Week 11 Lectures 31-34

Lecture 31: Initial Value Problem

Solution– a la Picard

The existence and unique of the solution of an Initial Value Problem(IVP)

$$y' = f(x, y), \quad y_(x_0) = y_0 \tag{19}$$

is of fundamental importance in several branches of mathematics, not just in the theory of Differential equations. However, it is not taught in any first course in differential equations, since the students do not have the required analysis background and then a student may never take a formal course in differential equations thereby totally 'missing' this beautiful theorem.

Observe that f is a given real valued function defined in a (rectangular) neighbourhood of the point $(x_0, y_0) \in \mathbb{R}^2$. By a solution of (19), we mean a once differentiable function ϕ defined in some neighbourhood of the point x_0 say $(x_0 - \delta, x_0 + \delta)$ satisfying,

$$\phi(x_0) = y_0, \quad \& \ \phi'(x) = f(x, \phi(x)), x \in (x_0 + \delta, x_0 + \delta). \tag{20}$$

By Fundamental Theorem of Riemann Integration, we can convert (19) into an integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \tag{21}$$

and it is in this form Picard came up with his classical solution of this problem, via the so called iteration method. Here we give a simple version of this great theorem. Before that, we would like to present the modern avatar of iteration principle:

Definition 6 Let X be a metric space. By a contraction map on X we mean a function $T : X \to X$ such that there exists a constant $0 < c < 1$ such that for $x, y \in X$ we have

$$d(T(x), T(y)) \leq c\, d(x, y).$$

Remark 10 It is easy to see that every contraction mapping is continuous. The map $f(x) = \lambda x$ on $\mathbb{R}^n$ is a contraction iff $|\lambda| < 1$. The most important property of contraction mapping is:

Theorem 18 Contraction Mapping Principle *On a complete metric space, every contraction mapping T has precisely one fixed point, i.e., there exists exactly one point $t_0 \in X$ such that $T(t_0) = t_0$.*

Proof: First let us prove the uniqueness. If $T(t_1) = t_1$ and $T(t_2) = t_2$ then we have

$$d(T(t_1), T(t_2)) \leq c\, d(t_1, t_2) = Dd(T(t_1), T(t_2))$$

which is absurd unles $t_1 = t_2$. Now starting with any point $t \in X$ define

$$t_1 = T(t), t_2 = T(t_1), \ldots, t_n = T(t_{n-1}.$$

Verfiy that $\{t_n\}$ is a Cauchy's sequence. Since X is a complete metric space, it follows that $t_n \to t_0$ say. Then

$$T(t_0) = T(\lim_n t_n) = \lim_n T(t_n) = \lim_n t_{n+1} = t_0.$$

This completes the proof of the theorem. ♠

Remark 11 This principle has the following wonderful interpretation. Take a map of a country which is 'to the scale' and throw it inside the country.

Then there is (exactly) one point on the map which lies exactly on the point in the country which it represents. You may wonder why it should be true for countries like USA which has several connected components but this is true!

Theorem 19 *Let* $R = [a, b] \times [c, d]$ *and* $f : R \to \mathbb{R}$ *be a continuous real valued valued function and let* M *be a constant such that* f *satisfies the following Lipschitz condition of first order:*

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|, \quad (x, y_j) \in [a, b] \times [c, d]. \tag{22}$$

Given $a < x_0 < b, c < y_0 < d$ *there exists a* $\delta >$ *and a unique function* ϕ *which satisfies (20).*

Proof: Put $K = sup\{|f(x, y)|, (x, y) \in R\}$. Choose $\delta > 0$ so that

$$M\delta < 1; \quad a < x_0 - \delta < x_0 + \delta < b \text{ and } c < y_0 - K\delta < y_0 + K\delta < d.$$

Consider the space $A = \mathcal{C}[x_0 - \delta, x_0 + \delta]$ of all continuous real valued functions on the closed interval. We know that this is a complete metric space. Now consider the subspace B of those $\phi \in A$ such that

$$|\phi(x) - y_0| \leq K\delta.$$

Then B is a closed subspace of A and hence is a complete metric space.

It is important to note that B is nonempty. (Why?)

We consider the map $T : B \to B$ defined by

$$T(\phi)(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt. \tag{23}$$

By theory of Riemann integration, it follows that $T(\phi)$ is continuous. For $x \in [x_0 - \delta, x_0 + \delta]$, we have,

$$|T(\phi)(x) - y_0)| \leq |\int_{x_0}^{x} f(t, \phi(t))dt| \leq K|x - x_0| \leq K\delta.$$

This implies that $T(\phi) \in B$.

Observe that $\phi \in B$ is a solution of (20) iff $T(\phi) = \phi$. Therefore, our aim is to prove that T is a contraction mapping. Given $\phi_j \in B$ consider

$$\begin{array}{rcl} |T(\phi_1)(x) - T(\phi_2)(x)| & = & |\int_{x_0}^{x}(f(t,\phi_1(t)) - f(t,\phi_2(t)))dt| \\ & \leq & M\int_{x_0}^{x}|\phi_1(t) - \phi_2(t)|dt \\ & \leq & M\delta d(\phi_1,\phi_2). \end{array}$$

and since this is true for all $x \in [x_0 - \delta, x_0 + \delta]$ we have
$d(T(\phi_1), T(\phi_2)) = sup\{|T(\phi_1)(x) - T(\phi_2)(x)| \ : \ x \in [x_0 - \delta, x_0 + \delta]\}$
$\leq M\delta d(\phi_1,\phi_2)$. This completes the proof of the theorem. ♠.

Lecture 32. Cantor set

Here we shall define an operator $\mathfrak{C}$ on the class of all closed intervals $[a,b], a < b \in \mathbb{R}$ to the class of compact subsets of $\mathbb{R}$. Given any closed interval $J = [a,b]$, let us define $\phi(J)$ to be the set obtained by deleting the middle-1/3 open interval of J from J. That is,

$$\phi(J) := J \setminus (a + \frac{b-a}{3}, a + 2\frac{b-a}{3}).$$

For any set A which is the finite union of disjoint closed interval $A = \cup_i^k [a_i, b_i]$, define

$$\phi(A) = \cup_i \phi([a_i, b_i])$$

Put $I_0 = [a,b]$ and inductively put $I_n = \phi(I_{n-1}), n \geq 1$. We then have a decreasing sequence of closed subsets

$$I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$$

Put

$$\mathfrak{C}[a,b] := \cap_n I_n.$$

The function $\mathfrak{C}$ is called the Cantor's construction. The set $C = \mathfrak{C}[0,1]$ is called the Cantor set.

The sets $\mathcal{C}[a,b]$ have some wonderful properties:
(a) $\mathcal{C}[a,b]$ is a non empty compact subset of $[a,b], a < b$
(b) If J is one of the connected components of I_n for some n, then $\mathcal{C}(J) \subset \mathcal{C}[a,b]$.
(c) $a, b \in \mathcal{C}[a,b]$.
(d) Let $f(x) = a + (b-a)x$. Then f induces a continuous bijection of $C = \mathcal{C}[0,1]$ with $\mathcal{C}[a,b]$.

From now onward we shall specialize to $C = \mathcal{C}[0,1]$. Each of the properties of C which we list below is carried over to an identical or similar property of $\mathcal{C}[a,b]$ by the similarity map f above.

(e) The end points of every component of $I_n, n \geq 0$ is in C.
(f) The set of all rationals of the form $\sum_1^n \frac{a_k}{3^k}$, where $a_k = 0$ or 2 is contained in C.
(g) C contains no open intervals.
(h) Every point of C is a limit point of C. (Such closed subset of $\mathbb{R}^n$ are called perfect sets.)
(i) C is uncountable.
(j) C is totally disconnected.
(k) C is of length zero.
Proof: (a)-(d) Obvious.
(e) This is an easy consequence of (b) and (c).
(f) This is just the restatement of (e).
(g) Let $J = (c,d)$ be any open interval contained in $[0,1]$. Choose n so that $d - c > 1/3^n$. Then for some i such that $0 \leq i < 3^n$, $J_1 := [\frac{i}{3^n}, \frac{i+1}{3^n}] \subset J$ It follows that I_{n+1} does not contain the middle 1/3 of J_1 and hence $J \not\subset I_{n+1}$.
(h) Let $x \in C$ and J be an interval around x. If n is chosen as above, there is a unique i such that $0 \leq i < 3^n$ such that $x \in [\frac{i}{3^n}, \frac{i+1}{3^n}] = J_1$. Now both the

end points of J_1 are in C. One of them is not equal to x and has to be inside J. Hence $J \cap C \neq \emptyset$.
(i) This can be deduced from the fact that C is a perfect set. Here is an easier way. From (f), since C is closed it follows that every number represented as an infinite sum

$$\sum_1^\infty \frac{a_k}{3^k}$$

belongs to C. Let A be the set of all sequences $\alpha : \mathbb{N} \to \{0, 2\}$. We know that A is uncountable. The assignment

$$(a_k) \mapsto \sum_k^\infty \frac{a_k}{3^k}$$

defines an injective mapping of A into C.
(j) Given any two points $x < y \in C$, since the interval $[x, y]$ is not contained in C, there exists $z \notin C$, such that $x < z < y$. Then $\{[0, z] \cap C, [z, 1] \cap C\}$ defines a separation of C.
(k) This follows by the fact that $\sum_1^\infty \frac{2^{n-1}}{3^n} = 1$.

Lecture 33

Definition 7 By a box R in $\mathbb{R}^n$ we mean a product of intervals

$$R = \Pi_{i=1}^n [a_i, b_i] \tag{24}$$

The volume of the box R is defined to be $\mu(R) := \Pi(b_i - a_i)$.

Definition 8 Let $A \subset \mathbb{R}^n$. We say A has *measure zero* if for every $\epsilon > 0$, there exists a countable cover $\{R_1, \ldots, R_k, \ldots\}$ of A by boxes such that $\sum_i \mu(R_i) < \epsilon$.

Remark 12
(a) If $B \subset A$ then A has measure zero implies so has B.
(b) Any countable subset of $\mathbb{R}^n$ has measure zero. More generally, a countable

union of measure zero sets has measure zero.
(c) In the definition, we can use open boxes instead of closed ones. We can even use closed balls or open balls instead of rectangles. All these give the same notion of measure zero.
(d) If A is a compact subset of measure zero, then for every $\epsilon > 0$, there exists a finite cover of A by closed boxes $\{R_1, \ldots, R_k\}$ such that $\sum_i \mu(R_i) < \epsilon$.
(e) For any finite cover of $[a, b]$ by closed intervals $[a_i, b_i]$, we have $\sum_i (b_i - a_i) \geq b - a$. Thus we are happy that the interval $[a, b]$ for $a < b$ is not of measure zero.
(f) The following lemma tells you that being 'measure-zero' is a kind of local property of sets.

Theorem 20 *The cantor set is of measure zero in* $\mathbb{R}$.

Lemma 5 $A \subset \mathbb{R}^n$ is of measure zero in $\mathbb{R}^n$ iff $A \cap U$ is of measure zero for every open subset in $\mathbb{R}^n$.

Proof: Cover A by a family of open sets in $\mathbb{R}^n$. Then by Lindelöff property for subsets of $\mathbb{R}^n$, there exists a countable subcover $\{U_i\}$ for A. Now each $A \cap U_i$ is of measure zero and hence for each $\epsilon > 0$, there is a countable cover of $A \cap U_i$ such that the total measure is $< \epsilon/2^i$. The collection of all these open sets forms a cover for A with the property that the total measure is $< \sum \epsilon/2^i = \epsilon$. The converse is obvious from remark 12(a). ♠

Theorem 21 *Let* $A \subset \mathbb{R}^n$ *be of measure zero and* $f : A \longrightarrow \mathbb{R}^n$ *be any smooth function. Then* $f(A)$ *is of measure zero.*

Proof: Given a point $x \in A$, choose a ball B around x so that $f|\bar{B} \cap A$ is the restriction of a smooth function $\hat{f} : \bar{B} \longrightarrow \mathbb{R}^n$. Since A can be covered by a countable union of such balls, we may as well assume that $A \subset B \subset \bar{B}$ and $f : \bar{B} \longrightarrow \mathbb{R}^n$ is a smooth map. By continuity and compactness of $\bar{B}$

we can now choose $M > 0$ such that $\|Df_x\| \leq M$ for all $x \in \bar{B}$. A simple application of weak mean value theorem **??** now yields that

$$\|f(x) - f(y)\| \leq M\|x - y\|, \quad \forall\, x, y \in \bar{B}.$$

Therefore, if D is a disc of radius r in $\bar{B}$, it follows that $f(D)$ is contained in a disc of radius rM. Now given $\epsilon > 0$ cover A by balls D_i of radius r_i such that $\sum vol_n(D_i) < \epsilon$. It follows that $f(A)$ is contained in a countable collection of balls of total volume $< M\epsilon$. ♠

Lecture 34

Theorem 22 *Every non constant polynomial in one variable with coefficients in* $\mathbb{C}$ *has at least one root in* $\mathbb{C}$.

This theorem is usually stated immediately after introducing complex numbers and is taken for granted and is heavily used. Most probably the first time a student comes across a proof of this is in a first course is complex analysis. Here is a proof of FTA which can be taught to any student who has taken a first course in real analysis. All the student needs to know is the Intermediate Value Theorem and Weierstrass's theorem that every continuous real valued function on a closed and bounded subset of $\mathbb{R}^2$ attains its infimum.

Consider the following statement:

Theorem 23 *A polynomial in 1-variable of odd degree and having all real coefficients has a real root.*

If you ask any school student to supply reason for this, it is most likely that you will get an answer similar to the following: '*Since the roots occur in pairs such that one is the complex conjugate of the other and since there are odd number of roots there must be one which is real'* The trouble with this reasoning is that it uses the FTA and many proofs of FTA use 23 which amounts to a circular argument. Even in the proof of FTA that you learn in complex analysis, the so called Intermediate Value Property is implicitly used, even though the above result 23 is not directly used.

Theorem 23 can be proved directly by using Intermediate Value Theorem:

Theorem 24 *Let $f : [a, b] \to \mathbb{R}$ be a continuous and let $f(a) < t < f(b)$. Then there exists $a < c < b$ such that $f(c) = t$.*

If $p(x)$ is an odd degree polynomial with its leading coefficient positive, we first observe that

$$\lim_{t\to\pm\infty} p(t) = \pm\infty$$

respectively. Thus there exist $a < 0$ and $b > 0$ such that $p(a) < 0$ and $p(b) > 0$. Now $p(a) < 0 < p(b)$ and from IVP, we conclude that there is $a < t < b$ such that $p(t) = 0$.

In this talk, we shall see how to prove FTA directly using IVP and the fundamental result due to Weierstrass:

Theorem 25 *Every continuous real valued function on a closed and bounded subset of $\mathbb{R}^2$ attains it infimum.*

Toward the proof of FTA, let us fix a polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n, a_j \in \mathbb{C}, a_n \neq 0, n \geq 1$.

Lemma 6 $|p(z)| \to \infty$ as $|z| \to \infty$.

Proof:

$$|p(z)| = |a_n z_n| \left| 1 + \frac{a_{n-1}}{a_n z_n} + \cdots + \frac{a_0}{a_n z_n} \right| \geq |a_n z^n| \left| \left(1 - \left| \frac{a_{n-1}}{a_n z} \right| \cdots - \left| \frac{a_0}{a_n z^n} \right| \right) \right|.$$

Now take the limit as $|z| \longrightarrow \infty$. ♠

Lemma 7 $|p| : \mathbb{C} \to \mathbb{R}$ attains its infimum.

Proof: We have to show that there exists $z_0 \in \mathbb{C}$ such that $|p(z_0)| \leq |p(z)|$ for all $z \in \mathbb{C}$.

In the above lemma we have seen that $|p(z)| \longrightarrow \infty$ as $|z| \longrightarrow \infty$. This means that there exists $R > 0$ such that $|p(z)| > |p(0)|$ for all $|z| > R$. It follows that

$$\text{Inf } \{|p(z)| \ : \ z \in \mathbb{C}\} = \text{Inf } \{|p(z)| \ : \ |z| \leq R\} \leq |p(0)|.$$

But the disc $\{z \ : \ |z| \leq R\}$ is closed and bounded. Since the function $z \mapsto |p(z)|$ is continuous, it attains its infimum on this disc. This completes the proof of the lemma. ♠

Slowly but surely, now an idea of the proof of FTA emerges: Observe that FTA is true iff the infimum z_0 obtained in the above lemma is a zero of p, i.e., $p(z_0) = 0$. Therefore in order to complete a proof of FTA, it is enough to assume that $p(z_0) \neq 0$ and arrive at a contradiction. (This idea is essentially due to Argand.)

Consider the polynomial $q(z) = p(z + z_0)$. Both the polynomials, p, q have the same value set and hence minimum of $|q(z)|$ is equal to minimum of $|p(z)|$ which is equal to $|p(z_0)| = |q(0)|$.

We shall assume that $q(0) \neq 0$ and arrive a contradiction.

Write $q(z) = q(0)\phi(z)$. Then $\phi(0) = 1$ and hence we can write

$$\phi(z) = 1 + wz^k + z^{k+1}f(z)$$

for some $w \in \mathbb{C} \setminus \{0\}$, some integer $k \geq 1$ and some polynomial $f(z)$. Observe that $|q(0)|$ is the minimum of $|q(z)|$ iff 1 is the minimum of $|\phi(z)|$. It is enough to prove that

Lemma 8 Argand's Inequality For any polynomial f, positive integer k, and any $w \in \mathbb{C} \setminus \{0\}$,

$$\text{Min}\{|1 + wz^k + z^{k+1}f(z)| \ : \ z \in \mathbb{C}\} < 1. \tag{25}$$

Choose $r > 0$ such that $r^k = |w|$. (By IVP, there is such a real number and indeed a unique one.) Now replace z by z/r^k in (25). Thus, we may assume $|w| = 1$ in (25).

At this stage, Argand's proof uses de Moivre's theorem, viz., for every complex number α of modulus 1 and every positive integer k, the equation $z^k = \alpha$ has a solution. Strictly speaking this is cannot be allowed, since the proof of de Moivre's theorem uses measurement of an angle, which needs to be establised rigorously first. We shall see how to avoid this in a easy way. However, first we continue with Argand's proof of this lemma , because of its simplicity.

Choose λ such that $\lambda^k = -w^{-1}$. Replace z by λz in (25) to reduce it to proving

$$\mathrm{Min}\{|1 - z^k + z^{k+1}g(z)| \ : \ z \in \mathbb{C}\} < 1. \tag{26}$$

Now restrict z to positive real numbers, $z = t > 0$. Since $g(t)$ is a polynomial, $tg(t) \to 0$ as $t \to 0$. So there exists $0 < t < 1$ for which $|tg(t)| < 1/2$. But then

$$|1 - t^k + t^{k+1}g(t)| < |1 - t^k| + \frac{t^k}{2} = 1 - t^k + \frac{t^k}{2} < 1$$

thereby completing the proof of (25). This also completes a proof of FTA.

Why do we want to avoid using de Moivre's Theorem? As I have pointed out earlier, the answer is that it depends heavily upon the intuitive concept of the angle which needs to be established rigorously. (It should also be noted that during Argand's time, one could not expect a rigorous proof of lemma 7, which Argand simply assumed.[2] There are several ways to make this rigorous. For instance, we can introduce the notion of arc-length and then define the angle to be the length of the arc in the unit circle with center

[2]For more learned comments, see R. Remmert's article on 'Fundamental Theorem of Algebra' in the book *Numbers* edited by H.-D. Ebbinghaus et al, and published by GTM-RIM 123, Springer-Verlag, 1990.

at the point of intersection of given two lines. We can then develop all the required basic properties of the 'angle' and the real trigonometric functions leading to a proof of De Moivre's theorem.

Instead, we now follow an idea due to Littlewood in which one proves a weaker form of De Moivre's theorem:

Lemma 9 For any integer $n \geq 1$, the four equations

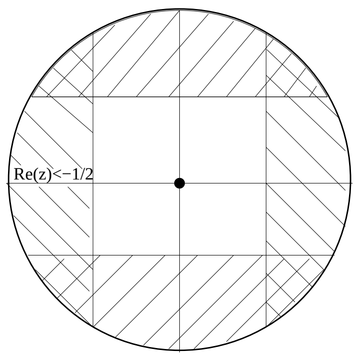
$$z^n = \pm 1; \quad z^n = \pm \imath; \tag{27}$$

have all solutions in $\mathbb{C}$.

The proof of this is achieved by two simple observations:

Lemma 10 Given any complex number w of modulus 1, one of the four numbers $\pm w, \pm \imath w$ has its real part less than $-1/2$.

Proof: [This is seen easily as illustrated in the Fig. 1. The four shaded regions which cover the whole of the boundary are got by rotating the region $\Re(z) < -1/2$. However, it is important to note that the following proof is completely independent of the picture.] Since $|w| = 1$, either $|\Re(w)|$ or $|\Im(w)|$ has to be bigger than 1/2. In the former case, one of $\pm w$ will have the required property. In the latter case, one of $\pm \imath w$ will do. ♠



The second observation is:

Lemma 11 Given any complex number $\alpha + \imath\beta$ there exists $z \in \mathbb{C}$ such that $z^2 = \alpha + \imath\beta$.

[**Proof:** Write $z = x + \imath y$. We must find real numbers x, y such that

$$x^2 - y^2 = \alpha; \quad 2xy = \beta.$$

Consider the case when $\beta = 0$. If $\alpha \geq 0$ take $x = \sqrt{\alpha}$ and $y = 0$. If $\alpha < 0$ then take $x = 0$ and $y = \sqrt{-\alpha}$. Now consider the case $\beta \neq 0$. It follows that $y \neq 0$ and we hence can eliminate y to obtain $4x^4 - 4x^2\alpha - \beta^2 = 0$. This can be solved for x^2 by method of completing the square: $(2x^2 - \alpha)^2 = \alpha^2 + \beta^2$ If $c = +\sqrt{\alpha^2 + \beta^2}$, then $c + \alpha > 0$. and so, we take $x = \sqrt{c + \alpha}$. Finally we put $y = \beta/2x$ to complete the solution.]

Now we can complete the proof of lemma 9.

Proof: of 9 Write $n = 2^k m$, where $m = 4l + 1$ or $4l + 3$. For $k \geq 0$, since we can take successive square-roots let $\alpha_k, \beta_k, \gamma_k$ be such that

$$\alpha_k^{2^k} = -1, \quad \beta_k^{2^k} = \imath, \quad \gamma_k^{2^k} = -\imath.$$

(For $k = 0$, this just means $\alpha_0 = -1; \beta_0 = \imath, \gamma_0 = -\imath$.)

Now let us take the four equations one by one:

(a) For $z^n = 1$, we can always take $z = 1$.

(b) For equation $z^n = -1$, take $z = \alpha_k$. Then $(\alpha_k)^n = (-1)^m = -1$.

(c) For the equation $z^n = \imath$: Take $z = \beta_k$, if $m = 4l+1$. Then $(\beta_k)^n = (\imath)^m = \imath$. If $m = 4l + 3$ then take $z = \gamma_k$ so that $(\gamma_k)^n = (-\imath)^m = (-\imath)^3 = \imath$.

(d) This case follows easily from (b) and (c). Choose z_1, z_2 such that $z_1^n = -1$ and $z_2^n = \imath$. Then $(z_1 z_2)^n = -\imath$. ♠

[At this stage, the proof given in literature first establishes de Moivre's theorem and then follows the arguments given above. Here, we shall directly derive Argand's inequality.]

Returning to the proof of lemma , choose $\tau = \pm 1, \pm\imath$ so that $\Re(\tau w) < -\dfrac{1}{2}$ (Lemma 10). Choose $\alpha \in \mathbb{C}$ such that $\alpha^k = \tau$ (Lemma 9).

Now, replace z by αz, so that we may assume that $w = a + \imath b$, where $a \le -1/2$ and $a^2 + b^2 = 1$.

Since f is continuous, it follows that $tf(t) \to 0$ as $t \to 0$. Restricting to just the real values of t, we can choose $0 < \delta < 1$ such that $|tf(t)| < 1/3$ for all $0 < t < \delta$. For such a choice of t, we have

$$|1 + wt^k + t^{k+1}f(t)| \le |1 + wt^k| + \frac{t^k}{3} = [(1 + at^k)^2 + b^2t^{2k}]^{1/2} + \frac{t^k}{3}.$$

We want to choose $0 < t < \delta$ such that this quantity is less than 1. For $a^2 + b^2 = 1$ and $t > 0$ we have

$$\begin{array}{lll} & [(1+at^k)^2 + b^2t^{2k}]^{1/2} + t^k/3 < 1 & \text{iff} \;\; [(1+at^k)^2 + b^2t^{2k}]^{1/2} < 1 - \dfrac{t^k}{3} \\ \text{iff} & (1+at^k)^2 + b^2t^{2k} < \left(1 - \dfrac{t^k}{3}\right)^2 & = \;\; 1 - \dfrac{2t^k}{3} + \dfrac{t^{2k}}{9} \\ \text{iff} & 1 + 2at^k + t^{2k} < 1 - \dfrac{2t^k}{3} + \dfrac{t^{2k}}{9} & \text{iff} \;\; \dfrac{8}{9}t^k < -\left(2a + \dfrac{2}{3}\right), \quad t > 0. \end{array}$$

This last condition can be fulfilled by choosing $t > 0$ such that $t^k < 3/8$, for then,

$$\frac{8}{9}t^k < \frac{1}{3} < -\left(2a + \frac{2}{3}\right).$$

Thus, for any $t > 0$ which is such that $t^k < \min\{3/8, \delta\}$ (IVP again), we have

$$|1 + wt^k + t^{k+1}f(t)| < 1.$$

This completes the proof of the lemma and thereby that of FTA. ♠