Third Week Lectures 7-9

Lecture 7

Series

Given two numbers, we can add them to get another number. Repeatedly carrying out this operation allows us to talk about sums of any finitely many numbers. We would like to talk about 'sum' of infinitely many numbers as well. A natural way to do this is to label the given numbers, take sums of first n of them and look at the 'limit' of the sequence of numbers so obtained.

Thus given a (countable) collection of numbers, the first step is to label them to get a sequence $\{s_n\}$. In the second step, we form another sequence: the sequence of partial sums $t_n = \sum_{k=1}^{n} s_k$. Observe that the first sequence $\{s_n\}$ can be recovered completely from the second one $\{t_n\}$. The third step is to assign a limit to the second sequence provided the limit exists. This entire process is coined under a single term 'series'. However, below, we shall stick to the popular definition of a series.

Definition 14 By a *series* of real or complex numbers we mean a formal infinite sum:

$$\sum_n s_n := s_0 + z_1 + \cdots + s_n + \cdots$$

Of course, it is possible that there are only finitely many non zero terms here. The *sequence of partial sums* associated to the above series is defined to be $t_n := \sum_{k=1}^{n} z_k$. We say the series $\sum_n s_n$ is *convergent* to the sum s if the associated sequence $\{t_n\}$ of partial sums is convergent to s. In that case, if s is the limit of this sequence, then we say s is the *sum* of the series and write

$$\sum_n z_n := s.$$

It should be noted that that even if s is finite, it is not obtained via an arithmentic operation of taking sums of members of $\{s_n\}$ but by taking the limit of the associated sequence $\{t_n\}$ of partial sums. Since displaying all elements of $\{t_n\}$ allows us to recover the original sequence $\{s_n\}$ by the formula $s_n = t_{n+1} - t_n$ results that we formulate for sequences have their counterpart for series and vice versa and hence in principle we need to do this only for one of them. For example, we can talk a series which is the sum of two series $\sum_n a_n, \sum_n b_n$ viz. $\sum_n (a_n + b_n)$ and if both $\sum_n a_n, \sum_n b_n$ are convergent to finite sums then the sum series $\sum_n (a_n + b_n)$ is convergent to the sum of the their sums.

Nevertheless, it is good to go through these notions. For example the Cauchy's criterion for the convergence of the sequence $\{t_n\}$ can be converted into

Theorem 14 *A series $\sum_n s_n$ is convergent to a finite sum iff for every $\epsilon > 0$ there exists n_0 such that $|\sum_{k=n}^{m} s_n| < \epsilon$, for all $m, n \geq n_0$.*

As a corollary we obtain

Corollary 1 *If $\sum s_n$ is convergent to a finite sum then $s_n \to 0$.*

Of course the converse does not hold as seen by the harmonic series $\sum_n \frac{1}{n}$.

Once again it is immediate that if $\sum_n z_n$ and $\sum_n w_n$ are convergent series then for any complex number λ, we have, $\sum_n \lambda z_n$ and $\sum_n (z_n + w_n)$ are convergent and

$$\boxed{\boxed{\sum_n \lambda z_n = \lambda \sum_n z_n; \quad \sum_n (z_n + w_n) = \sum_n z_n + \sum_n w_n.}} \tag{5}$$

Theorem 15 *A series of positive terms $\sum_n a_n$ is convergent iff the sequence of parial sums is bounded.*

Theorem 16 Comparison Test
(a) If $|a_n| \leq c_n$ for all $n \geq n_0$ for some n_0, and $\sum_n c_c$ is cgt then $\sum_n a_n$ is convergent.
(b) If $a_n \geq b_n \geq 0$ for all $n \geq n_0$ for some n_0 and $\sum_n b_n$ diverges implies $\sum_n a_n$ diverges.

The geometric series is the mother of all series:

Theorem 17 Geometric Series *If $0 \leq |x| < 1$ then $sum_n x^n = \frac{1}{1-x}$. If $|x| > 1$, then the series diverges.*

Here the partial sum sequence is given by $t_n = \frac{1-x^{n+1}}{1-x} \to \frac{1}{1-x}$.

Theorem 18 *The series $\sum_n \frac{1}{n!}$ is cgt and its sum is denoted by e. We have, $2 < e < 3$.*

Proof: For $n \geq 2$, we have,

$$2 < t_n = 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}} < 1+\frac{1}{1-1/2} = 3.$$

Theorem 19 $\lim_n \left(1+\frac{1}{n}\right)^n = e.$

Proof: Put $t_n = \sum_{k=0}^{n} \frac{1}{k!}, r_n = \left(1+\frac{1}{n}\right)^n$. Then

$$r_n = 1 + 1 + \frac{n-1}{2!n} + + \cdots + \frac{(n-1)!}{n!n^{n-1}} < t_n.$$

Therefore $\limsup r_n \leq e$. On the other hand, for a fixed m if $n \geq m$, we have

$$r_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

Therefore

$$\liminf_n r_n \geq t_m.$$

Since this true for all m, we get $\liminf_n r_n \geq 3$. ♠

Remark 15 The rapidity with which this sequence converges is estimated by considering:

$$e - t_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!}[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots] = \frac{1}{n!n}.$$

Thus

$$0 < e - t_n < \frac{1}{n!n}.$$

Corollary 2 *e is irrational.*

Proof: Assume on the contrary that $e = \frac{p}{q}$. Then $q!e$ and $q!t_q$ are both integers. On the other hand $0 < q!e - q!t_q < \frac{1}{q}$ which is absurd. ♠

Definition 15 A series $\sum_n a_n$ is said to be absolutely convergent if the series $\sum_n |a_n|$ is convergent to a finite limit.

Theorem 20 *Suppose $\{a_n\}$ is a decreasing sequence of positive terms, then $\sum_{n=0}^{\infty} a_n$ is cgt iff $\sum_k 2^k a_{2^k}$ is cgt.*

Proof: Put $t_n = \sum_{k=1}^{n} a_k, T_n = \sum_{k=1}^{n} 2^k a_{2^k}$. Check that $t_n \leq t_{2^n} \leq T_n + a_0$ and $2a_0 + a_1 + T_n \leq 2t_{2^n-1}$.

Theorem 21 $\sum_n \frac{1}{n^p} < \infty$ *iff* $p > 1$.

Corollary 3 *The harmonic series is divergent.*

Theorem 22 *The series* $\sum_{n=2}^{\infty} \frac{1}{(n \ln n)^p}$ *is convergent iff* $p > 1$.

Theorem 23 **Ratio Test:** *If* $\{a_n\}$ *is a sequence of positive terms such that*

$$\limsup_n \frac{a_{n+1}}{a_n} = r < 1,$$

then $\sum_n a_n$ *is convergent. If* $\frac{a_{n+1}}{a_n} \geq 1$ *for all* $n \geq n_0$ *for some* n_0, *then* $\sum_n a_n$ *is divergent.*

Proof: To see the first part, choose s so that $r < s < 1$. Then there exists N such that $\frac{a_{n+1}}{a_n} < s$ for all $n \geq N$. This implies $a_{N+k} < a_N s^k, k \geq 1$. Since the geometric series $\sum_k s^k$ is convergent, the convergence of $\sum_n a_n$ follows. The second part is obvious since a_n cannot converge to 0. ♠

Theorem 24 **Root Test** *For sequence* $\{a_n\}$ *of positive terms, put* $l = \limsup_n \sqrt[n]{a_n}$. *Then*
(a) $l < 1 \Longrightarrow \sum_n a_n < \infty$.
(b) $l > 1 \Longrightarrow \sum_n a_n = \infty$.
(c)$l = 1$ *the series* $\sum_n a_n$ *can be finite or infinite.*

Proof: Choose $l < r < 1$ and then an integer N such that $\sqrt[n]{a_n} < r$ for all $n \geq N$. Therefore $a_n < r^n$ and we can now compare with the geometric series. The proof of (b) is also similar. (c) is demostrated by the series $\sum_n \frac{1}{n}$ and $\sum_n \frac{1}{n^2}$. ♠

Remark 16 As compared to ratio test, root test is more powerful, in the sense, whereever ratio test is conclusive so is root test. Also there are cases when ration test fails but root test holds. However, ratio test is easier to apply.

Example 3 Put $a_{2n+1} = \frac{1}{2^{n+1}}, a_{2n} = \frac{1}{3^n}$. Then
$\liminf_n \frac{a_{n+1}}{a_n} = \lim_n \frac{2^n}{3^n} = 0; \quad \liminf_n \sqrt[n]{a_n} = \lim_n \sqrt[2n]{\frac{1}{3^n}} = \sqrt{\frac{1}{3}}$.

$\limsup_n \frac{a_{n+1}}{a_n} = \lim_n \left(\frac{3}{2}\right)^n = \infty$; $\limsup_n \sqrt[n]{a_n} = \lim_n \sqrt[2n+1]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}}$. The ratio test cannot be applied. The root test gives the convergence.

The following theorem proves the claim that we have made in the above remark.

Theorem 25 *For any sequence $\{a_n\}$ of positive terms,*

$$\liminf_n \frac{a_{n+1}}{a_n} \leq \liminf_n \sqrt[n]{a_n} \leq \limsup_n \sqrt[n]{a_n} \leq \limsup_n \frac{a_{n+1}}{a_n}.$$

Lecture 8

Example 4

1. Let $z_n = x_n + \imath y_n, n \geq 1$. Show that $z_n \to z = x + \imath y$ iff $x_n \to x$ and $y_n \to y$.

2. **Telescoping:** Given a sequence $\{x_n\}$ define the difference sequence $a_n := x_n - x_{n+1}$. Then show that the series $\sum_n a_n$ is convergent iff the sequence $\{x_n\}$ is convergent and in that case, $\sum_n a_n = x_0 - \lim_{n \longrightarrow \infty} x_n$.

Definition 16 A series $\sum_n z_n$ is said to be absolutely convergent if the series $\sum_n |z_n|$ is convergent.

Again, it is easily seen that an absolutely convergent series is convergent, whereas the converse is not true as seen with the standard example $\sum_n (-1)^n \frac{1}{n}$. The notion of absolute convergence plays a very important role throughout the study of convergence of series.

Theorem 26 *Let $\sum_n z_n$ be an absolutely convergent series. Then every rearrangement $\sum_n z_{\sigma_n}$ of the series is also absolutely convergent, and hence convergent. Moreover, each such rearrangement converges to the same sum.*

Proof: Recall that a rearrangement $\sum_n z_{\sigma_n}$ of $\sum_n z_n$ is obtained by taking a bijection $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$.) Let $\sum_n z_n = z$. The only thing that needs a proof at this stage is that $\sum_n z_{\sigma(n)} = z$. Let us denote the partial sums $s_n = \sum_{k=0}^n a_k \; t_n = \sum_{k=0}^n a_{\sigma(k)}$. Since $\sum_n a_n$ is absolutely convergent given $\epsilon > 0$ there is a N such that $\sum_{k=n}^m |a_k| < \epsilon$ for all $m \geq n \geq N$. Pick up N_1 large enough so that

$$\{1, 2, \ldots, N\} \subset \{\sigma(1), \sigma(2), \ldots \sigma(N_1)\}.$$

Then for $n \geq N_1$, we have $|s_n - t_n| \leq \sum_{k=N+1}^n |a_n| < \epsilon$. Therefore, $\lim_n s_n = \lim_n t_n$. ♠

Riemann's rearrangment Theorem: Let $\sum a_n$ be a convergent series of real numbers which is **not** absolutely convergent. Given $-\infty \leq \alpha \leq \beta \leq \infty$, there exists a rearrangements $\sum_n a_{\tau(n)}$ of $\sum_n a_n$ with partial sums t_n such that

$$\liminf_n t_n = \alpha; \quad \limsup_n t_n = \beta.$$

We are not going to prove this. See [R] for a proof.

Example 5 1. Let $\{z_n\}$ be a bounded sequence and $\sum_n w_n$ is an absolutely convergent series. Show that $\sum_n z_n w_n$ is absolutely convergent.

2. **Abel's Test:** For any sequence of complex numbers $\{a_n\}$, define $S_0 = 0$ and $S_n = \sum_{k=1}^n a_k, \; n \geq 1$. Let $\{b_n\}$ be any sequence of complex numbers.
(i) Prove **Abels' Identity**:

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} S_k(b_k - b_{k+1}) - S_{m-1} b_m + S_n b_n, \; 1 \leq m \leq n.$$

$(LHS = \sum (S_k - S_{k-1}) b_k = \sum_m^n S_k b_k - \sum_{m-1}^{n-1} S_k b_{k+1} = RHS.)$
(ii) Show that $\sum_n a_n b_n$ is convergent if the series $\sum_k S_k(b_k - b_{k+1})$

is convergent and $\lim_{n\longrightarrow\infty} S_n b_n$ exits.(Put $m = 1$.)

(iii) **Abel's Test:** Let $\sum_n a_n$ be a convergent series and $\{b_n\}$ be a bounded monotonic sequence of real numbers. Then show that $\sum_n a_n b_n$ is convergent.

(Solution:$\sum_n (b_n - b_{n+1})$ is convergent by Telescoping and absolutely, since $\{b_n\}$ is monotonic. The series $\sum_n a_n$ is convergent and hence $\{S_n\}$ is bounded. By the previous exercise, the product series is convergent. Since both S_n and b_n are convergent $S_n b_n$ is convergent. Therefore, (ii) applies.

3. **Dirichlet's Test:** Let $\sum_n a_n$ be such that the partial sums are bounded and let $\{b_n\}$ be a monotonic sequence tending to zero. Then show that $\sum_n a_n b_n$ is convergent.

   (Arguements are already there in above eaxmple)

4. Derive the following **Leibniz's test** from Dirichlet's Test: If $\{c_n\}$ is a monotonic sequence converging to 0 then the alternating series $\sum_n (-1)^n c_n$ is convergent.

   (Take $a_n = (-1)^n$ and $b_n = c_n$ in Dirichlet's test.)

5. Generalize the Leibniz's test as follows: If $\{c_n\}$ is a monotonic sequence converging to 0 and ζ is complex number such that $|\zeta| = 1$, $\zeta \neq 1$, then $\sum_n \zeta^n a_n$ is convergent.

Exercise 8 Show that if $\sum_n a_n$ is convergent then the following sequences are all convergent.

(a) $\displaystyle\sum_n \frac{a_n}{n^p}, p > 0$; (b) $\displaystyle\sum_n \frac{a_n}{log_p n}$; (c) $\sum_n \sqrt[n]{n} a_n$; (d) $\displaystyle\sum_n \left(1 + \frac{1}{n}\right)^n a_n$;

Show that for any $p > 0$, and for every real number x, $\displaystyle\sum_n \frac{\sin nx}{n^p}$ is convergent.

Lecture 9

Definition 17 Given two series $\sum_n a_n, \sum_n b_n$, the Cauchy product of these two series is defined to be $\sum_n c_n$, where $c_n = \sum_{k=0}^n a_k b_{n-k}$.

Theorem 27 *If $\sum_n a_n, \sum_n b_n$ are two absolutely convergent series then their Cauchy product series is absolutely convergent and its sum is equal to the product of the sums of the two series:*

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right). \tag{6}$$

Proof: Consider the remainder after $n-1$ terms of the corresponding absolute series:

$$R_n = \sum_{k \geq n} |a_k|; \quad R'_n = \sum_{k \geq n} |b_k|.$$

Clearly,

$$\sum_{0 \geq k \leq n} |c_k| \leq (\sum_{k \leq n} |a_k|)(\sum_{l \leq n} |b_l|) \leq R_0 T_0.$$

Thus the partial sums of the series $\sum_n |c|k_k$ forms a monotonically increasing sequence which is bounded above. Therefore the series $\sum_n c_n$ is absolutely convergent. Further,

$$\left|\sum_{k \leq 2n} c_k - \left(\sum_{k \leq n} a_k\right)\left(\sum_{k \leq n} b_k\right)\right| \leq R_0 T_{n+1} + T_0 R_{n+1},$$

since the terms that remain on the LHS after cancellation are of the form $a_k b_l$ where either $k \geq n+1$ or $l \geq n+1$. Upon taking the limit as $n \longrightarrow \infty$, we obtain (6). ♠

Remark 17 This theorem is true even if one of the two series is absolutely convergent and the other is convergent. For a proof of this, see [R].

Definition 18 By a *formal power series in one variable* t over $\mathbb{K}$, we mean a sum of the form

$$\sum_{n=0}^{\infty} a_n t^n, \quad a_n \in \mathbb{K}.$$

Note that for this definition to make sense, the sequence $\{a_n\}$ can be inside any set. However, we shall restrict this and assume that the sequences are taken inside field $\mathbb{K}$. Let $\mathbb{K}[[t]]$ denote the set of all formal power series $\sum_n a_n t^n$ in t with coefficients $a_n \in \mathbb{K}$. Observe that when at most a finite number of a_n are non zero the above sum gives a polynomial. Thus, all polynomials in t are power series in t, i.e., $\mathbb{K}[t] \subset \mathbb{K}[[t]]$.

Just like polynomials, we can add two power series 'term-by-term' and we can also multiply them by scalars, viz.,

$$\sum_n a_n t^n + \sum_n b_n t^n := \sum_n (a_n + b_n) t^n; \quad \alpha(\sum_n a_n t^n) := \sum_n \alpha a_n t^n.$$

Verified that the above two operations make $\mathbb{K}[[t]]$ into a vector space over $\mathbb{K}$.

Further, we can even multiply two formal power series:

$$\left(\sum_n a_n t^n\right)\left(\sum_n b_n t^n\right) := \sum_n c_n t^n,$$

where, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. This product is called the *Cauchy product.*

One can directly check that $\mathbb{K}[[t]]$ is then a commutative ring with the multiplicative identity being the power series

$$1 := \sum_n a_n t^n$$

where, $a_0 = 1$ and $a_n = 0, n \geq 1$. Together with the vector space structure, $\mathbb{K}[[t]]$ is actually a $\mathbb{K}$-algebra.) Observe that the ring of polynomials in t forms a subring of $\mathbb{K}[[t]]$. What we are now interested in is to get nice functions out of power series.

Observe that, if $p(t)$ is a polynomial over $\mathbb{K}$ then by the **method of substitution,** it defines a function $a \mapsto p(a)$, from $\mathbb{K}$ to $\mathbb{K}$. It is customary to denote this map by $p(t)$ itself. However, due to the infinite nature of the sum involved, given a power series P and a point $a \in \mathbb{K}$, the substitution $P(a)$ may not make sense in general. This is the reason why we have to treat power series with a little more care, via the notion of convergence.

Definition 19 A formal power series $P(t) = \sum_n a_n z^n$ is said to be *convergent at* $z_0 \in \mathbb{C}$ if the sequence $\{s_n\}$, where, $s_n = \sum_{k=0}^{n} a_k z_0^k$ is convergent. In that case we write $P(z_0) = \lim_{n\to\infty} s_n$ for this limit. Putting $t_n = a_n z_0^n$, this just means that the series of complex numbers $\sum_n t_n$ is convergent.

Remark 18 Observe that every power series is convergent at 0.

Definition 20 A power series is said to be a convergent power series, if it is convergent at some point $z_0 \neq 0$.

The following few theorems, which are attributed to Cauchy-Hadamard[1] and Abel[2], are most fundamental in the theory of convergent power series.

Theorem 28 Cauchy-Hadamard Formula: *Let* $P = \sum_{n\geq 0} a_n t^n$ *be a power series over* $\mathbb{C}$. *Put* $L = \limsup_n \sqrt[n]{|a_n|}$ *and* $R = \frac{1}{L}$ *with the*

[1]Jacques Hadamard(1865-1963) was a French Mathematician who was the most influential mathematician of his days, worked in several areas of mathematics such as complex analysis, analytic number theory, partial differential equations, hydrodynamics and logic.

[2]Niels Henrik Abel (1802-1829) was a Norwegian, who died young under deprivation. At the age of 21, he proved the impossibility of solving a general quintic by radicals. He did not get any recognition during his life time for his now famous works on convergence, on so called abelian integrals, and on elliptic functions.

convention $\frac{1}{0} = \infty$; $\frac{1}{\infty} = 0$. *Then*
(a) *for all* $0 < r < R$, *the series* $P(t)$ *is absolutely and uniformly convergent in* $|z| \leq r$ *and*
(b) for all $|z| > R$ *the series is divergent.*

Proof: (a) Let $0 < r < R$. Choose $r < s < R$. Then $1/s > 1/R = L$ and hence by property (Limsup-I), we must have n_0 such that for all $n \geq n_0$, $\sqrt[n]{|a_n|} < 1/s$. Therefore, for all $|z| \leq r$, $|a_n z^n| < (r/s)^n$, $n \geq n_0$. Since $r/s < 1$, by Weierstrass majorant criterion, (Theorem **??**), it follows that $P(z)$ is absolutely and uniformly convergent.
(b) Suppose $|z| > R$. We fix s such that $|z| > s > R$. Then $1/s < 1/R = L$, and hence by property (Limsup-II), there exist infinitely many n_j, for which $\sqrt[n_j]{|a_{n_j}|} > 1/s$. This means that $|a_{n_j} z^{n_j}| > (|z|/s)^{n_j} > 1$. It follows that the n^{th} term of the series $\sum_n a_n z^n$ does not converge to 0 and hence the series is divergent. ♠

Definition 21 Given a power series $\sum_n a_n t^n$,

$$R = \sup\{|z| : \sum_n a_n z^n < \infty\}$$

is called the *radius of convergence* of the series. The above theorem gives you the formula for R.

Remark 19 Observe that if $P(t)$ is convergent for some z, then the radius of convergence of P is at least $|z|$. The second part of the theorem gives you the formula for it. This is called the **Cauchy-Hadamard formula.** It is implicit in this theorem that the the collection of all points at which a given power series converges consists of an open disc centered at the origin and perhaps some points on the boundary of the disc. This disc is called the disc of convergence of the power series. Observe that the theorem does not say anything about the convergence of the series at points on the boundary $|z| = R$. The examples below will tell you that any thing can happen.

Example 6 The series $\sum_n t^n, \sum_n \frac{t^n}{n}, \sum_n \frac{t^n}{n^2}$ all have radius of convergence 1. The first one is not convergent at any point of the boundary of the disc of convergence $|z| = 1$. The second is convergent at all the points of the boundary except at $z = 1$ (Dirichlet's test) and the last one is convergent at all the points of the boundary (compare with $\sum_n \frac{1}{n^2}$). These examples clearly illustrate that the boundary behavior of a power series needs to be studied more carefully.

Assignement 3

Solutions to be submitted on 1st Sept. Wednesday morning.

1. Let $\sum_n z_n$ be a convergent series of complex numbers such that the real part $\Re(z_n) \geq 0$ for all n. If $\sum_n z_n^2$ is also convergent, show that $\sum_n |z_n|^2$ is convergent.

2. For $0 \leq \theta < 2\pi$ and for any $\alpha \in \mathbb{R}$, define the *closed sector* $S(\alpha, \theta)$ *with span* θ by

$$S(\alpha, \theta) = \{rE(\beta) \; : \; r \geq 0 \,\&\, \alpha \leq \beta \leq \alpha + \theta\}.$$

   Let $\sum_n z_n$ be a convergent series. If $z_n \in S(\alpha, \theta), n \geq 1$, where $\theta < \pi$, then show that $\sum_n |z_n|$ is convergent.

3. Give a direct proof of the fact that $\sum_{n \geq 0} \frac{z^n}{n!}$ is convergent for all $z \in \mathbb{C}$. Use this to prove that

$$\limsup_n \sqrt[n]{n!} = \infty.$$

4. If $P(t), Q(t)$ are two convergent power series with radius of convergence r and s respectively, show that the radius of convergence of $(P(t) + Q(t)$ is at least $min\{r, s\}$.

5. Let $p(t) = a_0 + a_1 t + \cdots$ be a formal power series with coefficients in $\mathbb{C}$. Show that there exists another power series $q(t) = b_0 + b_1 t + \cdots$ such that $p(t)q(1) = 1$ iff $a_0 \neq 0$. In this case, show that q is unique. (q is called the multiplicative inverse of p.) Write down formula for b_n in terms of $a_0, a_1, \cdots$

6. **Hemachandra Numbers** For any positive integer n, let H_n denote the number of patterns you may be able to produce on a drum in a fixed duration of n beats. For instance, $Dha - dhin - dhin$ the first $Dha-$ takes two syllables whereas the following two Dhin's take one syllable each. Clearly $H_1 = 1$ and $H_2 = 2$. Hemachandra [3] noted that since the last syllable is either of one beat or two beats it follows that $H_n = H_{n-1} + H_{n-2}$ for all $n \geq 3$. These numbers were known to Indian poets, musicians and percussionists as Hemachandra numbers.

   Define $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 2$. Note that $F_n = H_{n-1}, n \geq 2$. These F_n are called **Fibonacci numbers.** [4] (Thus the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.)

   Form the formal power series

$$F(z) = \sum_{n=0}^{\infty} F_n z^n \tag{7}$$

   (a) Show thaT $(1 - t - t^2)F(t) = t$.
   (b) Put $S_w(t) := 1 + wt + w^2t^2 + \cdots$. Find $\alpha, \beta \in \mathbb{R}$ such that

$$F(t) = S_\alpha(t) S_\beta(t) t.$$

[3] Hemachandra Suri (1089-1175) was born in Dhandhuka, Gujarat. He was a Jain monk and was an adviser to king Kumarapala. His work in early 11 century is already based on even earlier works of Gopala.

[4] Leonardo Pisano (Fibonacci) was born in Pisa, Italy (1175-1250) whose book Liber abbaci introduced the Hindu-Arabic decimal system to the western world. He discovered these numbers at least 50 years later than Hemachandra's record.

(c) Show that

$$F_{n+1} = \sum_{j=0}^{n} \alpha^j \beta^{n-j} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

7. *Summability* Let

$$F = \{P_\alpha(t) = \sum_n a_{\alpha,n} t^n, \ : \ \alpha \in \Lambda\} \subset \mathbb{C}[[t]]$$

be a family of formal power series in one variable with complex coefficients. We say F is summable if for every $n \geq 0$ the set $\Lambda_n = \{\alpha \in Lambda \ : \ A_{\alpha,n} \neq 0\}$ is finite. In this case, we define the sum of this family to be the element

$$p(t) = \sum_n \left(\sum_{\alpha \in \Lambda} a_{\alpha,n} \right) t^n.$$

Let now $A(t) = \sum_n a_n t^n, B(t) = \sum_n b_n t^n$ be any two power series. Prove or disprove the following statements.

(a) The Cauchy product AB is the sum of the family $\{a_n b_m t^{m+n} \ : \ m, n \geq 0\}$.

(b) If $\{A_j(t)\}$ is a summable family then for any B the family $\{A_j B\}$ is summable.

(c) If $b_0 \neq 0$ then the family $\{a_n B^n\}$ is summable.