientable iff it is transverse orientable.
(ii) On a simply connected manifold, every vector bundle is orientable and hence every foliation is both orientable and transverse orientable.

(iii) It is possible to have a foliation on an orientable manifold, which is neither orientable nor transverse orientable. Begin with $M = \mathbb{S}^1 \times \mathbb{S}^1$. Any rank 1 sub-bundle of TM is integrable and hence gives 1-foliation. We have only to choose it so that it is not orientable. Therefore at each point $(e^{\imath\theta_1}, e^{\imath\theta_2}) \in \mathbb{S}^1 \times \mathbb{S}^1$ consider the line spanned by $(\imath e^{\imath\theta_1} \sin(\theta_2/2), \imath e^{\imath\theta_2} \cos(\theta_2/2))$. This line field defines a non orientable sub-bundle of $T(\mathbb{S}^1 \times \mathbb{S}^1)$. (iv) For any vector bundle (E, π, M) over a manifold M, there is the orientable double cover constructed as follows: Let (U_i, ϕ) be an atlas for E, i.e., $\{U_i\}$ is an open covering of M and $\phi_i : \pi^{-1}(U_i) \to U_i\mathbb{R}^d$ diffeomorphisms etc. For each i, let $V_{i,\pm}$ denote two copies of $U_i \times \mathbb{R}^d$. On $\coprod_i (V_{i,+} \cup V_{i,-})$ introduce an equivalence relation as follows. For each i, j, and $x \in U_i \cap U_j$ suppose $\alpha_{i,j} = \phi_i \circ \phi_j^{-1} : \{x\} \times \mathbb{R}^d \to \{x\} \times \mathbb{R}^d$ is orientation preserving: then identify $(x, v) \in V_{i,\pm}$ with $\alpha_{i,j}(x, v) \in V_{j,\pm}$. Otherwise identify $(x, v) \in V_{i,\pm}$ with $\alpha_{ij}(x, v) \in V_{j,\mp}$. Let $\tilde{E}$ denote the quotient space. Verify that the first projection factors to give a rank d-vector bundle projection $\tilde{\pi} : \tilde{E} \to \tilde{M}$, where $\tilde{M} \to M$ itself is a double covering. (This double covering may be a trivial one.)

In particular, given a foliation $(M, \mathcal{F})$, the orientation double cover $\tilde{E}(\mathcal{F})$ of the d-dim. bundle $(\mathcal{F})$ is a foliation on $\tilde{M}$. This is called the orientable double cover of $(M, \mathcal{F})$ and is denoted by $(\tilde{M}, \tilde{F})$.

For instance, if $(M, \mathcal{F})$ is orientable, then this orientation double cover will be just the disjoint union of two copies of $(M, \mathcal{F})$.

Likewise one can talk about the transverse orientation double cover of $(M, \mathcal{F})$ also, viz., the orientation double cover of the quotient bundle $TM/E(\mathcal{F})$. However, one has to check that the bundle so obtained actually corresponds to a foliation viz., it is the normal bundle of a transverse orientable foliation $(M', \mathcal{F}')$. So, suppose (E', π', M') is the orientation double cover of the normal bundle $TM/E(\mathcal{F})$. One first shows that this is the quotient bundle of TM', $\eta : TM' \to E'$ be the quotient map, and the take $ker\, \eta$. It follows that $\ker \eta$ is completely integrable and hence defines a foliation on M' which is obviously transverse orientable.

8 Integral Manifolds: Leaves

Definition 8.1 Let $\mathcal{F}$ be a d-foliation of a manifold X. By an *integral manifold* of $\mathcal{F}$ we mean a pair (Y, f), where Y is a smooth manifold and $f : Y \to X$ is an injective immersion such that $Df_y(T_yY) \subset E_{f(y)}(\mathcal{F})$. If $dim\ Y = 1$ then (Y, f) called an integral curve for the foliation.

On the collection of all connected integral manifolds of $\mathcal{F}$, there is an obvious partial ordering viz., $(Y_1, f_1) \leq (Y_2, f_2)$ iff $Y_1 \subset Y_2$ and $f_1 = f_2|_{Y_1}$. A connected integral manifold of dimension d, maximal w.r.t. this ordering is called a *leaf* of $\mathcal{F}$.

Remark 8.1

(i) Often we allow ourselves to confuse image $f(Y)$ of f with the leaf (Y, f). In general the image of a leaf need not be a submanifold of the foliated manifold M.

(ii) If $\mathcal{F}$ is the foliation of a fibration then of course the leaves are submanifolds and every

submanifold of the fibre is an integral manifold. For any foliation $\mathcal{F}$, if $(U, f) \in \mathcal{F}$ then any connected component of $f^{-1}(f(x)) \subset U$ passing through x is an integral manifold which is also a submanifold called a *a slice* of $\mathcal{F}$ at x.
(iii) It is not difficult to see that every leaf is a union of slices. Also, note that the dimension of a slice is always equal to d. (where $\mathcal{F}$ is a $d-$foliation) Therefore, it follows that through every point $p \in M$, there exists a leaf of $\mathcal{F}$ of dimension d. Indeed, there a unique leaf and its dimension has to be necessarily d. But that need to be proved carefully.

Lemma 8.1 Let $(U, f) \in \mathcal{F}$ and S be a slice in U.
(a) Let (Y, g) be a connected integral manifold of $\mathcal{F}$ such that $g(Y) \subset U$ and $g(Y) \cap S \neq \emptyset$. Then $g(Y) \subset S$.
(b) For any $(V, h) \in \mathcal{F}$, at most countably many slices of V will intersect S.

Proof: Since $D(f \circ g) = D(f) \circ D(g) = 0$, it follows that $f \circ g$ is locally constant. Since Y is connected, this implies $f \circ g$ is a constant. This implies (a).

Let S' be another slice of $\mathcal{F}$. Let W be a connected component of $S \cap S'$. Then $\iota : W \to S$ is an immersion and since both have same dimension, it follows that W is open in S. We conclude that $S \cap S'$ is open in both S and S'.

Therefore, $\{S \cap S'\}_{S'}$ forms a disjoint family Ω of open sets in S as S' varies. Now let S' vary over all slices of (V, h). Clearly, two distinct slices of V do not intersect. Therefore, by II-countability of S, Ω has only countably many members. Therefore, only countably many of S' could intersect S. ♠

Lemma 8.2 Every slice is contained in a $d-$dimensional leaf.

Proof: Let L be a slice and $\mathcal{L}$ be the family of all d-dimensional connected integral manifolds (Y', f') which contain L. If $\{(Y_k, f_k)\}$ is a chain in $\mathcal{L}$, then we can take $Y = \cup_k Y_k$ and $f : Y \to X$ to be such that $f|Y_k = f_k$. Being a countable union of an increasing family of smooth manifolds, Y is a smooth manifold, in which each Y_k is open. It follows easily that f is an injective immersion also. Therefore, every chain in $\mathcal{L}$ has an upper bound. Apply Zorn's lemma to conclude that $\mathcal{L}$ has a maximal element, (Y, f), which is easily seen to be a leaf. ♠

Theorem 8.1 *Let $\mathcal{F}$ be a $d-$foliation on X. Then X is the disjoint union of $d-$dimensional leaves of $\mathcal{F}$.*

Proof: The only thing that we need to see is that any two distinct d-dimensional leaves are disjoint which follows easily from lemma 8.1. ♠

Remark 8.2

1. The existence of leaves is used in the theory of Lie groups, in the proof of the fact that to each subalgebra of the Lie algebra of a Lie group, there corresponds a 'virtual subgroup' of G.

2. The leaves of a foliated manifold $(M, \mathcal{F})$ define a partition on M. Let $\pi : M \to M/\sim$ denote the quotient map and the space. A central problem in Foliation Theory is to understand the topology of this quotient space.

3. Suppose S is a subset of M which is the union of leaves of $\mathcal{F}$. Then its closure in M, $\bar{S}$ is also a union of leaves. To see this, we take any leaf L of $\mathcal{F}$ and show that $L \cap \bar{S}$ is open in L. Since L is connected this will imply that either $L \cap \bar{S} = \emptyset$ or $= L$ and thereby prove our claim. So, choose $p \in L \cap \bar{S}$ and choose a product nbd $U = \mathbb{R}^d \times real^{m-d}$ around p from $\mathcal{F}$. It follows that $S = \mathbb{R}^d \times A$ for some $A \subset \mathbb{R}^{m-d}$. Now $p \in \bar{S}$ implies that there is a sequence $(x_n, y_n) \in S$ converging to $p = (x, y)$. This implies that $y_n \to y$. Therefore for each $z \in \mathbb{R}^d$ the sequence $(z, y_n) \in S$ converges to the point (z, y). This means that the slice $real^d \times y$ is in the closure of S. But this slice is contained in L since L is a leaf through p. Therefore $L \cap barS$ is open as claimed.

4. If S is a union of leaves then $int(S)$ is also so. This follows from the the above observation: $M \setminus S$ is a union of leaves and hence $\overline{M \setminus S}$ is also so. Therefore $int(S) = M \setminus \overline{M \setminus S}$ is also so.

5. If U is any open set in M then the union of all leaves intersecting U is an open set. For, if S is this set, then by the above observation $int(S)$ is the union of leaves. Since $U \subset S$ is open $U \subset int(S)$. Therefore $S \subset int(S)$.

6. The projection map $\pi : M \to M/\sim$ is an open mapping. This follows immediately from the previous observation.

7. The Hausdorffness is the most misbehaved property under quotients. Here is a simple example to illustrate that the leaf-topology can be easily non-Hausdorff.

Example 8.1 Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and $\pi : M \to \mathbb{R}$ be the first projection. Then π is a submersion and the leaves of the corresponding foliation are vertical line $x = c, c \neq 0$ and the positive and the negative y-axis. The quotient topology on $M/\sim$ is nothing but the real line with the double origin.

Remark 8.3 Of course, under stringent conditions, we can make sure that the quotient topology is Hausdorff. For instance, first prove that if a leaf L is compact, then show that the nbds of L which are unions of leaves form a fundamental system of nbds for L. Using this we can show that if all leaves are compact then $M/\sim$ is Hausdorff.

Our next task will be to see under what suitable condition the leaf space $M/\sim$ is a manifold. This brings us to another interesting concept in Foliation Theory which we shall take up in the next section.

Remark 8.4 Given any codim. q immersion $f : X \to Y$ of smooth manifolds, we can define the normal bundle to f as a vector bundle on X of rank q as follows: Assume Y is embedded $\mathbb{R}^N$ for large N. Take

$$\nu(f) = \{(x, \mathbf{v}) \in X \times TY \ : \ \mathbf{v} \perp df_x(T_xX)\}.$$

Then it is not difficult to verify that the projection to the first factor $\nu(f) \to X$ defines a vector bundle of rank q. The normal bundle apparently depends upon on the choice of the 'metric' on Y, (in our case on the embedding of $Y \subset \mathbb{R}^N$. But the isomorphism class of the bundle depends only on the immersion f.

Remark 8.5 Let $(M, \mathcal{F})$ be a d-foliated manifold and L be a leaf of $\mathcal{F}$. Let E be the integrable sub-bundle of TM corresponding to the foliation. Consider the pull-back of the quotient bundle TM/E on L is called the normal bundle to the leaf L. Verify that the this bundle is the same as the normal bundle to the immersion $L \to M$ as defined in the above remark.

Lecture 6

9 Leaf Holonomy

Definition 9.1 Given two manifolds M, N and points $x \in M, y \in N$ we consider smooth maps $f : U \to V$ where U, V are nbds of x, y respectively, such that $f(x) = y$. Two such maps $f_1 : U_1 \to V_1, f_2 : U_2 \to V_2$ are said to be equivalent if there exists a nbd W of x such that $W \subset U_1 \cap U_2$ $f_1|W = f_2|W$. It is easily verified that this actually an equivalence relation. The equivalence classes are classed germs of smooth functions at x on M. Every germ is of course represented by some function f and then we write $germ_x f : (M, x) \to (N, y)$.

Remark 9.1
(i) The composition of maps factors down to define composition of germs as well. Thus $germ_x f : (M, x) \to (N, y)$ and $germ_y g : (N, y) \to (P, z)$ can be composed to yield $germ_x(g \circ f|f^{-1} dom\ g)$.
(ii) There is the $germ_x(Id)$, the germ of the identity map which acts as a two sided identity for this composition. We can speak about invertible germs, viz. germs of local diffeomorphism at x. Each such germ has in inverse $germ_y f^{-1}$. Thus the set of all germs of diffeomorphisms at $(M, x) \to (M, x)$ forms a group denoted by $\mathrm{Diff}_x(M)$.
(iii) There is an obvious group homomorphism $\mathrm{Diff}_x(M) \to Aut(T_xM)$ from $\mathrm{Diff}_x(M)$ to the group of linear automorphisms of the tangent space at x to M viz., $germ_x f \mapsto D_x f$. This is easily seen to be surjective. But it is far from being injective. For example, consider the case when $M = \mathbb{R}$ and $x = 0$. Consider $f(x) = x + g(x)$ where g is any smooth map with $g'(0) = 0$. Then by inverse function theorem, f is a local diffeomorphism at 0. By choosing germs of g at 0 to be different we get different elements of $\mathrm{Diff}_0(\mathbb{R})$ whereas all of them are mapped to Id under derivation.

Definition 9.2 A $(m-d)$-dimensional submanifold S passing through a point $x \in L$ is said to be transversal section to L if S intersects L transversally at x. It follows that in a small nbd of x, S will be transversal to all the leaves near L.

Example 9.1 If $(U, \phi) \in \mathcal{F}$ is a foliation chart, it follow that $\phi^{-1}(x \times \mathbb{R}^{m-d})$ is a transversal section to all the leaves emerging from U.

Definition 9.3 We are going to take a number slow steps in the following definition:
(0) Fix a leaf L of a d-foliation $(M, \mathcal{F})$. Given $(U\phi) \in \mathcal{F}$ and two points p, q belonging to the same component of $L \cap U$ we define a diffeomorphism $HOL^{q,p} : S_p \to S_q$ from the transversal section S_p at p to S_q at q as follows: For $w \in S_p$ consider the slice L_w (plaque) through w in U at take $HOL^{q,p}(w) = L_w \cap S_q$. Under the diffeomorphism $\phi : U \to \mathbb{R}^d \times \mathbb{R}^{m-d}$ this map corresponds to the map $x \times \mathbb{R}^{m-d} \to y \times \mathbb{R}^{m-d}$ given by $(x, t) \mapsto (y, t)$ and therefore is a diffeomorphism. The germ of this diffeomorphism will be denoted by $\mathrm{hol}_U^{q,p}$.
(i) Observe that the germ is independent of the chart (U, ϕ) that you choose provided both p, q belong to the same component of $L \cap U$ since it is defined purely set theoretically.

Next choose a path ω in L starting from p and ending say at q. Cover it with a finite 'chain' of foliation charts $(U_i, \phi_i), 1 \leq i \leq r$ such that $U_i \cap U_{i+1} \neq \emptyset$. Choose $p = p_1, \ldots, p_r = q$ such that $p_{i+1} \in U_i \cap U_{i+1} \cap \omega$. It follows that we can compose the diffeomorphism germs in that order and obtain a germ

$$\mathrm{hol}_\omega^{q,p} = \mathrm{hol}_r^{p_r, p_{r-1}} \circ \cdots \circ \mathrm{hol}_1^{p_2, p_1}.$$

(ii) The composite germ is independent of the choice of the chain that covers it. For we can work the common refinement of the two partitions of the path ω and use (i) repeatedly.
(iii) If ω and τ are two paths in L, one form p to q and the other from q to r, we have,

$$\mathrm{hol}_\tau^{r,q} \circ \mathrm{hol}_\omega^{q,p} = \mathrm{hol}_{\omega * \tau}^{r,p}. \tag{4}$$

(iv) If τ is another path lying in L and passing through the same points $p = p_1, p_r = q$ and if the segments of τ from p_i to p_{i+1} are covered by the same chart U_i for each i, then $\mathrm{hol}^{p_{i+1}, p_i}(\omega) = \mathrm{hol}^{p_{i+1}, p_i}(\tau)$ since the definition on either side has nothing to do with the paths involved. (v) Standard arguments with homotopy now yield:

Proposition 9.1 If ω and τ are two paths in L path homotopic to each other then $\mathrm{hol}_\omega = \mathrm{hol}_\tau$.

Corollary 9.1 : There is a well defined homomorphism $\mathrm{hol} : \pi_1(L, p) \to \mathrm{Diff}_p(S_p)$.

(vi) If S_p and S'_p are two transversal sections through the same point, we may assume that both are contained in a single chart U. Let $\mathrm{hol} : \pi_1(L, p) \to \mathrm{Diff}_p(S_p), \mathrm{hol}' : \pi_1(L, p) \to \mathrm{Diff}_p(S'_p)$ be the corresponding holonomy homomorphisms. Let $f_p = f(S_p, S'_p) : S_p \to S'_p$

be the be the germ of the diffeomorphism from S_p to S'_p as given by the chart U. Then clearly

$$\mathrm{hol}(\alpha) = f_p^{-1}\mathrm{hol}'(\alpha) f_p. \tag{5}$$

(vii) Fixing a diffeomorphism $g : S_p \to \mathbb{R}^{m-d}$ such that $g(p) = 0$, it follows that there is a well defined homomorphism

$$\mathrm{hol} : \pi_1(L, x) \to \mathrm{Diff}_0(\mathbb{R}^{m-d}) \tag{6}$$

unique up to inner conjugation. By general consideration about base points, the statement holds, when we change the base point of the fundamental group of L. Therefore we conclude that up to inner conjugation, there is a well defined homomorphism

$$\mathrm{hol} : \pi_1(L) \to \mathrm{Diff}_0(\mathbb{R}^{m-d}) \tag{7}$$

This homomorphism is called the holonomy of L.
(viii) By composing this with the derivation

$$d : \mathrm{Diff}_0(\mathbb{R}^{m-d}) \to GL(m-d, \mathbb{R}),$$

we get another homomorphism

$$d\mathrm{hol} : \pi_1(L) \to GL(m-d, \mathbb{R}). \tag{8}$$

Example 9.2
(i) If a loop ω is contained in a single chart, then it is easily seen that $\mathrm{hol}(\omega) = 1$.
(ii) Consider a simple foliation associated to a submersion $f : M \to N$. Let ω be a loop at p in a leaf $L \subset f^{-1}(q)$ for some $q \in N$. Now we can cover the loop by finitely many open sets U_i such that $f(U_i) = V$ where is a (single) open nbd of q in N and choose diffeomorphisms $\phi_i : U_i \to \mathbb{R}^n$ and a (single) diffeomorphism $\alpha : V \to \mathbb{R}^{m-d}$ such that $\pi_2 \circ \phi_i(z) = \alpha \circ f$ as in the example 2.1 It follows that corresponding h_{ij} are identity maps, which in turn implies that $hol_\omega = Id$. Thus the holonomy of any leaf in a simple foliation is trivial.
(iii) For the foliation of the Möbius band as given in (4) of Examples 1 the holonomy is trivial for all circles except the central one, where it is the quotient homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.
(iv) Consider the Reeb foliation of $\mathbb{S}^3$ as in (5) of Examples 1. First of all for the solid torus, all leaves except the boundary torus are simply connected (actually diffeomorphic to $\mathbb{R}^2$). On the boundary, along the latitudes the the holonomy is trivial. Along the longitudes, the diffeomorphism $f : [0, \infty) \to [0, \infty)$ is of the form $f(t) < t$ for every $t > 0$. Thus we have $\mathrm{hol} : \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \to \mathrm{Diff}_0[0, \infty)$ takes one of the generator to the trivial element and the other one to an element of infinite order.

We can now determine the holonomy of the Reeb foliation of the $\mathbb{S}^3$ as follows: Let us denote the two generators of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ by α, β. Then $\mathrm{hol} : \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \to$

$\mathrm{Diff}_0(\mathbb{R})$ has the property:

$$\mathrm{hol}(\alpha)(t)\begin{cases} < t, & t > 0 \\ = t & t \leq 0. \end{cases} \qquad \mathrm{hol}(\beta)(t)\begin{cases} = t, & t \geq 0 \\ < t & t < 0. \end{cases} \tag{9}$$

Thus, hol is an isomorphism onto its image. However not that $d$hol is trivial.

10 Some Applications

Proposition 10.1 Let $(M, \mathcal{F})$ be a foliated manifold. Suppose for each point $p \in M$ there exists a transverse section T such that $L \cap T$ is at most a singleton for each leaf L. Then $M/\sim$ is a smooth manifold (perhaps non Hausdorff) in such a way that the canonical projection $\pi : M \to M/\sim$ is a submersion.

Proof: Fix a point $p \in M$ and choose a transverse T such that the above condition is satisfied. Observe that T is may be assumed to be diffeomorphic to an open subset of $\mathbb{R}^{m-d}$. Taking a smaller T if necessary we may assume that T is contained in a product nbd U of p. It then follows that for every open subset V of T the union of all slices inside U which intersect V is an open set W in M and hence the union of all leaves which intersection V is an open set W' in M. Therefore $\pi(V) = \pi(W')$ is open in $M/\sim$ and is a nbd of $\pi(p)$. Thus $\pi|_T : T \to \pi(T)$ is an open map. Already this is a one mapping. Hence defines a homeomorphism onto a nbd of $\pi(p)$. If $\pi(p) = \pi(q)$ and if ω is any path from p to q contained in the leaf $\pi(p)$ then the holonomy map $hol_\omega^{q,p}$ is such that $\pi \circ hol_\omega^{q,p} = \pi$. Since the holonomy maps are diffeomorphisms, this clearly defines an atlas for a smooth structure for $M/\sim$ and moreover for this smooth structure π is smooth and defines a local diffeomorphism on suitably chosen transverses. ♠

Theorem 10.1 *Let $(M, \mathcal{F})$ be a d-foliated manifold, L be a compact leaf with trivial holonomy. Then there exists a nbd V of L in M and a leaf preserving diffeomorphism $\lambda : L \times S \to V$ where S is a transversal section to L.*

Proof: Not done in the lectures.

Theorem 10.2 Structure Theorem for Simple Foliations
(i) In a simple foliation every leaf has trivial holonomy.
(ii) In a strictly simple foliation, the leaf space is Hausdorff.
(iii) Conversely, if
(a) each leaf has trivial holonomy,
(b) the leaf space is Hausdorff and
(c) each leaf has finitely generated fundamental group,
then $\mathcal{F}$ is a simple foliation. Indeed the projection $M \to M/\sim$ to the leaf space itself is a submersion.

Remark 10.1 In particular, if each leaf is compact, both conditions (b) and (c) hold. Also, the proof of the converse part is easier in this case.

Proof: We have seen (i).
In (ii), let $(M, \mathcal{F})$ be the strictly simple foliation corresponding to a submersion $f : M \to N$. Then the map f factors through a homeomorphism $\bar{f} : M/\sim \to N$.
(iii) Here we want to show that the leaf space $M \sim$ has a structure of a smooth Hausdorff manifold such that the quotient map $M \to M/\sim$ itself is a submersion. From Proposition 10.1, it suffices to show that at each point $p \in M$ there is a transverse which meets at each leaf at most once. Fix a set finite set of loops $\omega_1, \ldots, \omega_k$ in L at p whose homotopy class generate the fundamental group $\pi_1(L, p)$. Let T be any transverse at p. Since the holonomy is trivial, it follows that there are transverses $T_j \subset T$ such that $holomorphic_{omega_j} : T_j \to T_j$ is actually represented by the identity map. Take $T_0 = \cap T_j$. It follows that for any loop ω at p $hol_\omega : T_0 \to T_0$ is represented by the identity map. Therefore, it follows that T_0 intersects every leaf through some point of T_0 only once. This completes the proof. ♠