Lecture Notes in Real Analysis 2010

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Lectures 1-3 (I-week)

Lecture 1

Why real numbers?

Example 1 Gaps in the rational number system. By simply employing the unique factorization theorem for integers, we can easily conclude that there is no rational number r such that $r^2 = 2$. So there are gaps in the rational number system in this sense. The gaps are somewhat subtle. To illustrate this fact let us consider any positive rational number p and put

$$q = \frac{2p+2}{p+2} = p - \frac{p^2-2}{p+2}. \tag{1}$$

Check that

$$q^2 - 2 = \frac{2(p^2-2)^2}{p+2} > 0 \tag{2}$$

Now if $p^2 < 2$ then check that $p < q$ and $q^2 < 2$. Similarly, if $p^2 > 2$ then check that $q < p$ and $2 < q^2$. This shows that there exist a sequence $r_1 > r_2 > r_3 > \cdots$ of rational numbers such that $r_k^2 > 2$ and a sequence of rational number $s_1 < s_2 < \cdots$ such that $s_k^2 < 2$. In other words, in the set of all positive rationals r such that $r^2 > 2$ there is no least element and similarly in the set of all positive rationals such that $s^2 < 2$ there is no greatest element. The real number system fulfills this kind of requirement that the rational number system is unable to fulfill.

Some Basic Set theory Membership, union, intersection, power set, De'Morgan's law, (the episode of RAMA and SITA), ordered pairs:$(x, y) := \{\{x\}, \{x, y\}\}$. Cartesian product $X \times Y$ as a subset of the power set of power set of $X \cup Y$. Relations, functions, cartesian product of arbitrary family of sets, cardinality, finiteness and infiniteness, countability of $\mathbb{Q}$

Notation:

$\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers.
$\mathbb{Z}$ = set of interger.
$\mathbb{Z}^+$ = the set of positive integers
$\mathbb{Q}$ = the set of rational numbers.
$\mathbb{R}$ = the set of real numbers.
$\mathbb{C}$ = the set of complex numbers.

Lecture 2

Definition 1 Let X be a set $R \subset X \times X$ be a relation in X. We shall write $x < y$ whenever $(x, y) \in R$. We say R is an **order** (**total order** or **linear order**) on X if the following conditions hold:
(i) Transitivity: $x < y, y < z \Longrightarrow x < z$ for any $x, y, z \in X$.
(ii) Law of Trichotomy: Given $x, y \in X$ either $x < y$ or $y < x$ or $x = y$.

We shall read $x < y$ as 'x is less than y'. We shall write $x \leq y$ to mean either $x < y$ or $x = y$. We shall also write $x > y$ to mean $y < x$ and $x \geq y$ to mean $y \leq x$. Note that $>$ becomes another order on the set X. However, these two orders on X are so closely related to each other, that we can recover any information on one of them from a corresponding information on the other.

Let now $A \subset X$. We say $x \in X$ is an **upper bound** of A if $a \leq x$

for all $a \in A$. If such an x exists we then say A is **bounded above.** Likewise we define **lower bounds** and **bounded below** as well.

An element $x \in X$ is called a **least upper bound** abbreviated as lub, (or **supremum** to be written as 'sup' of A if x is an upper bound of A and if y is an upper bound of A then $x \leq y$. Similarly we define a **greatest lower bound** (glb or 'inf'= infimum).

Remark 1
(i) The set of integers is an ordered set with the usual order $<$. The subset of positive integers is bounded below but not bounded above. Also it has greatest lower bound viz., 1. Indeed the set of rational numbers is also an ordered set with the natural order and this way we can view $\mathbb{Z}$ as an ordered subset of $\mathbb{Q}$. Note that $\mathbb{Z}^+$ is not bounded above even in $\mathbb{Q}$.
(ii) Even if a set A is bounded above, there may not be a least upper bound as seen in the example 1. However, if it exists then it is unique.
(iii) Let $A = \emptyset$ be the emptyset of an ordered set X. Then every member of X is an upper bound for A. Therefore, least upper bound for A would exist iff X has a least element.
(iv) Let $A = X$. Then an upper bound for A is nothing but the greatest element of X if it exists and hence the lub of X is also equal to this element.

Definition 2 An ordered set X is said to be **order complete** if for every nonempty subset A of X which is bounded above there is a least upper bound for A in X.

Definition 3 By a binary operation on a set X we mean a function $\cdot : X \times X \to X$

Remark 2 It is customary to denote $\cdot(a, b)$ by $a \cdot b$ or some other conjunction symbol between the two letters a and b. if there is no scope for confusion, by ab. Typical example of binary operations are addition and multiplication defined on the set of integers (rational numbers) etc..

Definition 4 A **field** $\mathbb{K}$ is a set together with two binary operations denoted by $+$ and $\cdot$ satisfying a number of properties called field axioms which we shall express in three different lists:

List(A) Axioms for addition:
(A1) Associativity: $x + (y + z) = (x + y) + z; \quad x, y, z \in \mathbb{K}$.
(A2) Commutativity: $x + y = y + x; \quad x, y \in \mathbb{K}$.
(A3) The zero element: There exists $0 \in \mathbb{K}$ such that $x + 0 = x; x \in \mathbb{K}$.
(A4) Negative: For $x \in \mathbb{K}$ there is a $y \in \mathbb{K}$ such that $x + y = 0$.
List (M) Axioms for multiplication:
(M1) Associativity: $x(yz) = (xy)z; \quad x, y, z \in \mathbb{K}$.
(M2) Commutativity: $xy = yx; \quad x, y \in \mathbb{K}$.
(M3) The unit element: There exists $1 \in \mathbb{K}, 1 \neq 0$ such that $1x = x; \quad x \in \mathbb{K}$.
(M4) Inverse: For each $x \in \mathbb{K}$ such that $x \neq 0$ there exists $z \in \mathbb{K}$ such that $xz = 1$.

List (D) Distributivity: $x(y + z) = xy + xz; \quad x, y, z \in \mathbb{K}$.

Remark 3 Note that the zero element is unique. Therefore (M3) and (M4) make sense. Moreover, the unit element is also unique. Further the negative and the inverse are also unique and are denoted respectively by $-x$ and $1/x$. Because of the associativity, we can drop writing down brackets at all. We also use the notation n to indicate the sum $1 + 1 + \cdots + 1$ (n times). Likewise we use the notation x^n in denote $xx \cdots x$ (n times). Thus all 'polynomial' expressions of elements of $\mathbb{K}$

make sense. That is to say, if $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, with $a_i \in \mathbb{K}$ then we can substitute for t any element $x \in \mathbb{K}$ and obtain a well defined element of $\mathbb{K}$. The most important example of a field for us now is the field of rational numbers $\mathbb{K} = \mathbb{Q}$.

Definition 5 An **ordered field** is a field $\mathbb{K}$ with an order $<$ satisfying the following axioms:
(O1) $x < y \Longrightarrow x + z < y + z$ for all $x, y, z \in \mathbb{K}$.
(O2) $x > 0, y > 0 \Longrightarrow xy > 0$ for all $x, y \in \mathbb{K}$.

Remark 4 Once again a typical example is the field of rational numbers with its usual order. All familiar rules for working with inequalities will be valid in any ordered field. For example the square of any element in an ordered field cannot be negative. Let us list a few of such properties which can be derived easily from the axioms:

Theorem 1 *Let $\mathbb{K}$ be an ordered field with the order $<$. Then the following properties are true for elements of $\mathbb{K}$:*
(a) $0 < x$ *iff* $-x < 0$.
(b) $0 < x, y < z \Longrightarrow xy < xz$.
(c) $x \neq 0 \Longrightarrow 0 < x^2$. *In particular,* $0 < 1$.
(d) $0 < x < y \Longrightarrow 0 < 1/y < 1/x$.

Exercise 1 Let $\mathbb{K}$ be an ordered field. Temporarily let us denote the identity elememt of $\mathbb{K}$ by $1_{\mathbb{K}}$ and $1_{\mathbb{K}} + \cdots + 1_{\mathbb{K}}$ (m times) by $m\dot{1}_{\mathbb{K}}$.
(a) Show that the mapping $m \mapsto m\dot{1}_{\mathbb{K}}$ defines an injective ring homomorphism of $\phi : \mathbb{Z} \to \mathbb{K}$ which is order preserving, viz.,

$$\phi(x + y) = \phi(x) + \phi(y); \quad \phi(xy) = \phi(x)\phi(y); \quad x < y \Longrightarrow \phi(x) < \phi(y).$$

(b) Show that ϕ extends to an injective field homomorphsim $\mathbb{Q} \to \mathbb{K}$

which is order preserving. In this way we can now say that every ordered field contains the field of rational numbers.

We shall now state a result which asserts the existence of real number system. We shall not prove this. Interested reader may read this from [R].

Theorem 2 *There is a unique ordered field* $\mathbb{R}$ *which contains the ordered field* $\mathbb{Q}$ *and is order complete.*

Remark 5 Note that a field $\mathbb{K}$ is order complete iff for every subset A of $\mathbb{K}$ which is bounded below there is a greatest lower bound. This follows easily by considering $-A$. The uniqueness of $\mathbb{R}$ has to be interpreted correctly in the sense that if there is another such $\mathbb{R}'$ then there is a bijection $\phi : \mathbb{R} \to \mathbb{R}'$ such that
(i) $\phi(r) = r, r \in \mathbb{Q}$
(ii) $\phi(x+y) = \phi(x) +' \phi(y), x, y \in \mathbb{R}$
(iii) $\phi(xy) = \phi(x).'\phi(y), x, y \in \mathbb{R}$.
(iv) $x < y \Longrightarrow \phi(x) <' \phi(y), x, y \in \mathbb{R}$.

Lecture 3

Theorem 3 $\mathbb{Z}^+$ *is not bounded above in* $\mathbb{R}$.

Proof: If $x \in \mathbb{R}$ is such that $n < x$ for all $n \in \mathbb{Z}^+$, then we can take the least upper bound $l \in \mathbb{R}$ for $\mathbb{Z}^+$. But then there must exist $n \in \mathbb{Z}^+$ such that $l - 1 < n$. This implies $l < n + 1$ which is absurd. ♠

Theorem 4 Archimedian Property
(A) For every $x \in \mathbb{R}$ *there exists* $n \in \mathbb{Z}^+$ *such that* $x < n$.
(B) $x, y \in \mathbb{R}, 0 < x \Longrightarrow$ *there exists* $n \in \mathbb{Z}^+$ *such that* $y < nx$.

Proof: : (A) This is just a restatement of the above theorem.
(B) Apply (A) to y/x. ♠

Theorem 5 *If S is a nonempty subset of $\mathbb{Z}$ which is bounded above then S has a maximum.*

Proof: (Recall that a set has a maximum iff the least upper bound exists and belong to the set.) Let $y \in \mathbb{R}$ be the least upper bound of S. We claim that $y \in S$. Suppose y is not in S. Now there exists $m \in S$ such that $y - 1/2 < m < y$. This implies $0 < y - m < 1/2$. Also, since $\frac{y+m}{2} < y$ there exists $n \in S$ such that $\frac{y+m}{2} < n < y$. This implies that $0 < \frac{y-m}{2} < n - m < y - m < 1/2$ which is absurd. ♠

Definition 6 We can now define the 'floor' and 'ceiling' functions on $\mathbb{R}$. Given any $x \in \mathbb{R}$ consider the set $Z_x = \{m \in \mathbb{Z} \;:\; m \leq r\}$. Clearly Z_x is bounded above and non empty (Archimedian property). From the above theorem, it has a maximum which is of course unique. We define this maximum to be $\lfloor x \rfloor$. Likewise $\lceil x \rceil$ is also defined.

Lemma 1 Let $x, y \in \mathbb{R}$ be such that $y - x > 1$. Then there exists $m \in \mathbb{Z}$ such that $x < m < y$.

Proof: By the definition of floor, it follows that $x < 1 + \lfloor x \rfloor$. Therefore,

$$1 < y - x \Longrightarrow x < 1 + \lfloor x \rfloor < y - x + \lfloor x \rfloor < y.$$

and by taking $m = 1 + \lfloor x \rfloor$, we are done. ♠

Theorem 6 Density of $\mathbb{Q}$ in $\mathbb{R}$. *Given $x < y \in \mathbb{R}$ there exist $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof: We have to find $r = m/n$ such that $x < m/n < y$ which is the same as finding integers $m, n, n > 0$ such that $nx < m < ny$. This is possible iff we can find $n > 0$ such that the interval (nx, ny) contains an integer iff $n(y-x) > 1$. This last claim follows from (B) of theorem 4. ♠

The last theorem may lead us to believe that the set of real numbers is not much too large as compared to the set of rational numbers. However the fact is indeed the opposite. This was a mild shock for the mathematical community in the initial days of invention of real numbers. We define an irrational number to be a real number which is not a rational number. To begin with we shall prove:

Theorem 7 *The set $\mathbb{R} \setminus \mathbb{Q}$ of irrationals is dense in $\mathbb{R}$.*

Proof: Given any two real numbers $x < y$ we must find an irrational number ϕ such that $x < \phi < y$. By the earlier theorem we can first choose rational numbers x_1, y_1 such that $x < x_1 < y_1 < y$ and then show that there is an irrational number ϕ such that $x_1 < \phi < y_1$. By clearing the denominators we can then reduce this to assuming that x, y are integers and then by taking the difference, we can further assume that $x = 0$. But then we can as well assume that $y = 1$. Thus it is enough to show that there is an irrational number between 0 and 1. If this were not true, by translation, it would follow that there are no irrational numbers at all! ♠

Remark 6 Pay attention to this argument which occurs in somewhat different forms in several places in mathematics. Later we shall show that $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. To begin with, at least, we can now be sure that there is a real number x such that $x^2 = 2$.

Theorem 8 *Given any positive real number y, there is a unique positive real x such that $x^2 = y$.*

Proof: Uniqueness is easy: $x_1^2 = x_2 \Longrightarrow (x_1 + x_2)(x_1 - x_2) = 0$ which in turn implies $x_1 - x_2 = 0$.

The existence will be left to you as an exercise for the time being. We shall prove this after awhile.

Exercise 2 Let S be a non empty subset of $\mathbb{R}$ which is bounded above. Let T be the set of all upper bounds of S. Show that $lub(S) = glb(T)$.

Exercise 3 Fix $x > 1$.

1. For positive integers p, q, m, n such that $q \neq 0, n \neq 0$ and $p/q = m/n = r$ show that $(x^m)^{1/n} = (x^p)^{1/q}$. This allows us to define $x^r = (x^m)^{1/n}$ unambiguously.

2. Show that for rational number r, s, $x^{r+s} = x^r x^s; \;\; (x^r)^s = x^{rs}$.

3. For any real number α, let

$$S(\alpha) = \{x^r \;:\; r \leq \alpha, r \in \mathbb{Q}\}.$$

   Show that if α is rational then $x^\alpha = lub(S(\alpha))$. This prompts us to define, for any real number α

$$x^\alpha := lub S(\alpha).$$

4. Prove that $x^\alpha x^\beta = x^{\alpha+\beta}$ for all real numbers α, β.