Fourth Week: Lectures 10-12

Lecture 10

The fact that a power series p of positive radius of convergence defines a function inside its disc of convergence via substitution is something that we cannot ignore any longer. Let us take the study of such functions. The sequence of partial sums of p, each being a polynomial, defines a function on the whole of the complex plane. (If all the coefficients of p are real we can view each of the partial sums as a real valued functions defined on $\mathbb{R}$.) However, the limit makes sense only inside the disc of convergence. More generally, we can talk about a sequence $\{f_n\}$ of functions defined on some subset of $A \subset \mathbb{C}$ such that at each point $z \in A$ the sequence is convergent. We then get a function $f : A \to \mathbb{C}$ as the limit function viz.,

$$f(z) = \lim_n f_n(z), z \in A.$$

Remember that this means for each $\epsilon > 0$ there exists $n_0(z)$ such that $n \geq n_0$ implies $|f_n(z) - f(z)| < \epsilon$. The number $n_0(z)$ may well vary drastically as we vary the point $z \in A$. In order that the limit function f retains some properties of the members of the sequence $\{f_n\}$ it is anticipated that there must be some control over the possible $n_0(z)$. This leads us to the notion of uniform convergence.

Definition 22 Let $\{f_n\}$ be a sequence of complex valued functions on a set A. We say that it is *uniformly convergent* on A to a function f if for every $\epsilon > 0$ there exists n_0, such that for all $n \geq n_0$, we have, $|f_n(x) - f(x)| < \epsilon$, for all $x \in A$.

Remark 20 Clearly, Uniform convergence implies pointwise convergence. The converse is easily seen to be false, by considering the sequence $f_n(x) = \frac{1}{1+nx^2}$. However, it is fairly easy to see that this is so if A is a finite set. Thus the interesting case of uniform convergence occurs only when A is an infinite set. The terminology is also adopted in an obvious way for series of functions via the associated sequences of partial sums. As in the case of ordinary convergence, we have Cauchy's criterion here also.

Theorem 29 *A sequence of complex valued functions* $\{f_n\}$ *is uniformly convergent iff it is* uniformly Cauchy *i.e., given* $\epsilon > 0$, *there exists* n_0 *such that for all* $n \geq n_0, p \geq 0$ *and for all* $x \in A$, *we have,*

$$|f_{n+p}(x) - f_n(x)| < \epsilon.$$

Example 7 The mother of all convergent series is the geometric series

$$1 + z + z^2 + \cdots$$

The sequence of partial sums is given by

$$1 + z + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}.$$

For $|z| < 1$ upon taking the limit we obtain

$$\boxed{\boxed{1+z+z_2+\cdots+z^n+\cdots = \frac{1}{1-z}.}} \tag{8}$$

In fact, if we take $0 < r < 1$, then in the disc $B_r(0)$, the series is uniformly convergent. For, given $\epsilon > 0$, choose n_0 such that $r^{n_0} < \epsilon(1-r)$. Then for all $|z| < r$ and $n \geq n_0$, we have,

$$\left|\frac{1-z^n}{1-z} - \frac{1}{1-z}\right| = \left|\frac{z^n}{1-z}\right| \leq \frac{|z^{n_0}|}{1-|z|} < \epsilon$$

There is a pattern in what we saw in the above example. This is extremely useful in determining uniform convergence:

Theorem 30 Weierstrass[5]M-test: *Let $\sum_n a_n$ be a convergent series of positive terms. Suppose there exists $M > 0$ and an integer N such that $|f_n(x)| < Ma_n$ for all $n \geq N$ and for all $x \in A$. Then $\sum_n f_n$ is uniformly and absolutely convergent in A.*

Proof: Given $\epsilon > 0$ choose $n_0 > N$ such that $a_n + a_{n+1} + \cdots + a_{n+p} < \epsilon/M$, for all $n \geq n_0$. This is possible by Cauchy's criterion, since $\sum_n a_n$ is convergent. Then it follows that

$$|f_n(x)| + \cdots + |f_{n+p}(x)| \leq M(a_n + \cdots + a_{n+p}) < \epsilon,$$

for all $n \geq n_0$ and for all $x \in A$. Again, by Cauchy's criterion, this means that $\sum f_n$ is uniformly and absolutely convergent. ♠

Remark 21 The series $\sum_n a_n$ in the above theorem is called a 'majorant' for the series $\sum_n f_n$. Here is an illustration of the importance of uniform convergence.

Theorem 31 *Let $\{f_n\}$ be a sequence of continuous functions defined and uniformly convergent on a subset A of $\mathbb{R}$ or $\mathbb{C}$. Then the limit function $f(x) = \lim_{n \longrightarrow \infty} f_n(x)$ is continuous on A.*

[5]Karl Weierstrass (1815-1897) a German mathematician is well known for his perfect rigor. He clarified any remaining ambiguities in the notion of a function, of derivatives, of minimum etc., prevalent in his time.

Proof: Let $x \in A$ be any point. In order to prove the continuity of f at x, given $\epsilon > 0$ we should find $\delta > 0$ such that for all $y \in A$ with $|y - x| < \delta$, we have, $|f(y) - f(x)| < \epsilon$. So, by the uniform convergence, first we get n_0 such that $|f_{n_0}(y) - f(y)| < \epsilon/3$ for all $y \in A$. Since f_{n_0} is continuous at x, we also get $\delta > 0$ such that for all $y \in A$ with $|y - x| < \delta$, we have $|f_{n_0}(y) - f_{n_0}(x)| < \epsilon/3$. Now, using triangle inequality, we get,

$$|f(y) - f(x)| \leq |f(y) - f_{n_0}(y)| + |f_{n_0}(y) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)| < \epsilon,$$

whenever $y \in A$ is such that $|y - x| < \delta$. ♠

Exercise 9 Put $f_n(z) = \frac{z^n}{1-z^n}$. Determine the domain on which the sum $\sum_n f_n(z)$ defines a continuous function.

Definition 23 Given a power series $P(t) = \sum_{n\geq 0} a_n t^n$, the derived series $P'(t)$ is defined by taking term-by-term differentiation: $P'(t) = \sum_{n\geq 1} n a_n t^{n-1}$. The series $\sum_{n\geq 0} \frac{a_n}{n+1} t^{n+1}$ is called the integrated series.

As an application of Cauchy-Hadamard formula, we derive:

Theorem 32 *A power series $P(t)$, its derived series $P'(t)$ and any series obtained by integrating $P(t)$ all have the same radius of convergence.*

Proof: Let the radius of convergence of $P(t) = \sum_n a_n t^n$, and $P'(t)$ be r, r' respectively. It is enough to prove that $r = r'$.

We will first show that $r \geq r'$. For this we may assume without loss of generality that $r' > 0$. Let $0 < r_1 < r'$. Then

$$\sum_{n\geq 1} |a_n| r_1^n = r_1 \left(\sum_{n\geq 1} n |a_n| r_1^{n-1} \right) < \infty.$$

It follows that $r \geq r_1$. Since this is true for all $0 < r_1 < r'$ this means $r \geq r'$.

Now to show that $r \leq r'$, we can assume that $r > 0$ and let $0 < r_1 < r$. Choose r_2 such that $r_1 < r_2 < r$. Then for each $n \geq 1$

$$nr_1^{n-1} \leq \frac{n}{r_1}\left(\frac{r_1}{r_2}\right)^n r_2^n \leq \frac{M}{r_1} r_2^n$$

where $M = \sum_{k\geq 1} k\left(\frac{r_1}{r_2}\right)^k < \infty$, since the radius of convergence of $\sum_k kt^k$ is at least 1 (See Example **??**.) Therefore,

$$\sum_{n\geq 1} n|a_n|r_1^{n-1} \leq \frac{M}{r_1}\sum_{n\geq 1}|a_n|r_2^n < \infty.$$

We conclude that $r' \geq r_1$ and since this holds for all $r_1 < r$, it follows that $r' \geq r$. ♠

Remark 22

(i) For any sequence $\{b_n\}$ of non negative real numbers, one can directly try to establish

$$\limsup_n \sqrt[n]{(n+1)b_{n+1}} = \limsup_n \sqrt[n]{b_n}$$

which is equivalent to proving theorem **??**. However, the full details of such a proof are no simpler than the above proof. In any case, this way, we would not have got the limit of these derived series.

(ii) A power series with radius of convergence 0 is apparently 'useless for us', for it only defines a function at a point. It should noted that in other areas of mathematics, there are many interesting applications of formal power series which need be convergent,

(iii) A power series $P(t)$ with a positive radius of convergence R defines a continuous function $z \mapsto p(z)$ in the disc of convergence $B_R(0)$, by theorem 31. Also, by shifting the origin, we can even get continuous functions defined in $B_R(z_0)$, viz., by substituting $t = z - z_0$.

(iv) One expects that functions which agree with a convergent power series in a small neighborhood of every point will have properties akin to those of polynomials. So, the first step towards this is to see that a power series indeed defines a $\mathbb{C}$-differentiable function in the disc of convergence.

(**The material below this was not actually discussed in the class**)

Theorem 33 Abel: *Let $\sum_{n\geq 0} a_n t^n$ be a power series of radius of convergence $R > 0$. Then the function defined by*

$$f(z) = \sum_n a_n (z - z_0)^n$$

is complex differentiable in $B_r(z_0)$. Moreover the derivative of f is given by the derived series

$$f'(z) = \sum_{n\geq 1} n a_n (z - z_0)^{n-1}$$

inside $|z - z_0| < R$.

Proof: Without loss of generality, we may assume that $z_0 = 0$. We already know that the derived series is convergent in $B_R(0)$ and hence defines a continuous function g on it. We have to show that this function g is the derivative of f at each point of $B_R(0)$. So, fix a point $z \in B_R(0)$. Let $|z| < r < R$ and let $0 \neq |h| \leq r - |z|$ so that $|z + h| \leq r$. Consider the difference quotient

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n\geq 1} u_n(h) \tag{9}$$

where, we have put $u_n(h) := \dfrac{a_n[(z+h)^n - z^n]}{h} - n a_n z^{n-1}$. We must show that given $\epsilon > 0$, there exists $\delta > 0$ such that for all $0 < |h| < \delta$, we have,

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| < \epsilon. \tag{10}$$

The idea here is that the sum of first few terms can be controlled by continuity whereas the remainder term can be controlled by the convergence of the derived series. Using the algebraic formula

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \sum_{k=0}^{n-1} \alpha^{n-1-k}\beta_k,$$

putting $\alpha = z + h, \beta = z$ we get

$$u_n(h) = a_n[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + (z+h)z^{n-2} + z^{n-1} - nz^{n-1}]. \tag{11}$$

Since $|z| < r$ and $|z+h| < r$, it follows that

$$|u_n(h)| \leq 2n|a_n|r^{n-1}. \tag{12}$$

Since the derived series has radius of convergence $R > r$, it follows that we can find n_0 such that

$$2\sum_{n \geq n_0} |a_n|nr^{n-1} < \epsilon/2. \tag{13}$$

On the other hand, again using (11), each $u_n(h)$ is a polynomial in h which vanishes at $h = 0$. Therefore so does the finite sum $\sum_{n<n_0} u_n(h)$. Hence by continuity, there exists $\delta' > 0$ such that for $|h| < \delta'$ we have,

$$\sum_{0<n<n_0} 2|a_n|nr^{n-1} < \epsilon/2. \tag{14}$$

Taking $\delta = min\{\delta', r - |z|\}$ and combining (13) and (14) yields (10). ♠

The exponential function

The exponential function plays a central role in analysis, more so in the case of complex analysis and is going to be our first example using the power series method. We define

$$\boxed{\exp z := e^z := \sum_{n\geq 0} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.} \tag{15}$$

By comparison test it follows that for any real number $r > 0$, the series $\exp(r)$ is convergent. Therefore, the radius of convergence of (15) is ∞. Hence from theorem 33, we have, $\exp$ is differentiable throughout $\mathbb{C}$ and its derivative is given by

$$\boxed{\exp'(z) = \sum_{n\geq 1} \frac{n}{n!} z^{n-1} = \exp(z)} \tag{16}$$

for all z. It may be worth recalling some elementary facts about the exponential function that you probably know already. Let us denote by

$$e := \exp(1) = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

Clearly, $exp(0) = 1$ and $2 < e$. By comparing with the geometric series $\sum_n \frac{1}{2^n}$, it can be shown easily that $e < 3$. Also we have,

$$\boxed{e = \lim_{n\longrightarrow\infty} \left(1 + \frac{1}{n}\right)^n.} \tag{17}$$

To see this, put $t_n = \sum_{k=0}^{n} \frac{1}{k!}$, $s_n = \left(1 + \frac{1}{n}\right)^n$, use binomial expansion to see that

$$limsup_n s_n \leq e \leq \liminf_n s_n.$$

Since $\overline{\sum_0^n \frac{z^k}{k!}} = \sum_0^n \frac{\bar{z}^k}{k!}$, by continuity of the conjugation, it follows that

$$\boxed{\boxed{\overline{\exp z} = \exp \bar{z},}} \tag{18}$$

Formula (16) together with the property $\exp(0) = 1$, tells us that exp is a solution of the initial value problem:

$$\boxed{\boxed{f'(z) = f(z); \quad f(0) = 1.}} \tag{19}$$

It can be easily seen that any analytic function which is a solution of (19) has to be equal to exp. (Ex. Prove this.)

We can verify that

$$\boxed{\boxed{\exp(a+b) = \exp(a)\exp(b), \quad \forall a, b \in \mathbb{C}}} \tag{20}$$

directly by using the product formula for power series. (Use binomial expansion of $(a+b)^n$.) This can also be proved by using the uniqueness of the solution of (19) which we shall leave it you as an entertaining exercise. (See ex. ??)

Thus, we have shown that exp defines a homomorphism from the additive group $\mathbb{C}$ to the multiplicative group $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$. As a simple consequence of this rule we have, $\exp(nz) = \exp(z)^n$ for all integers n. In particular, we have, $\exp(n) = e^n$. This is the justification to have the notation

$$e^z := \exp(z).$$

Combining (18) and (20), we obtain,

$$|e^{\imath y}|^2 = e^{\imath y}\overline{e^{\imath y}} = e^{\imath y}e^{-\imath y} = e^0 1.$$

Hence,

$$\boxed{\boxed{|e^{\imath y}| = 1, \quad y \in \mathbb{R}.}} \tag{21}$$

Example 8 Trigonometric Functions. Recall the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots ;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots ,$$

valid on the entire of $\mathbb{R}$, since the radii of convergence of the two series are ∞. Motivated by this, we can define the complex trigonometric functions by

$$\boxed{\boxed{\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots ; \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots .}} \tag{22}$$

Check that

$$\boxed{\boxed{\sin z = \frac{e^{\imath z} - e^{-\imath z}}{2\imath}; \quad \cos z = \frac{e^{\imath z} + e^{-\imath z}}{2}.}} \tag{23}$$

It turns out that these complex trigonometric functions also have differentiability properties similar to the real case, viz., $(\sin z)' = \cos z; (\cos z)' = -\sin z$, etc.. Also, from (23) additive properties of sin and cos can be derived.

Other trigonometric functions are defined in terms of sin and cos as usual. For example, we have $\tan z = \dfrac{\sin z}{\cos z}$ and its domain of definition is all points in $\mathbb{C}$ at which $\cos z \neq 0$.

In what follows, we shall obtain other properties of the exponential function by the formula

$$\boxed{\boxed{e^{\imath z} = \cos z + \imath \sin z.}} \tag{24}$$

In particular,

$$\boxed{\boxed{e^{x+\imath y} = e^x e^{\imath y} = e^x(\cos y + \imath \sin y).}} \tag{25}$$

It follows that $e^{2\pi\imath} = 1$. Indeed, we shall prove that $e^z = 1$ iff $z = 2n\pi\imath$, for some integer n. Observe that $e^x \geq 0$ for all $x \in \mathbb{R}$ and that if $x > 0$ then $e^x > 1$. Hence for all $x < 0$, we have, $e^x = 1/e^{-x} < 1$. It follows that $e^x = 1$ iff $x = 0$. Let now $z = x + \imath y$ and $e^z = 1$. This means that $e^x \cos y = 1$ and $e^x \sin y = 0$. Since $e^x \neq 0$ for any x, we must have, $\sin y = 0$. Hence, $y = m\pi$, for some integer m. Therefore $e^x \cos m\pi = 1$. Since $\cos m\pi = \pm 1$ and $e^x > 0$ for all $x \in \mathbb{R}$, it follows that $\cos m\pi = 1$ and $e^x = 1$. Therefore $x = 0$ and $m = 2n$, as desired.

Finally, let us prove:

$$\boxed{\boxed{\exp(\mathbb{C}) = \mathbb{C}^\star.}} \tag{26}$$

Write $0 \neq w = r(\cos\theta + \imath \sin\theta), r \neq 0$. Since e^x is a monotonically increasing function and has the property $e^x \longrightarrow 0$, as $x \longrightarrow -\infty$ and $e^x \longrightarrow \infty$ as $x \longrightarrow \infty$, it follows from Intermediate Value Theorem that there exist x such that $e^x = r$. (Here x is nothing but $\ln r$.) Now take $y = \theta, z = x + \imath\theta$ and use (25) to verify that $e^z = w$. This is one place, where we are heavily depending on the intuitive properties of the angle and the corresponding properties of the real sin and cos functions. We remark that it is possible to avoid this by defining sin and cos by the formula (23) in terms of exp and derive all these properties rigorously from the properties of exp alone.

Remark 23 One of the most beautiful equations:

$$\boxed{\boxed{e^{\pi\imath} + 1 = 0}} \tag{27}$$

which relates in a simple arithmetic way, *five* of the most fundamental numbers, made Euler[6] believe in the existence of God!

Example 9 Let us study the mapping properties of tan function. Since $\tan z = \frac{\sin z}{\cos z}$, it follows that tan is defined and complex differentiable at

[6]See E.T. Bell's book for some juicy stories

all points where $\cos z \neq 0$. Also, $\tan(z + n\pi\imath) = \tan z$. In order to determine the range of this function, we have to take an arbitrary $w \in \mathbb{C}$ and try to solve the equation $\tan z = w$ for z. Putting $e^{\imath z} = X$, temporarily, this equation reduces to $\dfrac{X^2 - 1}{\imath(X^2 + 1)} = w$. Hence $X^2 = \dfrac{1 + \imath w}{1 - \imath w}$. This latter equation makes sense, iff $w \neq -\imath$ and then it has, in general two solutions. The solutions are $\neq 0$ iff $w \neq \imath$. Once we pick such a non zero X we can then use the ontoness of $\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$, to get a z such that that $e^{\imath z} = \pm X$. It then follows that $\tan z = w$ as required. Therefore we have proved that the range of tan is equal to $\mathbb{C} \setminus \{\pm\imath\}$. From this analysis, it also follows that $\tan z_1 = \tan z_2$ iff $z_1 = z_2 + n\pi\imath$.

Likewise, the hyperbolic functions are defined by

$$\boxed{\sinh z = \frac{e^z - e^{-z}}{2}; \qquad \cosh z = \frac{e^z + e^{-z}}{2}.} \tag{28}$$

It is easy to see that these functions are $\mathbb{C}$-differentiable. Moreover, all the usual identities which hold in the real case amongst these functions also hold in the complex case and can be verified directly. One can study the mapping properties of these functions as well, which have wide range of applications.

Remark 24 Before we proceed onto another example, we would like to draw your attention to some special properties of the exponential and trigonometric functions. You are familiar with the real limit

$$\lim_{x\to\infty} \exp(x) = \infty.$$

However, such a result is not true when we replace the real x by a complex z. In fact, given any complex number $w \neq 0$, we have seen that there exists z such that $\exp(z) = w$. But then $\exp(z + 2n\pi\imath) = w$

for all n. Hence we can get z' having arbitrarily large modulus such that $\exp(z') = w$. As a consequence, it follows that $\lim_{z \longrightarrow \infty} \exp(z)$ does not exist. Using the formula for sin and cos in terms of exp, it can be easily shown that sin and cos are both surjective mappings of $\mathbb{C}$ onto $\mathbb{C}$. In particular, remember that they are not bounded unlike their real counter parts.

Lecture 11

Mainly devoted to the discussion of the Assignment 3 problems.

Lecture 12

Definition 24 By a *metric* or a *distance function* on a set X we mean a function $d : X \times X \to \mathbb{R}$ such that
(a) $d(x, y) \geq 0$ for all (x, y) and $= 0$ iff $x = y$.
(b) $d(x, y) = d(y, x)$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$. A set X together with a chosen metric on it is called a metric space.

Example 10
1. The simplest and most important examples of metric spaces are the Euclidean spaces $\mathbb{R}^n$ with $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. In case of $n = 1$ this also takes the form $d(x, y) = |x - y|$. So, we also use this notation in the general case.
2. A metric on X automatically restricts to a metric on any subset of X and thus, it makes sense to talk about subspaces of metric spaces. For instance, if we consider $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ then the standard metric on $\mathbb{R}^n$ is seen to be the restriction of that on $\mathbb{R}^{n+1}$.
3. For any set X consider the function

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$$

Verify that this is a distance function. It is called the discrete metric.
4. On $\mathbb{R}^n$ define

$$d_{max}(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

5. On $\mathbb{R}^n$ define

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

6. On the set of square summable sequences of real numbers, define

$$d_2(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$$

7. On the set of bounded continuous real valued functions on an interval J, define

$$d_s(f, g) = sup\{|f(x) - f(y)|, x \in J\}.$$

Definition 25 Let (X, d) be a metric space, $x \in X, \delta > 0$. We shall denote

$$B_\delta(x) := \{y \in X \ : \ d(x, y) < \delta\}$$

and call it the open ball of radius δ and center x.

Exercise 10 Draw a picture of the unit ball in the $\mathbb{R}^n$ in each of the various metrics that we have seen above.

Definition 26 Let (X, d) be a metric space.
1. By an open subset in X we mean a subset $U \subset X$ which is the union of some open balls in X.
2. A set $U \subset X$ is called a *neighbourhood* of $x \in X$ if $x\imath U$ and U is open.
3. A subset F in X is closed in X if $X \setminus F$ is open in X.
4. A point $x \in X$ is called a limit point $A \subset X$ if every nbd U of x contains a point of A not equal to x i.e., $(A \setminus \{x\}) \cap U \neq \emptyset$. The set of all limit points of A is denoted by $l(A)$.
6. The set $A \cup l(A)$ is calld teh clsouse of A and is denoted by $\bar{A}$. (Ex. prove that $\bar{A}$ is a closed set in X for any subset $A \subset X$. 7. If $x \in A$ is not a limit point of A then it is called an isolated point of A.

8. $x \in A$ is called an interior point if there exists an open set U in X such that $x \in U \subset A$. The set of all interior points of A is called the interior of A and is denoted by $int\, A$. 9. $A \subset X$ is called bounded if there exists $M > 0$ and $p \in X$ such that such that $A \subset B_M(p)$.
10. $A \subset X$ is called dense in X if every point of $X \setminus A$ is a limit point of X. This is the same as saying $\bar{A} = X$.

Theorem 34 *Let $\{U_j\}$ be a family of open sets in X. Then the union $U = \cup_j U_j$ is open. Also intersection of any two open sets is open.*

Remark 25 The empty set and the whole set X are open.

Theorem 35 *A set is closed iff it contains all its limit points.*

Definition 27 The closure $\bar{A}$ of a set A is defined to be the union of A with all its limit points.

Definition 28 Let X be a set and $\mathcal{T}$ be a family of subsets of X. We say $\mathcal{T}$ is a topology on X and $(X, \mathcal{T})$ a topological space, if the following condtions are satisfied.
(i) $\emptyset \in \mathcal{T}$.
(ii) If $U_i \in \mathcal{T}$, then $\cup_i U_i \in \mathcal{T}$, i.e., union of a family of memebrs of $\mathcal{T}$ is again a member of $\mathcal{T}$.
(iii) If $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.
(iv) $X \in \mathcal{T}$.

In this situations, members of $\mathcal{T}$ are called open subsets in the topologicval space $(X < \mathcal{T})$.

Example 11 Given a metric space (X, d) if we take $\mathcal{T}$ to be the set of all open sets in this metric space then $\mathcal{T}$ satisifies the conditions of the above defintion and hence is a topology on X. This topology is called the topology induced by the metric d. We shall be studying only such topologies in this course.

Assigment 4 (1) Put $f_n(z) = \frac{z^n}{1-z^n}$. Determine the domain on which the sum $\sum_n f_n(z)$ defines a continuous function. (2) Draw a picture of the unit ball in the $\mathbb{R}^2, \mathbb{R}^3$ in the euclidean metric, the supremum metrc d_{max} and l_1-metric d_1 as defined in the example 10.(1,4, and 5 respectively) above.
(3) Let F be a closed subset of a metric space. Consider $f(x) = d(x, F) = \inf\{d(x,y) \ : \ y \in F\}$. Show that f is continuous.
(4) Let $f : X \to Y$ be any function, $x_0 \in X$. Prove that the FAE:
(a) f is continuous at x_0.
(b) For every sequence $\{x_n\}$ in X which converges to x_0 the sequence $\{f(x_n)\}$ converges to $f(x_0)$.
(5) Let $f, g : X \to \mathbb{R}$ be any two continuous functions. Define $Max\{f,g\}, min\{f,g\}$ by the formulae:

$$Max\{f,g\}(x) = \max\{f(x), g(x)\}; \quad Min\{f,g\}(x) = min\{f(x), g(x)\}.$$

Show that $Max\{f,g\}, Min\{f,g\}$ are both continuous.