Topology of Lie Groups

Lecture 1

In this seminar talks, to begin with, we plan to present some of the classical results on the topology of Lie groups and (homogeneous spaces).

The basic assumption is that the participants are familiar with the 'algebra' of Lie group theory. However, in order to take care of those who are not, most of the time, we shall concentrate on Matrix Groups.

As a sample of the several results to come, let me begin with:

Theorem 1.1 *$GL(n;\mathbb{C})$ is connnected.*

Proof: We shall give a completely elementary proof of that $GL(n,\mathbb{C})$ path connected.

Consider the simplest case when $n = 1$. In order to show that $GL(1,\mathbb{C}) = \mathbb{C}^*$ is connected, the best thing is to say that given any two points in $\mathbb{C}^*$ there is a circle in $\mathbb{C}^*$ passing through these two points (rather than using straight line segments). And this is going to help us in the general case as well.

We now appeal to Gauss Elimination Method, which reduces a given matrix A to its row-reduced echelon form $R(A)$. We claim that given $A \in GL(n,\mathbb{C})$, there exists a path γ in $GL(n,\mathbb{C})$ from A to $R(A)$. Applying this to $R(A)^t$, given a path γ' from $R(A)^t$ to its row-reduced echelon form $R(R(A)^t) = Id$. Then $\gamma \circ \gamma'$ is a path in $GL(n,\mathbb{C})$ joining A to Id.

GEM consists of a finite sequence of steps of row-operations of one of the three types.
(1) Interchaning k^{th} row and l^{th} row: For this step we can choose the path $\theta \mapsto \omega(\theta)A$ where ω is defined by as follows:

$$\omega(\theta)_{ij} = \begin{cases} \cos\theta & (i,j) = (k,k) \\ \sin\theta & (i,j) = (k,l) \\ -\sin\theta & (i,j) = (l,k) \\ \cos\theta & (i,j = (l,l) \\ 0 & \text{otherwise} \end{cases}$$

(2) Dividing a row by $\lambda \neq 0$. Here we choose a circular path $\tau(s)$ in $\mathbb{C}^*$ joining 1 with λ and let $\gamma(s)$ be the diagonal matrix with all the diagonal entries equal to 1 except the $(k,k)^{th}$ entry which is equal to $1/\tau(s)$. We then take the path $s \mapsto \gamma(s)A$.
(3) Adding a multiple of k^{th} row to l^{th} row. We shall leave this step as an exercise to the reader.
♠

Let us fix up some notation. $\mathbb{K}$ will denote one of the (skew-)fields $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and $M_(n(\mathbb{K})$ will denote the space of all $n \times n$ matrices with entries in $\mathbb{K}$. There are the forgetful functors, $\mathcal{R} : \mathbb{C} \to \mathbb{R}^2$ and $\mathcal{Q} : \mathbb{H} \to \mathbb{C}^2$, given by

$$\mathcal{R} : x + \imath y \rightsquigarrow (x, y);$$

and

$$\mathcal{Q} : a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \rightsquigarrow (a_0 + a_1\imath, (a_2 + a_3\imath).$$

We treat $\mathbb{K}^n$ as a right $\mathbb{K}$-module and $M_n(\mathbb{K})$ as the space of right endomorphisms $\mathbb{K}^n \to \mathbb{K}^n$. The above forgetful functors then yield identifications

$$\mathbb{C}^n \rightsquigarrow \mathbb{R}^{2n}; \quad \mathbb{H}^n \rightsquigarrow \mathbb{C}^{2n}$$

given by

$$(x_1 + \imath y_1, \ldots, x_n + \imath y_n) \rightsquigarrow (x_1, y_1, \ldots, x_n, y_n)$$

and

$$(z_1 + w_1\mathbf{j}, \ldots, z_n + w_n\mathbf{j}) \rightsquigarrow (z_1, w_1, \ldots, z_n, w_n).$$

These, in turn, yield imbeding

$$\mathcal{R} : M_n(\mathbb{C}) \to M_{2n}\mathbb{R}); \quad \mathcal{Q} : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$$

For example, when $n = 1$ we have

$$x + \imath y \rightsquigarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix}; \quad z + w\mathbf{j} \rightsquigarrow \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

Verify that the following Cauchy-Riemann equations:
(1) $\mathcal{R}(M_n(\mathbb{C})) = \{A \in M_{2n}(\mathbb{R}) \ : \ AJ_{2n} = J_{2n}A\}$ and
(2) $\mathcal{Q}(M_n(\mathbb{H})) = \{A \in M_{2n}(\mathbb{C}) \ : \ AJ_{2n} = J_{2n}\bar{A}\}$.

Let $GL(\mathbb{K}^n)$ denote the space of all auotomorphisms of $\mathbb{K}^n$ as a right $\mathbb{K}$-module. Then $GL(\mathbb{K}^n)$ is a Lie group of dimension $(cn)^2$ where $c = 1, 2$ or 4 according as $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Theorem 1.2 *$\mathcal{C}_n : GL(n, \mathbb{C}) \longrightarrow GL(2n, \mathbb{R})$ and $\mathcal{Q}_n : GL(n, \mathbb{H}) \longrightarrow GL(2n, \mathbb{C})$ define monomorphisms of groups.*

Definition 1.1 We define the *determinant function on $M(n, \mathbb{H})$* by the formula

$$\det A = \det (\mathcal{Q}_n(A)). \tag{1}$$

Exercise 1.1

1. Show that $A \in GL(n, \mathbb{H})$ iff $\det A \neq 0$.

2. Show that $\det A \in \mathbb{R}$ for all $A \in M(n, \mathbb{H})$.

3. Show that $GL(n, \mathbb{K})$ is an open subspace of $M(n, \mathbb{K})$.

4. Show that $GL(n, \mathbb{K})$ is path connected for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ and has two connected components for $\mathbb{K} = \mathbb{R}$. (Hint: Use appropriate modifications in the proof of 1.1).

5. Prove that $\det A > 0$ for all $A \in GL(n, \mathbb{H})$.

The Orthogonal Groups

We fix the standard inner product $\langle\, ,\, \rangle_{\mathbb{K}}$ on $\mathbb{K}^n$ as follows:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{K}} = \sum_r a_r \bar{b}_r$$

where $\mathbf{a} := (a_1, \ldots, a_n) \in \mathbb{K}^n$ etc.. We define the norm on $\mathbb{K}^n$ by

$$\|\mathbf{a}\|_{\mathbb{K}} = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_{\mathbb{K}}}.$$

Note that for $\mathbf{a} \in \mathbb{C}^n$, $\|\mathbf{a}\|_{\mathbb{C}} = \|\mathcal{C}_n(\mathbf{a})\|_{\mathbb{R}}$. Similarly, for $\mathbf{b} \in \mathbb{H}^n$, $\|\mathbf{b}\|_{\mathbb{H}} = \|\mathcal{Q}_n(\mathbf{b})\|_{\mathbb{C}}$. Thus the embeddings $\mathcal{C}_n$ and $\mathcal{Q}_n$ are norm preserving. For this reason, we may soon drop the subscript $\mathbb{K}$, from $\| - - \|_{\mathbb{K}}$ unless we want to draw your attention to it.

Check that $\langle\, ,\, \rangle_{\mathbb{K}}$ is sesqui-linear, conjugate symmetric, non degenerate and positive definite. (In case $\mathbb{K} = \mathbb{R}$, the conjugation is identity and hence it is bi-linear and symmetric.) Orthogonality, orthonormal basis etc. are taken with respect to this norm.

Theorem 1.3 *$\mathcal{C}_\backslash$ preserves the inner product. In particular, $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset \mathbb{C}^n$ is an orthonormal basis for $\mathbb{C}^n$ iff*
$\{\mathcal{C}_n(\mathbf{x}_1), \mathcal{C}_n(\imath\mathbf{x}_1), \ldots, \mathcal{C}_n(\mathbf{x}_n), \mathcal{C}_n(\imath\mathbf{x}_n)\}$ *is an orthonormal basis for $\mathbb{R}^{2n}$.*

For any $A \in M(n, \mathbb{K})$ we shall use the notation $A^* := \overline{A^T} = \bar{A}^T$. Observe that the identity

$$\langle \mathbf{x}A, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y}A^* \rangle, \quad \mathbf{x}, \mathbf{y} \in \mathbb{K}^n$$

defines A^*.

Theorem 1.4 *For $A \in M(n, \mathbb{K})$ the following conditions are equivalent:*
(i) $\langle \mathbf{x}A, \mathbf{y}A \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
(ii) *R_A takes orthonormal sets to orthonormal sets.*
(iii) *The row-vectors of A form an orthonormal basis for $\mathbb{K}^n$.*
(iv) $AA^* = Id$.
(v) $A^*A = Id$.

Definition 1.2 $\mathcal{O}_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) \ : \ AA^* = I\}$ is called the *orthogonal group* of the standard inner product on $\mathbb{K}^n$. For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, it is also denoted by $O(n), U(n), Sp(n)$ respectively and is known by the names *orthogonal group, unitary group* and *symplectic group* respectively.

Observe that $\mathcal{C}_n(A^*) = (\mathcal{C}_n(A))^*$ for all $A \in M(n, \mathbb{C})$. Similarly, $\mathcal{Q}_n(B^*) = (\mathcal{Q}_n(B))^*$ for all $B \in M(n, \mathbb{H})$. The following theorem is then an easy consequence.

Theorem 1.5 *For each $n \geq 1$, we have:*
(i) $\mathcal{C}_n(U(n)) = O(2n) \cap \mathcal{C}_n(GL(n, \mathbb{C}))$;
(ii) $\mathcal{Q}_n(Sp(n)) = U(2n) \cap \mathcal{Q}_n(GL(n, \mathbb{H}))$;
(iii) $\mathcal{C}_{2n} \circ \mathcal{Q}_n(Sp(n)) = O(4n) \cap \mathcal{C}_{2n} \circ \mathcal{Q}_n(GL(n, \mathbb{H}))$.

In other words, the two other types of orthogonal transformations of $\mathbb{R}^m$ ($m = 2n$ or $4n$) are all real orthogonal transformations satisfying certain Cauchy-Riemann equations. Hence they are closed subgroups of $O(m)$ and hence compact.

Corollary 1.1 *$A \in \mathcal{O}_n(\mathbb{K})$ iff R_A is norm preserving.*

Proof: For $\mathbb{K} = \mathbb{R}$, this is a consequence of the fact that the norm determines the inner product via the **polarization identity:**

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2).$$

For $\mathbb{K} = \mathbb{C}, \mathbb{H}$ we can now use the above theorem and the fact that $\mathcal{C}_n$ and $\mathcal{Q}_n$ are norm preserving. ♠

More generally, fix two non negative integers p, q such that $p + q = n$. Let $I_{p,q}$ denote the $n \times n$ diagonal matrix with p of them equal to -1 and q of them equal to 1. We can then replace statement (v) of theorem 1.4 by $A * I_{p,q} A = I_{p,q}$ and obtain the following subgroups:

$$\mathcal{O}_{p,q}(\mathbb{K}) = \{A \in GL(n, \mathbb{K}) \ : \ A^* I_{p,q} A = I_{p,q}\}.$$

It is easily verified that these are closed subgroups of $GL(n, \mathbb{K})$. We use the notation $O(p, q), U(p, q), Sp(p, q)$ respectively for $\mathcal{O}_{p,q}(\mathbb{K})$ when $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. It is clear that

$$\mathcal{O}_{n,0}(\mathbb{K}) = \mathcal{O}_n(\mathbb{K}) = \mathcal{O}_{0,n}(\mathbb{K})$$

and

$$\mathcal{C}_n(U(p, q)) = O(2p, 2q) \cap \mathcal{C}_n(GL(n, \mathbb{C}), \quad \mathcal{Q}_n(Sp(p, q)) = U(2p, 2q) \cap \mathcal{Q}_n(GL(n, \mathbb{H})).$$

Exercise 1.2

1. Show that $|\det A| = 1$ for $A \in \mathcal{O}_n(\mathbb{K})$.

2. Show that $\mathcal{O}_n(\mathbb{K})$ is compact.

3. Show that Gram-Schmidt process is valid in a finite dimensional vector space over $\mathbb{H}$.

4. Show that $Sp(n, \mathbb{C}) := \{A \in M(2n, \mathbb{C}) \ : \ A^t J_{2n} A = J_{2n}\}$ forms a subgroup of $GL(2n, \mathbb{C})$. This is called the complex symplectic group of order n.

5. Show that $Sp(n, \mathbb{C}) \cap U(2n) = Sp(n, \mathbb{C}) \cap \mathcal{Q}_n(M(n, \mathbb{H})) = \mathcal{Q}_n(Sp(n))$.

The Special Orthogonal Group

Definition 1.3 $SL(n; \mathbb{K}) = \{A \in M(n, \mathbb{K}) \ : \ \det A = 1\}$. This forms a subgroup of $GL(n, \mathbb{K})$. We define $SO(n) = SL(n, \mathbb{R}) \cap O(n); SU(n) = SL(n, \mathbb{C}) \cap U(n)$. These are called *special orthogonal* and *special unitary group* respectively. Likewise, we have the definitions of the groups $SO(p, q) = SO(n) \cap O(p, q), SU(p, q) = SO(2n) \cap U(p, q)$ etc.

Remarks 1.1 We do not need to define the special symplectic groups. Why?

Exercise 1.3

1. $O(1) = \{\pm 1\} \approx \mathbb{Z}_2$ and $SO(1) = (1)$ is the trivial group. It is not difficult to see that

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; : \; \theta \in [0, 2\pi) \right\}$$

   and

$$0(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ \pm\sin\theta & \mp\cos\theta \end{pmatrix} \; : \; \theta \in [0, 2\pi) \right\}$$

   Thus $SO(2)$ can be identified with the group of unit complex numbers under multiplication. Topologically this is just the circle. Similarly, $O(2)$ just looks like disjoint union of two circles. However, the emphasis is on the group structure rather than just the underlying topological space.

2. Again, it is easy to see that $U(1)$ is just the group of complex numbers of unit length and hence is isomorphic to $SO(2)$. Indeed, it is not hard to see that $\mathcal{C}_1$ defines this isomorphism. Clearly, $SU(1) = (1)$.

3. The group $Sp(1)$, is nothing but the group of unit quaternions that we have met. As a topological space, it is $\mathbb{S}^3$. Thus, we have now got three unit spheres viz., $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$ endowed with a group multiplication. A very deep result in topology says that spheres of other dimensions have no such group operations on them.

4. We also have $\mathcal{Q}_1 : Sp(1) \longrightarrow SU(2)$ is an isomorphism. It is easily seen that $\mathcal{Q}_1(Sp(1)) \subset SU(2)$. The surjectivity of $\mathcal{Q}_1$ is the only thing that we need to prove. This we shall leave as an exercise.

Exercise 1.4 Write down the Lie algebra of each of the matrix group you have come across above.

Exercise 1.5

1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a *rigid motion* (*isometry*) if

$$d(f(x), f(y)) = d(x, y) \; \forall \; x, y \in \mathbb{R}^n.$$

   (Recall that the Euclidean distance is defined by $d(x, y) = \|x - y\|$.) Show that for $A \in O(n)$, $R_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry.

2. Show that every isometry of $\mathbb{R}^n$ is continuous and injective. Can you also show that it is surjective and its inverse is an isometry?

3. Show that composite of two isometries is an isometry.

4. Show that $x \mapsto x + \mathbf{v}$ is an isometry for any fixed $\mathbf{v} \in \mathbb{R}^{\mathbf{n}}$.

5. Let f be an isometry of $\mathbb{R}^n$. If $\mathbf{v}, \mathbf{u} \in \mathbb{R}^{\mathbf{n}}$ are such that $f(\mathbf{v}) = \mathbf{v}$ and $f(\mathbf{u}) = \mathbf{u}$ then show that f fixes the entire line L passing through $\mathbf{v}$ and $\mathbf{u}$ and keeps every hyperplane perpendicular to L invariant.

6. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an isometry such that $f(0) = 0$. If $f(e_i) = e_i, i = 1, 2, \ldots, n$, where e_i are the standard orthonormal basis for $\mathbb{R}^n$, then show that $f = Id$.

7. Given any isometry f of $\mathbb{R}^n$ show that there exists a vector $\mathbf{v} \in \mathbb{R}^{\mathbf{n}}$ and $A \in O(n)$ such that $f(x) = xA + \mathbf{v}, \mathbf{x} \in \mathbb{R}^{\mathbf{n}}$. Thus, isometries of $\mathbb{R}^n$ are all 'affine transformations'. (In particular, this answers exercise 2 completely.)

8. Show that the set of all $A \in GL(n+1, \mathbb{R})$ which keep the subspace $\mathbb{R}^n \times \{1\}$ invariant forms a subgroup isomorphic to the group of affine transformations of $\mathbb{R}^n$. Is it a closed subgroup?

9. Show that $GL(n, \mathbb{K})$ is isomorphic to a closed subgroup of $GL(n+\ 1, \mathbb{K})$.

10. Show that $A \in O(3)$ is an element of $SO(3)$ iff the rows of A form a right-handed orthonormal basis, i.e., $R_A(e_3) = R_A(e_1) \times R_A(e_2)$.

11. Show that every element of $A \in O(n)$ is the product $A = BC$ where $B \in SO(n)$ and R_C is either Id or the reflection in the hyperplane $x_n = 0$, i.e.,

$$R_C(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, -x_n).$$

12. Show that eigenvalues of any $A \in O(n)$ are of unit length.

13. Show that one of the eigen values of $A \in SO(3)$ is equal to 1.

14. Given $A \in SO(3)$, fix $\mathbf{v} \in \mathbb{S}^{\mathbf{2}}$ such that $\mathbf{vA} = \mathbf{v}$. Let P be the plane perpendicular to $\mathbf{v}$.
(a) Show that $R_A(P) = P$.
(b) Choose any vector $\mathbf{u} \in \mathbf{P}$ of norm 1. Let $\mathbf{w} = \mathbf{v} \times \mathbf{u}$. Show that every element of P can be written as $(\cos\theta)\mathbf{u} + (\sin\theta)\mathbf{w}$, for some θ.
(c) Show that there exists ϑ such that

$$R_A(\cos\theta\mathbf{u} + \sin\theta\mathbf{w}) = \cos(\theta + \vartheta)\mathbf{u} + \sin(\theta + \vartheta)\mathbf{w}.$$

Thus every element of $SO(3)$ is a rotation about some axis.

Lecture 2

The Exponential Map and Polar Decomposition

We can endow $M(n,\mathbb{K})$ with various norms. The Euclidean norm is our first choice. If we view $M(n,\mathbb{K})$ as the space of linear maps, then the so called L^2-norm becomes quite handy. There are other norms such as row-sum norm, column-sum norm, maximum norm etc.. For the discussion that follows, you can use any one of them. But let us concentrate on the Euclidean norm.

Lemma 2.1 *For any $x, y \in M(n,\mathbb{K})$, we have $\|xy\| \le \|x\|\|y\|$.*

Proof: Straight forward. ♠

Definition 2.4 By a *formal power series in one variable T with coefficients in* $\mathbb{K}$ we mean a formal sum $\sum_{r=0}^{\infty} a_r T^r$, with $a_r \in \mathbb{K}$. In an obvious way, the set of all formal power series in T forms a module over $\mathbb{K}$ denoted by $\mathbb{K}[[T]]$. We define the *Cauchy product* of two power series $p = \sum_r a_r T^r, q = \sum_r b_r T^r$ to be another power series $s = \sum_r c_r T^r$ where $c_r = \sum_{l=0}^{r} a_l b_{r-l}$. (Note that except for being non-commutative, $\mathbb{H}[[T]]$ has all other arithmetic properties of $\mathbb{C}[[T]]$ or $\mathbb{R}[[T]]$.]

Theorem 2.6 *Suppose for some $k > 0$, the series $\sum_r |a_r| k^r$ is convergent. Then for all $A \in M(n,\mathbb{K})$ with $\|A\| < k$, the series $\sum_r a_r A^r$ is convergent.*

Proof: The convergence of the series is the same as the convergence of each of n^2 series $\sum_r a_r A^r_{ij}$, formed by the entries. Since $|A^r_{ij}| \le \|A^r\| \le \|A\|^r < k^r$, we are through. ♠

Definition 2.5 Taking $p(T) = exp(T) = 1 + T + \dfrac{T^2}{2!} + \cdots$ which is absolutely convergent for all $T \in \mathbb{R}$, we define the *exponential of* $A \in M(n,\mathbb{K})$ to be the value of the convergent sum

$$\exp(A) = Id + A + \frac{A}{2!} + \cdots$$

Lemma 2.2 *For any invertible $B \in M(n,\mathbb{K})$ we have $B \exp(A) B^{-1} = \exp(BAB^{-1})$.*

Proof: Check this first on the partial sums. ♠

Lemma 2.3 *If $AB = BA$, then $\exp(A + B) = \exp(A)\exp(B)$.*

Proof: In this special case, the Cauchy product becomes commutative and hence binomial expansion holds for $(A + B)^n$. The rest of the proof is similar to the case when the matrix is replaced by a complex number. ♠

Corollary 2.2 *For all $A \in M(n,\mathbb{K})$, $\exp(A) \in GL(n,\mathbb{K})$.*

Proof: We have $\exp(A)\exp(-A) = \exp(A - A) = \exp(0) = Id.$ ♠

Theorem 2.7 *The function* $\exp : M(n,\mathbb{K}) \longrightarrow GL(n,\mathbb{K})$ *is smooth (indeed, real analytic). The derivative at* 0 *is the identity transformation.*

Proof: The analyticity follows since the n^2-entries are all given by convergent power series. To compute the derivative at 0, we fix a 'vector' $A \in M(n,\mathbb{K})$ and take the directional derivative of exp in the direction of A : viz.,

$$D(\exp)_0(A) = \lim_{t\longrightarrow 0} \frac{\exp(tA) - \exp(0)}{t} = A.$$

Corollary 2.3 *The function* exp *defines a diffeomorphism of a neighbourhood of* 0 *in* $M(n,\mathbb{K})$ *with an neighbourhood* $Id \in GL(n,\mathbb{K})$.

Proof: Use inverse function theorem.

Lemma 2.4 *Given any* $A \in M(n,\mathbb{C})$, *there exists* $U \in U(n)$ *such that* UAU^{-1} *is a lower triangular matrix.*

Proof: If λ is a root of the characteristic polynomial of A there exists a unit vector $\mathbf{v_1} \in \mathbb{C}^\mathbf{n}$ such that $\mathbf{v_1 A} = \lambda \mathbf{v_1}$. Gram-Schmidt process allows us to complete this vector to an orthonormal basis $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$. Take U to be the matrix with these as row vectors. Then $\mathbf{e}_1 UAU^{-1} = \mathbf{v_1 A U^{-1}} = \lambda \mathbf{e_1}$. Hence, UAU^{-1} is of the form

$$\begin{pmatrix} \lambda, & 0 \\ \star & B \end{pmatrix}.$$

Now a simple induction completes the proof. ♠

Definition 2.6 We say A is *normal* if $AA^* = A^*A$. A square matrix A is called symmetric, (skew-symmetric, Hermitian, skew-Hermitian, respectively) if $A = A^T$ ($A = -A^T$; $A = A^*$; $A = -A^*$).

Corollary 2.4 *If* $A \in M(n,\mathbb{C})$ *is normal, then there exists* $U \in U(n)$ *such that* UAU^{-1} *is a diagonal matrix.*

Proof: If A is normal then so is UAU^{-1} for any $U \in U(n)$. On the other hand a lower triangular normal matrix is a diagonal matrix. ♠

Remarks 2.2 In particular, if A is hermitian, symmetric, skew symmetric etc., then it is diagonalizable. The entries on the diagonal are necessarily the characteristic roots of the original matrix. Moreover, if A is real symmetric matrix, then all its eigenvalues are real with real eigen vectors and hence U can be chosen to be inside $O(n)$.

Definition 2.7 A Hermitian matrix A defines a sesqui-linear (Hermitian) form on $\mathbb{C}^n$. Recall that A^* satisfies the property

$$\langle \mathbf{uA}, \mathbf{v}\rangle_{\mathbb{C}} = \langle \mathbf{u}, \mathbf{vA}^*\rangle_{\mathbb{C}}.$$

Therefore, for a Hermitian matrix A, $\langle \mathbf{uA}, \mathbf{u}\rangle_{\mathbb{C}}$ is always a real number. We say A is *positive semi-definite (positive definite)* if $\langle \mathbf{uA}, \mathbf{u}\rangle_{\mathbb{C}} \geq \mathbf{0}$ for all $\mathbf{u}$ (respectively, > 0 for all non zero $\mathbf{u}$.)

Lemma 2.5 *A is positive semi-definite (positive definite) iff all its eigenvalues are non negative (positive).*

Lemma 2.6 *If A is Hermitian so is,* $\exp(A)$.

Theorem 2.8 *The space of all $n \times n$ complex Hermitian matrices is a real vector space of dimension n^2. Exponential map defines a diffeomorphism of this space onto the space of all positive definite Hermitian matrices.*

Proof: The first part is obvious. Given a positive definite Hermitian matrix B let U be a unitary matrix such that

$$UBU^{-1} = diag\,(\lambda_1, \ldots, \lambda_n).$$

Then we know that $\lambda_j > 0$ and hence we can put $\mu_j := log\,\lambda_j$. Put

$$A := U^{-1} diag\,(\mu_1, \ldots, \mu_n) U.$$

Then $\exp(A) = B$. This shows exp is surjective. For the injectivity of Exp on the space of Hermitian matrices, see the exercise below. ♠

Definition 2.8 A subgroup G of $GL(n, \mathbb{R})$ is said to be pseudo-algebraic if there exists a set P of polynomials in n^2 variables such that $g \in GL(n, \mathbb{R})$ belongs to G iff the coefficients g_{ij} of g satisfy $p(g_{ij}) = 0$ for all $p \in P$. A subgroup G on $GL(n, \mathbb{C})$ (or $GL(n, \mathbb{H})$) is said to be pseudo-algebraic if $\mathcal{C}_n(G)$ (or $\mathcal{C}_{2n} \circ \mathcal{Q}_n(G)$ is a pseudo-algebraic in $GL(2n, \mathbb{R})$ (in $GL(4n, \mathbb{R})$).

Remarks 2.3

(1) All the matrix groups considered so far above, are pseudo-algebraic.
(2) If G is pseudo-algebraic, then so is uGu^{-1} for any invertible matrix u.
(3) Also note that each of the above subgroups G has the property that $g \in G$ implies $g^* \in G$. This remark is going to be useful soon.

Lemma 2.7 *Let G be a pseudo algebraic subgroup of $GL(n, \mathbb{C})$ and H be a $n \times n$ Hermitian matrix such that* $\exp H \in G$. *Then for all $t \in \mathbb{R}$,* $\exp tH \in G$.

Proof: By spectral theorem and the remerk above, we may as well assume that

$$H = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n).$$

Now, $\exp H \in G$ implies that for all integers k, $e^{k\lambda_1}, \ldots, e^{k\lambda_n}$ satisfy a set of polynomials. By simple application of Vander Monde, this implies that $e^{t\lambda_1}, \ldots, e^{t\lambda_n}$ satisfy the same set of polynomials for all $t \in \mathbb{R}$. This in turn implies $\exp tH \in G$ for all $t \in \mathbb{R}$. ♠

Corollary 2.5 *Let G be a pseudo-algebraic subgroup of $GL(n, \mathbb{C})$. If H is any Hermitian matrix such that* $\exp H \in G$ *then $H \in \mathfrak{g}$, the Lie algebra of G.*

Proof: We have seen that the curve $t \mapsto \exp tH$ takes values in G. Taking the derivative at $t = 0$ we coclude the result. ♠

Theorem 2.9 Polar Decomposition *Every element A of $GL(n, \mathbb{C})$ can be written in a unique way as a product $A = UH$ where U is unitary and H is positive definite Hermitian. The decomposition defines a diffeomorphism of $\varphi : GL(n, \mathbb{C}) \to U(n) \times P_n$. Furthermore, if G any pseudo-algebraic subgroup which is closed under conjugate transpose, then φ restricts to a diffeomorphism $G \to (U(n) \cap G) \times (G \cap P_n)$ and $G \cap P_n$ is diffeomorphic to $\mathbb{R}^d$ for some d.*

In particular, φ restricts to a diffeomorphism $\varphi : GL(n, \mathbb{R}) \to O(n) \times \mathbb{R}^{n(n+1)/2}$.

Proof: Consider the matrix $B = A^*A$ which is Hermitian. Since A is invertible, so is A^*. Therefore, for a non zero vector $\mathbf{v}$, $\langle \mathbf{v}\mathbf{A}^*\mathbf{A}, \mathbf{v}\rangle = \langle \mathbf{v}\mathbf{A}^*, \mathbf{v}\mathbf{A}^*\rangle > \mathbf{0}$, which shows that B is positive definite. Choose $C \in GL(n, \mathbb{C})$ such that $CBC^{-1} = diag(\lambda_1, \ldots, \lambda_n)$ and put $H = C^{-1}diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})C$. Then H is clearly a positive definite Hermitian matrix and $H^2 = B$. Put $U = AH^{-1}$. Then $A = UH$ and we can directly verify that U is unitary.

Finally, if $A = U_1H_1$ where U_1 is unitary and H_1 is positive definite Hermitian, then $X = U^{-1}U_1 = HH_1^{-1}$ is both unitary and positive definite Hermitian. Therefore, X has all its eigenvalues of unit length, as well as, positive. Thus all its eigenvalues are 1. Since it is diagonalizable also, it follows that $X = Id_n$.

The construction of H from A is indeed a smooth process, though this is not clear from the way we have done this. But we can simply write

$$H = \exp(\frac{1}{2} log\, A^*A)$$

where log is the inverse map of Exp in the theorem 2.8. Thus $\varphi(A) = (AH^{-1}, H)$ where H is given as above. It follows that φ is smooth. The inverse map is clearly smooth. This proves the first part.

So far, we have kept the discussion of this proof elementary. However, to see the latter half, we need to bring in the the exponential map and the corollary 2.5 proved above. Let $\mathfrak{h}(n)$ denote the real vector space of all $n \times n$ Hermitian matrices.

Now, let G be a pseudo-algebraic subgroup of $GL(n, \mathbb{C})$ which is closed under transpose conjugation. If $A \in G$ then it follows that $A^*A \in G$ and we have seen that there is unique $C \in \mathfrak{h}(n)$ such that $\exp C = H$, and $H^2 = A^*A$. From the corollary above, it follows that $H \in \mathfrak{g}$ and $H \in G$. Thereofore $AH^{-1} \in U(n) \cap G$ and $H \in P_n \cap G$. Therefore exp restricts to a diffeomorphism $G \to (G \cap U(n) \times G(\cap P_n)$. It remains to see that $G \cap P_n$ is diffeomorphic to $\mathbb{R}^d$.

Indeed we already know that $\exp : \mathfrak{g} \cap \mathfrak{h}(n) \to G \cap P_n$ is injective. As seen above the corollary implies that this map is surjective.

♠

Exercise 2.6 Show that exp is injective on the space of Hermitian matrices by solving the following sequence of exercises.

(1) Let $D = \mathrm{diag}\,(\lambda_1 I_{k_1}, \lambda_r I_{k_r})$, where $\lambda_i \neq \lambda_j$, for $i \neq j$. If $AD = DA$ then show that A is also a block matrix of the form $\mathrm{diag}\,(\mathrm{A_{k_1}}, \ldots, \mathrm{A_{k_r}})$, where each A_{k_j} is a $k_j \times k_j$ matrix. (2) Let $D = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n), D' = \mathrm{diag}\,(\lambda'_1, \ldots, \lambda'_n)$, where $\lambda_i \geq \lambda_{i+1}, \lambda'_i \geq \lambda'_{i+1}$, for all $1 \leq i \leq n-1$. If U is an invertible matrix such that $U(\exp D)U^1 = \exp D'$ then show that $D = D'$ and $UDU^{-1} = D$.
(3) Let A, B be hermitian matrices such that $\exp A = \exp B$. Show that $A = B$.

The Fundamental Group

Here we assume that the audience is familiar with the notion of fundamental group of a topological space. We begin with an algebraic lemma:

Lemma 2.8 *Let G be a set with two binary operations $*$ and $\circ$ having a common tw-sided identity element e. Suppose the two operations satisfy the following mutually distributive property:*

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

for all $a, b, c, d \in G$. Then the two operations coincide and are associative and commutative.

Proof: Put $b = e = c$ to see that $a \circ d = a * d$. Next put $a = e = d$ to see the commutativity and finally put $b = e$ to see the associativity. ♠

Theorem 2.10 *$\pi_1(G)$ is abelian for any connected topological group G.*

Proof: Let $\omega_1 * \omega_2$ denote the composition of loops based at the identity element $e \in G$. For any two loops ω_1, ω_2 define $(\omega_1 \circ \omega_2)(t) = \omega_1(t)\omega_2(t)$, wherein on the RHS we have used the multiplication in the topological group G. It is easily checked that

$$(\omega_1 * \omega_2) \circ (\tau_1 * \tau_2) = \omega_1 \circ \tau_1) * (\omega_2 \circ \tau_2).$$

Paasing onto homotopy classes of loops we still have the same mutually distributive relation. Moreover, the class of the constant loop at e now serves as a common two-sided identity. The lemma now completes the proof. ♠

Remarks 2.4 A more popular proof of the above theorem is via covering space theory. Every connected Lie group G admits a universal covering $p : \tilde{G} \to G$, i.e., there exists connected, simply connected Lie group $\tilde{G}$ and a surjective homomorphism $p : \tilde{G} \to G$ which is a covering projection. In particular, this implies that $\mathrm{Ker}\, p$ is a discrete subgroup of $\tilde{G}$ and is isomorphic to $\pi_1(G)$. A standard result on topological groups now implies that $\mathrm{Ker}\, p$ is central in $\tilde{G}$ and hence, in particular, is abelian. The lemma above will come handy again for us later.

By the general theory of manifolds, the fundamental group of any compact manifold is finitely generated. By polar decomposition theorem, it follows that the fundamental group of every matrix group is finitely gnerated.

One can strengthen the above result and the observation as follows:

Theorem 2.11 *Let G be a connected Lie group and T be a maximal torus in G. Then the inclusion induced homomorphism $\eta_\# : \pi_1(T) \to \pi_1(G)$ is surjective.*

Here is the sketch of the proof.

An element $g \in G$ is called regular if it is a generator of a maximal torus in G. Let G_{reg} denote the set of all regular elements of G. Let T be a maximal torus in G and $T_{reg} = G_{reg} \cap T$. Consider the map $\phi : T_{reg} \times (G/T) \to G_{reg}$ given by $\phi(t, gT) = gtg^{-1}$.

Lemma 2.9 *G_{reg} is path connected.*

Lemma 2.10 *The map ϕ is a covering projection.*

Lemma 2.11 *There is an exact sequence of groups:*

$$\pi_1(T) \to \pi_1(G) \to \pi_1(G/T) \to (1).$$

Using these lemmas, the proof of theorem 2.11 is completed as follows. Note that the statement of the theorem is equivalent to showing that $\pi_1(G/T) = (1)$. Choose $t_0 \in T_{reg}$ and define $f_0 : G/T \to G$ by $f_0(gT) = gt_0g^{-1}$. By taking a path from t_0 to $e \in G$ it is clear that f_0 is homotopic to the constant map in G. On the other hand, f_0 is the composite of

$$G/T \to T_{reg} \times G/T \to G_{reg}$$

$$gT \mapsto (t_0, gT) \mapsto \phi(t_0, gT).$$

Being a coordinate inclusion, the first map induces a monomorphism on π_1. Being a covering projection the second map induces a monomorphism on π_1. Therefore f_0 induces a monomorphism on π_1. Being null-homotopic, it induces the trivial homomorphism. Therefore $\pi_1(G/T) = (1)$.

The proofs of the above lemmas take us a little deeper into the Lie theory, viz., the root systems.

Proof of lemma 2.10 Let $W = N(T)/T$ denote the Weyl group. Then we know that W acts on G/T freely, via $(nT, gT) \mapsto gn^{-1}T$. Consider the action of W on $T_{reg} \times G/T$ given by

$$(nT, t, gT) \mapsto (ntn^{-1}, gn^{-1}T).$$

It follows that the quotient map

$$q : T_{reg} \times G/T) \to W\backslash T_{reg} \times G/T$$

is a covering projection with number of sheets $|W|$. The map ϕ clearly factors through q to define a map $\psi : W\backslash T_{reg} \times G/T \to G_{reg}$. The claim is that ψ is a homeomorphism.

Since ϕ is surjective, so is ψ. Since ϕ and q are local homeomorphisms so is ψ. It can be shown that there is a dense subset of $W\backslash T_{reg} \times G/T$ on which ψ is injective. Therefore, it is injective all over. ♠

Remarks 2.5 Lemma 2.9 follows from a much stronger result:

Theorem 2.12 *G_{reg} is the complement of a finite union of closed submanifolds of G of codiemn-sion at least* 3.

Similarly, lemma 2.11 is a consequence of a much general result viz., The mapping $G \to G/T$ is submersion with compact fibres and hence is a locally trivial fibre bundle. We shall discuss this in the next meeting.

Lecture 3

Throughout today's discussion, let G denote a connected Lie group. In the previous meeting we discussed the surjectivity of $\pi_1(T) \to \pi_1(G)$ where $T \subset G$ is a maximal torus. One of the fact that we used is the exact sequence

$$\pi_1(T) \to \pi_1(G) \to \pi_1(G/T) \to (1)$$

as a consequence of the fact that the quotient map $G \to G/T$ is a fibration. Today, let us take a closer look at this, in a more general situation.

Theorem 3.13 *Let G be a Lie group, $H \subset G$ be a compact subgroup and $q : G \to G/H$ be the quotient map.Then G/H is has a structure of a smooth manifold such that the quotient map $G \to G/H$ is a smooth (locally trivial) fibre bundle.*

Proof: Let $\mathfrak{h} \subset \mathfrak{g}$ denote the respective Lie algebras, and let $\mathfrak{h}'$ be a vector space complement, viz., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Consider the map $\Psi : H \times \mathfrak{h}' \to G$ given by

$$\Psi(h, \mathbf{v}) = \mathbf{h}(\exp \mathbf{v}).$$

Since the derivative of the product is the sum of the derivatives, it follows by inverse function theorem, Ψ is a local diffeomorphism. Observe that $\Psi|_{H\times 0}$ is injective. Therefore, from standard arguments in differential topology, it follows that there is a there are disc neighbourhood of $0 \in \mathfrak{h}'$ such that the map $\Psi : H \times U' \to \Psi(H \times U') \subset G$ is diffeomorphism onto an open neighbourhoood of $H \subset G$ which is identity on $H \times 0$. Passing on to the quotient, we get a diffeomorphism of U' onto an open neighbouhood $V = q(\Psi(H \times U'))$ of H in G/H. Now for any $gH \in G/H$, we can consider the translates $\Psi \circ L_g$ to give neighbourhoods of gH. It is easily checked that this gives a smooth atlas $\{gV\}$ for G/H and a trivialization of the quotient map over each member gV of this atlas. ♠

Definition 3.9 A map $p : E \to B$ is said to have homotopy lifting property with to a space X if given maps $H : X \times [0,1] \to B$ and $f : X \times 0 \to E$ such that $p \circ f = H_0$, there exists a map $G : X \times [0,1] \to E$ such that $G_0 = f$ and $p \circ G = H$. If p has homotopy lifting property for all spaces then it is called a fibration.

Remarks 3.6 Clearly any coordinate projection $B \times B' \to B$ is a fibration. Typical non trivial example of a fibration is a covering projection. It is a deep theorem in algebraic topology (see Spanier) that every locally trivial fibre bundle over a paracompact space is a fibration. A surjective submersion of manifolds with compact fibres is a locally trivial fibre bundle and therefore is a fibration. We shall come back to discuss the homotopy theoretic aspects of a fibration a little later.

Recall that A subgroup T of G is called a torus if it is isomorphic to the porduct of finitely many copies of $\mathbb{T} = \mathbb{S}^1$. This is equivalent to demand that T is a connected abelian subgroup. T is called a maximal torus in G if for all tori T' such that $T \subset T' \subset G$ we have $T = T'$. Since the closure of a connected abelian subgroup in G is also connected abelian, it follows that a maximal torus is also a closed subgroup. Also, a connected closed subgroup T in G is a torus iff its Lie algbera is abelian and it is maximal iff its Lie algebra is a maximal abelian subalgebra. From this it follows that every compact connected Lie group has a maximal torus.

An element $g \in T$ is called a generator or 'generic' if the closure of the cyclic sugroup generated by g is equal to T. It is well-known that each torus has plenty of generic elements.

Clearly conjugates gTg^{-1} of any maximal torus is maximal.

Moreover,

Theorem 3.14 *Let T be a maximal torus in a connected Lie group G. Given any $x \in G$ there exists a $g \in G$ such that $gxg^{-1} \in T$.*

In other words, conjugates of any given maximal torus cover the entire group. As an easy consequence we obtain:

Corollary 3.6 *Any two maximal tori in a connected Lie group are conjugate to each other.*

Proof: Taking $x \in T'$ to be a generic element, and applying the above theorem, we get $g \in G$ such that $gxg^{-1} \in T$ which implies that $gT'g^{-1} \subset T$. Since $gT'g^{-1}$ is also maximal, equality holds.

Toward a proof of the above theorem, consider the diffeomorphism $L_x : G \to G$ given by $g \mapsto xg$. This, in turn, induces a diffeomorphism $L_x : G/T \to G/T$. Theorem above is equivalent to show that L_x has a fixed point. This is where we are going to use some algebraic topology, viz., we appeal to Lefschetz fixed point theorem which says:

Proposition 3.1 Any continuous map $f : X \to X$ of a compact polyhedron has fixed point if its Lefschetz number is not zero.

So, let us show that

Theorem 3.15 *The Lefschetz number of*: $L_x : G/T \to G/T$ *is equal to* $|W|$ *where* $W = N(T)/T$ *is the Weyl group.*

We observe that Letschetz number is a homological invariant and hence a homotopy invariant. Since G is path connected, it follows easily that L_x is homotopic to $L_{x'}$ for any other $x' \in G$. Therefore, we can as well assume that x is a generic element of T. But then gT is a fixed point of L_x iff $L_x(gT) = gT$ iff $g^{-1}Tg = T$ iff $g \in N(T)$ iff $gT \in W$.

Now the Lefschetz number of a map $f : X \to X$ which has finitely many fixed points is equal to the sum of multiplicities of all the fixed points. If $x \in X$ is an isolated fixed point then f maps a punctured disc-neighbourhood D of x into a punctured neighbourhood of x and the multiplicity of x is nothing but the winding number of $f_{\partial D}$ around x.

In our situation, $f = L_x$ is a diffeomorphism and hence it follows easily that the winding number is equal to ± 1. That it is actually equal to 1 follows from the fact that L_x preserves the local orientation, being homotopic to the identity (keeping the point T fixed).

Now, for $n \in N(T)$, the right-multiplication R_n induces a diffeomorphism $R_n : D \to D'$ via $gT \mapsto gnT$, where D is a disc -neighbourhood of T in G/T and D' is a neighbouhood of nT. Moreover, it commutes with L_x. Therefore, it follows that the multiplicity of L_x at T is the same that at nT for all $n \in W$. Therefore, the Lefschetz number of L_x is equal to $|W|$. ♠

Remarks 3.7 Incidentally, we have also proved that the Euler characteristic $\chi(G/T) = |W|$, since the Euler characteristic is the same as the Lefschetz numebr of the identity map.

Let us now recall a few general properties of homotopy groups. You are welcome to browse through Chapter 10 of my book.