Lecture 15

We have seen that a sequence of continuous functions which is uniformly convergent produces a limit function which is also continuous. We shall strengthen this result now.

Theorem 1 *Let $f_n : X \to \mathbb{R}$ or $(\mathbb{C})$ be a sequence of continuous functions. Let $A \subset X$ on which $\{f_n\}$ converges uniformly. Then $\{f_n\}$ converges on the closure $\bar{A}$ of A to a function f which is continuous.*

Proof: Let us fix a point $x_0 \in \bar{A}$. We must first of all show that the sequence $\{f_n(x_0)\}$ is convergent. Enough to show it is Cauchy. Given $\epsilon > 0$ there exist n_0 such that $n, m > n_0$ implies

$$|f_n(x) - f_m(x)| < \epsilon/3$$

for all $x \in A$. By continuity of f_n and f_m we can find $\delta > 0$ such that $d(x, x_0) < \delta$ implies that

$$|f_m(x) - f_m(x_0)| + |f_n(x) - f_n(x_0)| < 2\epsilon/3.$$

Now since $x_0 \in \bar{A}$, there exists $x \in B_\delta(x_0) \cap A$. With the help of this x, we have

$$|f_n(x_0) - f_m(x_0)| \leq |f_m(x) - f_m(x_0)| + |f_n(x) - f_n(x_0)| + |f_n(x) - f_m(x)| < \epsilon.$$

Therefore, we have got a function $f : \bar{A} \to \mathbb{R}$ which is the limit of $\{f_n\}$ and the convergence is uniform on A.

We now want to show that f is continuous at x_0.

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

Given $\epsilon > 0$ we can choose N_1 such that $n > N_1$ implies

$$|f(x) - f_n(x)| + |f_n(x_0) - f(x_0)| < 2\epsilon/3, \quad \text{for all } x \in A.$$

Fix one such n. Then by continuity of f_n we can find $\delta > 0$ such that $d(x, x_0) < \delta$ implies $|f_n(x) = f(n(x_0)| < \epsilon/3$. Once again since $B_\delta(x_0) \cap A \neq \emptyset$, has to be used to conclude the continuity of f at x_0. ♠

Remark 1 What about differentiabilty under uniform convergence? We should be careful here as illustrated by the example: $f_n(x) = \frac{x}{1+nx^2}$ on $[0, 1]$. This sequence converges uniformly to the function which is identically 0. However the derived sequence $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$ converges to a function which is not even continuous. It is also true that a uniform limit of a sequence of smooth functions can be continuous but not differentiable, or differentiable but not continuously differentiable or ... and so on.

On the positive side, we shall now see that by controling the limiting process of the derived sequence itself we get better results:

Theorem 2 *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions such that f_n' converges uniformly in $[a, b]$ to a function g. Also suppose for some $x_0 \in [a, b]$, the sequence $\{f_n(x_0)\}$ is convergent. Then the sequence f_n converges uniformly to a function f and $f' = g =$* $\lim_{n\to\infty} f_n'$.

Proof: First we want to show that f_n is uniformly convergent and for this it is enough to show that it is uniformly Cauchy, i.e., given $\epsilon > 0$ we must find n_0 such that $n, m > n_0$ implies

$$|f_n(x) - f_m(x)| < \epsilon, \quad x \in [a, b] \tag{1}$$

Using the hypothesis we get n_1 such that $n, m > n_1$ implies

$$|f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)}, \quad x \in [a, b]. \tag{2}$$

Put $\phi_{mn} = f_n - f_m$. Therefore by Mean Value theorem applied to ϕ_{mn}, we have

$$\left| \frac{\phi_{mn}(x_1) - \phi_{mn}(x_2)}{x_1 - x_2} \right| < \frac{\epsilon}{2(b-a)}, x_1, x_2 \in [a,b], m, n > n_1. \tag{3}$$

This is the same as

$$|f_n(x_1) - f_m(x_1) - f_m(x_2) + f_n(x_2)| < \frac{|x_1 - x_2|}{2(b-a)} \leq \epsilon/2. \tag{4}$$

We now use the fact that $f_n(x_0)$ is convergent and hence find n_2 such that $n, m > n_2$ implies

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2. \tag{5}$$

Combining the above two inequalities we conclude that f_n is uniformly Cauchy,

$$|f_n(x) - f_m(x)| < \epsilon, \quad m, n > max\{n_1, n_2\} \tag{6}$$

as required. Let now $f(x) = \lim_{n\to\infty} f_n(x)$. To show that $f' = g$: Now fix a $x_2 \in [a,b]$ and put $h_n(x_1) = \frac{f_n(x_1) - f_n(x_2)}{x_1 - x_2}$. Then (11) implies that h_n is uniformly Cauchy in $[a,b] \setminus \{x_2\}$ and hence converges to a continuous function $h(x_1)$ which is nothing but

$$\lim_{n\to\infty} \frac{f_n(x) - f_n(x_2)}{x_1 - x_2} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Therefore the limit function is continuous on the closure of $[a,b] \setminus \{x_2\}$ which is $[a,b]$. We can now interchange the takiong limit with respect to n with limit with respect to x, i.e.,

$$g(x_1) = \lim_{n\to\infty} f_n'(x_1) = \lim_{n\to\infty} \lim_{x_2\to x_1} \frac{f_n(x_2) - f_n(x_1)}{x_2 - x_1}$$

$$= \lim_{x_2\to x_1} \lim_{n\to\infty} \frac{f_n(x_2) - f_n(x_1)}{x_2 - x_1} = \lim_{x_2\to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1).$$

♠

Lecture 16 Cesaro Summability

Given a sequence $\{a_n\}$ of complex numbers, a method T first associates another sequence $\{t_n\}$ to it and then takes the limit of $\{t_n\}$. If this limit exists and is equal to L then we say $\{a_n\}$ is T-convergent to the T-limit L and write

$$Tlim_n a_n = L; \text{ OR } \lim_n a_n = L(T).$$

Example 1 Series summation is such a method in which t_n is just the n^{th} partial sum of the given sequence. Another method is called $(C,1)$-summable (Cesero-1) in which $t_n = \frac{a_n}{n} = \frac{a_1+\cdots+a_n}{n}$. Note that if the sequence $a_n \to L$, then it is $(C,1)$-summable to the sum L.

Proof: $a_n \to L$, which is the same as saying $\lim_n(a_n - L) = 0$. Given $\epsilon > 0$ there is a N_0 such that $|a_n - L| < \epsilon/2$ for $n \geq N_0$. Also, the sequence $\{a_n - L\}$ is bounded and so there is $M > 0$ such that $|a_n - L| < M$ for all n. Therefore $|t_n - L| = |\frac{a_1+\cdots+a_n-nL}{n}| \leq \frac{N_0M+(n-N_0)\epsilon/2}{n} \leq \frac{N_0M}{n} + \epsilon/2$ and so on. ♠

Another example is $a_n = (-1)^n$. Of course the sequence is not convergent. But it is $(C,1)$-summable to 0. The $(C,1)$-limit is a good representation of the average.

Example 2 More generally, given $k \geq 1$, we define a sequence $\{a_n\}$ to be (C,k)-summable to L if the sequence

$$t_n = \frac{1}{\binom{n+k-1}{n-1}} \sum_{j=1}^{n} \binom{n+k-1-j}{n-j} a_j \to L$$

It is not hard to check that if $\{a_n\}$ is (C,k)-summable to L then it is $(C,k+1)$-summable to L. Also, there are sequences which are $(C,k+1)$-summable but not (C,k)-summable. For instance the sequence $1,-1,2,-2,3,-3,\ldots,$ is not $(C,1)$ summable but is $(C,2)$. Similarly the sequence $1,-2,3,-4,5,-6,\ldots$ is not $(C,2)$ summable but $(C,3)$.

Example 3 (General Weighted Averages) Even more generally, given a sequence of positive real numbers $\mathcal{P} = \{p_1, p_2, \ldots, p_n, \ldots\}$, we put $P_n = \sum_{j=1}^n p_j$ and we define $\mathcal{P}$-summability of a sequence $\{a_n\}$ if the sequence

$$t_n = \frac{\sum_{j=1}^n a_j p_{n-j}}{P_n}$$

converges to a limit L and say $\mathcal{P} \lim a_n = L$. Check that each (C, k) in indeed a $\mathcal{P}$ method for some sequence $\mathcal{P}$. Thus each Cesaro sum can be thought of as a combinatorial (binomial) average.

Definition 1 We say a summability method T is regular if whenever $\lim_n a_n = L$ then $Tlim_n a_n = L$.

What we have seen above is that each (C, k) is regular. On the other hand the series method is not regular.

Theorem 3 *$\mathcal{P}$ is regular iff for each k,*

$$\lim_n \frac{p_{n-k}}{P_n} = 0. \tag{7}$$

Proof: Suppose $\mathcal{P}$ is regular. Take $a_n = 0, n \neq k$ and $= 1$ for $n = k$ to see (7). Conversely, suppose (7) holds and let $a_n \to L$. WLOG we may assume that $L = 0$. Given $\epsilon > 0$ find N_0 such that $|a_n| < \epsilon$ for $n \geq N_0$. Then for each $k \leq N_0$ find N_k such that $|\frac{p_{n-k}}{P_n}| < \epsilon/N_0$ for $n \geq N_k$. Take $N = max\{N_0, \ldots, N_{N_0}\}$. Then for $n \geq N$ we have $|t_n| < \epsilon(M + 1)$, where M is a bound $\{|a_n|\}$.

Remark 2 In this sense series is not a regular summability, whereas, all Cesaro summabilities are.

Definition 2 Given a series $\sum_n a_n$ with partial sums $\{s_n\}$, we say that $\sum_n a_n$ is $(C, 1)$-summable to S if

$$\lim_n s_n = S(C, 1).$$

And then we write

$$\sum_n a_n = S. \quad (C, 1).$$

Example 4 Consider the series $\sum_{n=0}^{\infty}(-1)^n$. Here the sequence of partial sums is $1, 0, 1, 0, 1, 0, \ldots$ which is (C,1)-converges to $1/2$. Therefore we write $\sum_n (-1)^n = 1/2 \quad (C, 1)$. Notice that the $(C, 1)$ limit of the sequence $\{(-1)^n\}$ is equal to 0. **So, you must pay attention to this definition properly.**

Definition 3 A sequence $\{a_n\}$ is called square summable OR is said to be of class ℓ^2 if $\sum_n a_n^2 \infty$. We can add two square summable sequences to get another such. Indeed square summable sequences form a vector space. $\{1/n\}$ is in ℓ^2 whereas $\{\sqrt{1/n}\}$ is not in ℓ^2.

Thus there are several important summation methods one can use depending upon ones requirement. We shall meet $(C, 1)$ summability again while studying Fourier series. You may consult Goldberg's book for an elementary exposition of this subject beyond what we have seen so far.

Connectedness

Definition 4 Let X be any topolgical space. We say X is connected if the only subsets $A \subset X$ which are both open and closed in X are X and $\emptyset$.

We say a subset $A \subset X$ is connected if the subspace A of X is connected.

Theorem 4 *Let X be a topological space. Then the following are equivalent:*

(a) X is connected.

(b) $A \cup B = X$, both A and B are open, $A \neq \emptyset \neq B$ then $A \cap B \neq \emptyset$.

(c) $A \cup B = X$, both A and B are closed, $A \neq \emptyset \neq B$ then $A \cap B \neq \emptyset$.

(d) $\emptyset \neq A \subset X$ is both open and closed then $A = X$.

Theorem 5 *A subset of $\mathbb{R}$ is connected iff it is an interval.*

Proof: Suppose $A \subset \mathbb{R}$ which is not an interval. This means there exist $x < z < y$ such that $x, y \in A$ but $z \notin A$. Put $F = A \cap (-\infty, z); G = A \cap (z, \infty)$. Then both F, G are open in A nonempty and the union is A. This is a contradiction.

Conversely, let A be an interval in $\mathbb{R}$, $A = F \cup G$, $x \in F, y \in G$ $x < y$. Assume that both F, G are closed in A. We shall show that $F \cap G \neq \emptyset$. Put $w = sup\, F \cap [x, y]$. Then $w \in A$ and since F is closed $w \in F$. Clearly, $w \leq y$. Now for any z such that $w < z \leq y$, then $z \notin F$ and hence $z \in G$. This means w is a limit point of G. Since G is closed $w \in G$. ♠

Theorem 6 *Let $f : X \to Y$ be a continuous function, $A \subset X$ is connected. Then $f(A)$ is connected.*

Proof: If not we can write $f(A) = U \cap V$ where U, V are both non empty open in $f(A)$ and $U \cap V = \emptyset$. But then $f^{-1}(U)$ and $f^{-1}(V)$ are non empty open in A and $A = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. THis means A is not connected. ♠

Theorem 7 Intermediate Value Property *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $f(a) < z < f(b)$. Then there exists $a < c < b$ such that $f(c) = z$.*

Proof: Since $[a, b]$ is connected this implies $f[a, b]$ is connected and hence is an interval. Therefore all real numbers between the two values $f(a)$ and $f(b)$ are also in this interval. ♠

Remark 3 IVP is equivalent to intervals being connected.

Example 5

(i) Every path is connected.

(ii) Every path connected space is connected. But converse is not true.

(iii) $\mathbb{R}^n$ is connected.

(iv) Every cell in $\mathbb{R}^n$ is connected.

(v) Complement of a countable set in $\mathbb{R}^n, n \geq 2$ is connected. (vi) Complement of a vector subspace of codimension ≥ 2 in $\mathbb{R}^n$ is connected.

(vii) Every convex subset is connected.

(viii) Spheres ellipsoids etc are connected. Not nec. hyperboloids.

Lecture 18 Uniform limits of functions

We have seen that a sequence of continuous functions which is uniformly convergent produces a limit function which is also continuous. We shall strengthen this result now.

Theorem 8 *Let X be any metric space. Let $f_n : X \to \mathbb{R}$ or $(\mathbb{C})$ be a sequence of continuous functions. Let $A \subset X$ on which $\{f_n\}$ converges uniformly. Then $\{f_n\}$ converges on the closure $\bar{A}$ of A to a function f which is continuous.*

Proof: Let us fix a point $x_0 \in \bar{A}$. We must first of all show that the sequence $\{f_n(x_0)\}$ is convergent. Enough to show that it is Cauchy.

To begin with we have the sequence is uniformly convergent on A. Therefore, given $\epsilon > 0$ there exist n_0 such that $n, m > n_0$ implies

$$|f_n(x) - f_m(x)| < \epsilon/3$$

for all $x \in A$. By continuity of f_n and f_m we can find $\delta > 0$ such that $d(x, x_0) < \delta$ implies that

$$|f_m(x) - f_m(x_0)| + |f_n(x) - f_n(x_0)| < 2\epsilon/3.$$

Now since $x_0 \in \bar{A}$, there exists $x \in B_\delta(x_0) \cap A$. With the help of this x, we have

$$|f_n(x_0) - f_m(x_0)| \leq |f_m(x) - f_m(x_0)| + |f_n(x) - f_n(x_0)| + |f_n(x) - f_m(x)| < \epsilon, \quad \forall \;\; n, m > n_0. \quad (8$$

Let us carefully examine what we have done now. We have got n_0 satisfying (8) without depending on what x_0 we have chosen in $\bar{A}$. This just means that the sequence $\{f_n\}$ is uniformly Cauchy on $\bar{A}$.

Therefore, we have got a function $f : \bar{A} \to \mathbb{R}$ which is the limit of $\{f_n\}$ and the convergence is uniform on A. Therefore the conclusion of the theorem follows. ♠

Remark 4 What about differentiability under uniform convergence? We should be careful here as illustrated by the example: $f_n(x) = \frac{x}{1+nx^2}$ on $[0, 1]$. This sequence converges uniformly to the function which is identically 0. (To see this find the maxima of f_n in $[0, 1]$.) However the derived sequence $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$ converges to a function which is not even continuous. It is also true that a uniform limit of a sequence of smooth functions can be continuous but not differentiable, or differentiable but not continuously differentiable or ... and so on.

On the positive side, we shall now see that by controlling the limiting process of the derived sequence itself we get better results:

Theorem 9 *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions such that f_n' converges uniformly in $[a, b]$ to a function g. Also suppose for some $x_0 \in [a, b]$, the sequence $\{f_n(x_0)\}$ is convergent. Then the sequence f_n converges uniformly to a function f and $f' = g =$* $\lim_{n\to\infty} f_n'$.

Proof: First we want to show that f_n is uniformly convergent and for this it is enough to show that it is uniformly Cauchy, i.e., given $\epsilon > 0$ we must find n_0 such that $n, m > n_0$ implies

$$|f_n(x) - f_m(x)| < \epsilon, \quad x \in [a, b] \tag{9}$$

Using the hypothesis we get n_1 such that $n, m > n_1$ implies

$$|f'_n(x) - f'_m(x)| < \frac{\epsilon}{2(b-a)}, \quad x \in [a, b]. \tag{10}$$

Put $\phi_{mn} = f_n - f_m$. Therefore by Lagrange Mean Value Theorem applied to ϕ_{mn}, we have

$$\left|\frac{\phi_{mn}(x_1) - \phi_{mn}(x_2)}{x_1 - x_2}\right\langle |\phi'_{mn}(c)| = |f'_n(x) - f'_m(x)| < \frac{\epsilon}{2(b-a)}, \ \forall\ x_1, x_2 \in [a, b], m, n > n_1. \tag{11}$$

This is the same as

$$|f_n(x_1) - f_m(x_1) - f_m(x_2) + f_n(x_2)| < \frac{|x_1 - x_2|}{2(b-a)} \leq \epsilon/2. \tag{12}$$

We now use the fact that $f_n(x_0)$ is convergent and hence find n_2 such that $n, m > n_2$ implies

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2. \tag{13}$$

Put $x_1 = x_0$ and $x_2 = x$ and combining the above two inequalities (12),(**??**), we conclude that f_n is uniformly Cauchy:

$$|f_n(x) - f_m(x)| < \epsilon, \quad m, n > max\{n_1, n_2\} \tag{14}$$

as required. Let now $f(x) = \lim_{n\to\infty} f_n(x)$. To show that $f' = g$: Now fix $x_2 \in [a, b]$ and put $h_n(x_1) = \frac{f_n(x_1) - f_n(x_2)}{x_1 - x_2}$. Then (11) implies that h_n is uniformly Cauchy in $[a, b] \setminus \{x_2\}$ and hence converges to a continuous function $h(x_1)$ which is nothing but

$$\lim_{n\to\infty} \frac{f_n(x) - f_n(x_2)}{x_1 - x_2} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Therefore the limit function is continuous on the closure of $[a,b]\setminus\{x_2\}$ which is $[a,b]$. We can now interchange the order of taking limit with respect to n with the limit with respect to x, i.e.,

$$g(x_1) = \lim_{n\to\infty} f_n'(x_1) = \lim_{n\to\infty}\lim_{x_2\to x_1} \frac{f_n(x_2)-f_n(x_1)}{x_2-x_1}$$

$$= \lim_{x_2\to x_1}\lim_{n\to\infty} \frac{f_n(x_2)-f_n(x_1)}{x_2-x_1} = \lim_{x_2\to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_1).$$

♠

Remark 5 There are certain properties of real valued functions defined on intervals, which is peculiar to the 1-variable functions only. For instance, let $f:[a,b]\to[c,d]$ be a bijection. Then it is not hard to see that f is continuous iff it is strictly monotone. (For instance, assume that f is order preserving. Then f^{-1} is also order preserving and show that image of any closed interval is a closed interval under f^{-1}. It would follow that image of an open interval is an open interval and hence f is continuous.) In particular it follow that if $f:[a,b]\to[c,d]$ is a continuous bijection then its inverse is also continuous. (That means f is homeomorphism.) There is nothing sacrosanct about taking closed intervals. The statement holds for open intervals and for the whole of $\mathbb{R}$ as well.

Remark 6 On the other and, there are some general results about topological spaces which have special properties such as compactness and connectedness. For example consider a continuous bijection $f:X\to Y$ where X is a compact space and Y is a metric space. We can prove that f^{-1} is contiguous very easily as follows: It is enough to show that f is an open mapping this is equivalent to show that f is a closed mapping (because f is a bijection). If F is a closed subset of X, F will be also compact. Therefore $f(F)$ is compact. Being a compact subset of metric space Y, $f(F)$ is closed.

Thus we get an alternative easy proof of the fact that a continuous bijection $[a, b] \to [c, d]$ is a homeomorphism.