Week 7 Riemann Stieltjes Integration: Lectures 19-21

Lecture 19

Throughout this section α will denote a monotonically increasing function on an interval $[a, b]$.

Let f be a bounded function on $[a, b]$.

Let $P = \{a = a_0 < a_1, \cdots, a_n = b\}$ be a partition of $[a, b]$. Put $\Delta\alpha_i = \alpha(a_i) - \alpha(a_{i-1})$.

$M_i = sup\{f(x) \ : \ a_{i-1} \leq x \leq a_i\}$.

$m_i = inf\{f(x) \ : \ a_{i-1} \leq x \leq a_i\}$.

$U(P, f) = \sum_{i=1}^{n} M_i \Delta\alpha_i; \quad L(P, f) = \sum_{i=1}^{n} m_i \Delta\alpha_i$.

$\overline{\int_a^b} f d\alpha = inf\{U(P, f) \ : \ P\}; \quad \underline{\int_a^b} f d\alpha = sup\{L(P, f) \ : \ P\}$.

Definition 1 If $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, then we say f is Riemann-Stieltjes (R-S) integrable w.r.t. to α and denote this common value by

$$\int_a^b f d\alpha := \int_a^b f(x) d\alpha(x) := \overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha.$$

Let $\mathcal{R}(\alpha)$ denote the class of all R-S integrable functions on $[a, b]$.

Definition 2 A partition P' of $[a, b]$ is called a refinement of another partition P of $[a, b]$ if, points of P are all present in P'. We then write $P \leq P'$.

Lemma 1 If $P \leq P'$ then $L(P) \leq L(P')$ and $U(P) \geq U(P')$.

Enough to do this under the assumption that P' has one extra point than P. And then it is obvious because if $a < b < c$ then

$$inf\{f(x)\ :\ a \leq x \leq c\} \leq min\{inf\{f(x)\ :\ a \leq x \leq b\}, inf\{f(x)\ :\ b \leq x \leq c\}$$

etc.

Theorem 1 $\overline{\int_a^b} f d\alpha \geq \underline{\int_a^b} f d\alpha.$

For first of all, because for every partition P we have $U(P, f) \geq L(P, f)$. Let P and Q be any two partitions of $[a, b]$. By taking a common refinement $T = P \cup Q$, and applying the above lemma we get

$$U(P; f) \geq U(T; f) \geq L(T; f) \geq L(Q; f)$$

Now varying Q over all possible partitions and taking the supremum, we get

$$U(P) \geq \underline{\int_a^b} f d\alpha.$$

Now varying P over all partitions of $[a, b]$ and taking the infimum, we get the theorem. ♠

Theorem 2 *Let f be a bounded function and α be a monotonically increasing function. Then the following are equivalent.*
(i) $f \in \mathcal{R}(\alpha)$.
(ii) *Given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(P, f) - L(P, f) < \epsilon.$$

(iii) *Given* $\epsilon > 0$, *there exists a partition* P *of* $[a, b]$ *such that for all refinements of* Q *of* P *we have*

$$U(Q, f) - L(Q, f) < \epsilon.$$

(iv) *Given* $\epsilon > 0$, *there exists a partition* $P = \{a_0 < a_1 < cdots < a_n\}$ *of* $[a, b]$ *such that for arbitrary points* $t_i, s_i \in [a_{i-1}, a_i]$ *we have*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon.$$

(v) *There exists a real number* η *such that for every* $\epsilon > 0$, *there exists a partition* $P = \{a_0 < a_1, < \cdots < a_n\}$ *of* $[a, b]$ *such that for arbitrary points* $t_i \in [a_{i-1}, a_i]$, *we have* $|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \eta| < \epsilon$.

Proof: (i) $\Longrightarrow$ (ii): By definition of the upper and lower integrals, there exist partitions Q, T such that

$$U(Q) - \overline{\int_a^b} f d\alpha < \epsilon/2; \quad \underline{\int_a^b} f d\alpha - L(T) < \epsilon/2.$$

Take a common refinement P to Q and T and replace Q, T by P in the above inequalities, and then add the two inequalities and use the hypothesis (i) to conclude (ii).
(ii) $\Longrightarrow$ (i): Since $L(P) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P)$ the conclusion follows.
(ii) $\Longrightarrow$ (iii): This follows from the previous theorem for if $P' \geq P$ then

$$L(P) \leq L(P') \leq U(P') \leq U(P).$$

(iii) $\Longrightarrow$ (ii): Obvious.
(iii) $\Longrightarrow$ (iv): Note that $|f(s_i) - f(t_i)| \leq M_i - m_i$. Therefore,

$$\sum_i |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_i (M_i - m_i)\Delta\alpha_i = U(P, f) - L(P, f) < \epsilon.$$

(iv) $\implies$ (iii): Choose points $t_i, s_i \in [a_{i-1}, a_i]$ such that

$$|m_i - f(s_i)| < \frac{\epsilon}{2n\Delta\alpha_i}, |M_i - f(t_i)| < \frac{\epsilon}{2n\Delta\alpha_i}.$$

Then $U(P,f) - L(P,f) - \sum_i (M_i - m_i)\Delta\alpha_i$
$\leq \sum_i [|M_i - f(t_i)| + |m_i - f(s_i)| + |f(t_i) - f(s_i)]\Delta\alpha_i < 2\epsilon.$
Thus so far, we have proved that (i) to (iv) are all equivalent to each other.
(i) $\implies$ (v): We first note that having proved that (i) to (iv) are all equivalent, we can use any one of them. We take $\eta = \int_a^b f d\alpha$. Given $\epsilon > 0$ we choose a partition P such that $|L(P) - \eta| < \epsilon/3$. and a partition Q such that (iv) holds with ϵ replaced by $\epsilon/3$. We then take a common refinement T of these two partitions for which again the same would hold because of (iii). We now choose $s_i \in [a_{i-1}, a_i]$ such that $|m_i - f(s_i)| < \frac{\epsilon}{3n\Delta\alpha_i}$ whenever $\Delta\alpha_i$ is non zero. (If $\Delta\alpha_i = 0$ we can take s_i to be any point.) Then for arbitrary points $t_i \in [a_{i-1}, a_i]$, we have

$$\begin{aligned} & |\textstyle\sum_i f(t_i)\Delta\alpha_i - \eta| \\ = & \left|\sum_i [(f(t_i) - f(s_i) + (f(s_i) - m_i) + m_i]\Delta\alpha_i - \eta\right| \\ \leq & \sum_i |f(s_i) - f(t_i)|\Delta\alpha_i + \sum_i |f(s_i) - m_i|\Delta\alpha_i + |L(P) - \eta| \\ \leq & \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

(v) $\implies$ (iv): Given $\epsilon > 0$ choose a partition as in (v) with ϵ replaced by $\epsilon/2$. ♠

Lecture 20

Fundamental Properties of the Riemann-Stieltjes Integral

Theorem 3 *Let f, g be bounded functions and α be an increasing function on an interval $[a,b]$.*
(a) Linearity in f : This just means that if $f, g \in \mathcal{R}(\alpha), \lambda, \mu \in \mathbb{R}$ then $\lambda f + \mu g \in \mathcal{R}(\alpha)$. Moreover,

$$\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f d\alpha + \mu \int_a^b f d\alpha.$$

(b) Semi-Linearity in α. This just means if $f \in \mathcal{R}(\alpha_j), j = 1, 2\ \lambda_j > 0$ then $f \in \mathcal{R}(\lambda_1\alpha_1 + \lambda_2\alpha_2)$ and moreover,

$$\int_a^b f d(\lambda_1\alpha_1 + \lambda_2\alpha_2) = \lambda \int_a^b f d\alpha_1 + \mu \int_a^b f d\alpha_2.$$

(c) Let $a < c < b$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if $f \in \mathcal{R}(\alpha)$ on $[a,c]$ as well as on $[c,b]$. Moreover we have

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

(d) $f_1 \le f_2$ on $[a,b]$ and $f_i \in \mathcal{R}(\alpha)$ then $\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$.
(e) If $f \in \mathcal{R}(\alpha)$ and $|f(x)| \le M$ then

$$\left| \int_a^b f d\alpha \right| \le M[\alpha(b) - \alpha(a)].$$

(f) If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$.
(g) $f : [a,b] \to [c,d]$ is in $\mathcal{R}(\alpha)$ and $\phi : [c,d] \to \mathbb{R}$ is continuous then $\phi \circ f \in \mathcal{R}(\alpha)$.
(h) If $f \in \mathcal{R}(\alpha)$ then $f^2 \in \mathcal{R}(\alpha)$.
(i) If, $f, g \in \mathcal{R}(\alpha)$ then $fg \in \mathcal{R}(\alpha)$.
(j) If $f \in \mathcal{R}(\alpha)$ then $|f| \in \mathcal{R}(\alpha)$ and

$$\left| \int_a^b f d\alpha \right| \le \int_a^b |f| d\alpha.$$

Proof: (a) Put $h = f + g$. Given $\epsilon > 0$, choose partitions P, Q of $[a, b]$ such that

$$U(P, f) - L(P, f) < \epsilon/2, \quad P(Q, g) - L(Q, g) < \epsilon/2$$

and replace these partitions by their common refinement T and then appeal to

$$L(T, f) + L(T, g) \le L(T, h) \le U(T, h) \le U(T, f) + U(T, g).$$

For a constant λ since

$$U(P, \lambda f) = \lambda U(P, f); \quad L(P, \lambda f) = \lambda L(P.f)$$

it follows that $\int_a^b \lambda f d\alpha = \lambda \int_a^b f d\alpha$. Combining these two we get the proof of (a).
(b) This is easier: In any partition P we have

$$\Delta(\lambda_1\alpha_1 + \lambda_2\alpha_2) = \lambda_1\Delta\alpha_1 + \lambda_2\Delta\alpha_2$$

from which the conclusion follows.
(c) All that we do is to stick to those partitions of $[a, b]$ which contain the point c.
(d) This is easy and
(e) is a consequence of (d).
(f) Given $\epsilon > 0$, put $\epsilon_1 = \frac{\epsilon}{\alpha(b)-\alpha(a)}$. Then by uniform continuity of f, there exists a $\delta > 0$ such that $|f(t) - f(s)| < \epsilon_1$ whenever $t, s \in [a, b]$ and $|t - s| < \delta$. Choose a partition P such that $\Delta\alpha_i < \delta$ for all i. Then it follows that $M_i - m_i < \epsilon_1$ and hence $U(P) - L(P) < \epsilon$.
(g) Given $\epsilon > 0$ by uniform continuity of ϕ, we get $\epsilon > \delta > 0$ such that $|\phi(t) - \phi(s)| < \epsilon$ for all $t, s \in [c, d]$ with $|t - s| < \delta$. There is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \delta^2.$$

The differences $M_i - m_i$ may behave in two different ways: Accordingly let us define

$$A = \{1 \leq i \leq n \ : \ M_i - m_i < \delta\}, \quad B = \{1, 2, \ldots, n\} \setminus A.$$

Put $h = \phi \circ f$. It follows that

$$M_i(h) - m_i(h) < \epsilon, \quad i \in A.$$

Therefore we have

$$\delta(\sum_{i \in B} \Delta\alpha_i) \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i < U(P, f) - L(P, f) < \delta^2.$$

Therefore we have $\sum_{i \in B} \Delta\alpha_i < \delta$. Now let K be a bound for $|\phi(t)|$ on $[c, d]$. Then
$U(P, h) - L(P, h) = \sum_i (M_i(h) - m_i(h))\Delta\alpha_i$
$= \sum_{i \in A} (M_i(h) - m_i(h))\Delta\alpha_i + \sum_{i \in B} (M_i(h) - m_i(h))\Delta\alpha_i$
$\leq \epsilon(\alpha(b) - \alpha(a)) + 2K\delta < \epsilon(\alpha(b) - \alpha(a) + 2K).$
Since $\epsilon > 0$ is arbitrary, we are done.
(h) Follows from (g) by taking $\phi(t) = t^2$.
(i) Write $fg = \frac{1}{4}[(f + g)^2 - (f - g)^2]$.
(j) Take $\phi(t) = |t|$ and apply (g) to see that $|f| \in \mathcal{R}(\alpha)$. Now let $\lambda = \pm 1$ so that $\lambda \int_a^b f d\alpha \geq 0$. Then

$$\left| \int_a^b f d\alpha \right| = \lambda \int_a^b f d\alpha = \int_a^b \lambda f d\alpha \leq \int_a^b |f| d\alpha.$$

This completes the proof of the theorem.

Theorem 4 *Suppose f is monotonic and α is continuous and monotonically increasing. Then $f \in \mathcal{R}(\alpha)$.*

Proof: Given $\epsilon > 0$, by uniform continuity of α we can find a partition P such that each $\Delta\alpha_i < \epsilon$.

Now if f is increasing, then we have $M_i = f(a_i), m_i = f(a_{i-1})$. Therefore,

$$U(P) - L(P) = \sum_i [f(a_i) - f(a_{i-1})]\Delta\alpha_i < f(b) - f(a))\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we are done. ♠

Lecture 21

Theorem 5 *Let f be a bounded function on $[a, b]$ with finitely many discontinuities. Suppose α is continuous at every point where f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.*

Proof: Because of (c) of theorem 3, it is enough to prove this for the case when $c \in [a, b]$ is the only discontinuity of f. Put $K = sup|f(t)|$. Given $\epsilon > 0$, we can find $\delta_1 > 0$ such that $\alpha(c + \delta_1) - \alpha(c - \delta_1) < \epsilon$. By uniform continuity of f on $[a, b] \setminus (c - \delta, c + \delta)$ we can find $\delta_2 > 0$ such that $|x - y| < \delta_2$ implies $|f(x) - f(y)| < \epsilon$. Given any partition P of $[a, b]$ choose a partition Q which contains the points c and whose 'mesh' is less than $\min\{\delta_1, \delta_2\}$. It follows that $U(Q) - L(Q) < \epsilon(\alpha(b) - \alpha(a)) + 2K\epsilon$. Since $\epsilon > 0$ is arbitrary this implies $f \in \mathcal{R}(\alpha)$. ♠

Remark 1 The above result leads one to the following question. Assuming that α is continuous on the whole of $[a, b]$, how large can be the set of discontinuities of a function f such that $f \in \mathcal{R}(\alpha)$? The answer is not within R-S theory. Lebesgue has to invent a new powerful theory which not only answers this and several such questions raised by Riemann integration theory but also provides a sound foundation to the theory of probability.

Example 1 We shall denote the unit step function at 0 by **U** which is defined as follows:

$$\mathbf{U}(x) = \begin{cases} 0, & x \le 0; \\ 1, & x > 0. \end{cases}$$

By shifting the origin at other points we can get other unit step function. For example, suppose $c \in [a, b]$. Consider $\alpha(x) = \mathbf{U}(x - c), x \in [a, b]$. For any bounded function $f : [a, b] \to \mathbb{R}$, let us try to compute $\int_a^b f d\alpha$. Consider any partition P of $[a, b]$ in which $c = a_k$. The only non zero $\Delta\alpha_i$ is $\Delta\alpha_k = 1$. Therefore $U(P) - L(P) = M_k(f) - m_k(f)$.

Now assume that f is continuous at c. Then by choosing a_{k+1} close to $a_k = c$, we can make $M_k - m_k \to 0$. This means that $f \in \mathcal{R}(\alpha)$. Indeed, it follows that $M_k \to f(c)$ and $m_k \to f(c)$. Therefore,

$$\int_a^b f d\alpha = f(c).$$

Now suppose f has a discontinuity at c of the first kind i.e, in particular, $f(c^+)$ exists. It then follows that $|M_k - m_k| \to |f(c) - f(c^+)|$. Therefore, $f \in R(\alpha)$ iff $f(c^+) = f(c)$.

Thus, we see that it is possible to destroy integrability by just disturbing the value of the function at one single point where α itself is discontinuous.

In particular, take $f = \alpha$. It follows that $\alpha \notin \mathcal{R}(\alpha)$ on $[a, b]$.

We shall now prove a partial converse to (c) of Theorem 3.

Theorem 6 *Let f be a bounded function and α an increasing function on $[a, b]$. Let $c \in [a, b]$ at which (at least) f or α is continuous. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on both $[a, c]$ and $[c, b]$; moreover, in that case,*

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

Proof: Assume α is continuous at c. If T_c is the translation function $T_c(x) = x - c$ then the functions $g_1 = \mathbf{U} \circ T$ and $g_2 = 1 - \mathbf{U} \circ T$ are both in $\mathcal{R}(\alpha)$ since they are discontinuous only at c. Therefore $fg_1, fg_2 \in \mathcal{R}(\alpha)$. But these respectively imply that $f \in \mathcal{R}(\alpha)$ on $[c, b]$ and on $[a, c]$.

We now consider the case when f is continuous at c. We shall prove that $f \in \mathcal{R}(\alpha)$ on $[a, c]$, the proof that $f \in \mathcal{R}(\alpha)$ on $[c, b]$ being similar.

Recall that the set of discontinuities of a monotonic function is countable. Therefore there exist a sequence of points c_n in $[a, c]$ (we are assuming that $a < c$) such that $c_n \to c$. By the earlier case $f \in \mathcal{R}(\alpha)$ on each of the intervals $[a, c_n]$. We claim that the sequence

$$s_n := \int_a^{c_n} f d\alpha$$

converges to a limit which is equal to $\int_a^c f d\alpha$. Let $K > 0$ be a bound for α. Given $\epsilon > 0$ we can choose $\delta > 0$ such that for $x, y \in [c - \delta, c + \delta]$, $|f(x) - f(y)| < \epsilon/2K$. If n_0 is big enough then $n, m \geq n_0$ implies that $|s_n - s_m| < \epsilon$. This means $\{s_n\}$ Cauchy and hence is convergent with limit equal to say, s. Now choose n so that $|s - s_n| < \epsilon$.

Put $\Delta = \alpha(c) - \alpha(c^-)$. Since $c_n \to c$, from the left, it follows that $\alpha(c_n) \to \alpha(c^-)$. Choose n large enough so that

$$|\alpha(c_n) - \alpha(c^-)| < \epsilon/L$$

where L is a bound for f.

Now, choose any partition Q of $[a, c_n]$ so that $|U(Q, f) - s_n| < \epsilon$. This is possible because $f \in \mathcal{R}(\alpha)$ on $[a, c_n]$. Put $P = Q \cup \{c\}$, $M = max\{f(x) \ : \ x \in [c_n, c]\}$. Then

$$\begin{aligned}
& |s + \Delta f(c) - U(P, f)| \\
\leq \ & |s - s_n| + |s_n - U(Q, f)| + |\Delta f(c) - (\alpha(c) - \alpha(c_n))M| \\
\leq \ & \epsilon + \epsilon + |\Delta(f(c) - M)| + |(\alpha(c_n) - \alpha(c^-))M| \\
\leq \ & 2\epsilon + \Delta \tfrac{\epsilon}{2K} + |M| \tfrac{\epsilon}{K} \leq 4\epsilon.
\end{aligned}$$

Theorem 7 *Let $\{c_n\}$ be a sequence of non negative real numbers such that $\sum_n c_n < \infty$. Let $t_n \in (a,b)$ be a sequence of distinct points in the open interval and let $\alpha = \sum_n c_n \mathbf{U} \circ T_{-t_n}$. Then for any continuous function f on $[a,b]$ we have*

$$\int_a^b f d\alpha = \sum_n c_n f(t_n).$$

Proof: Observe that for any $x \in [a,b], 0 \leq \sum_n \mathbf{U}(x - t_n) \leq \sum_n c_n$ and hence $\alpha(x)$ makes sense. Also clearly it is monotonically increasing and $\alpha(a) = 0$ and $\alpha(b) = \sum_n c_n$. Given $\epsilon > 0$ choose n_0 such that $\sum_{n > n_0} c_n < \epsilon$. Take

$$\alpha_1 = \sum_{n \leq n_0} \mathbf{U} \circ T_{-t_n}, \qquad \alpha_2 = \sum_{n > n_0} \mathbf{U} \circ T_{-t_n}.$$

By (b) of theorem 3, and from the example above, we have

$$\int_a^b f d\alpha_1 = \sum_{n \leq n_0} c_n f(t_n).$$

If K is bound for $|f|$ on $[a,b]$ we also have

$$\left| \int_a^b f d\alpha_2 \right| < K(\alpha_2(b) - \alpha_2(a)) = K \sum_{n > n_0} c_n = M\epsilon.$$

Therefore,

$$\left| \int_a^b f d\alpha - \sum_{n \leq n_0} c_n f(t_n) \right| < K\epsilon.$$

This proves the claim. ♠

Theorem 8 *Let α be an increasing function and $\alpha' \in \mathcal{R}(x)$ on $[a,b]$. Then for any bounded real function on $[a,b]$, $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}(x)$. Furthermore, in this case,*

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx.$$

Proof: Given $\epsilon > 0$, since α' is Riemann integrable, by (iv) of theorem 2, there exists a partition $P = \{a = a_0 < a_1 < \cdots < a_n = b\}$ of $[a, b]$ such that for all $s_i, t_i \in [a_{i-1}, a_i]$ we have,

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \epsilon.$$

Apply MTV to α to obtain $t_i \in [a_{i-1}, a_i]$ such that $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. Put $M = sup|f(x)|$. Then

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i.$$

Therefore,

$$\left|\sum_{i=1}^{n} f(s_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| < \sum_{i} |f(s_i)||\alpha'(s_i) - \alpha'(t_i)|\Delta x_i > M\epsilon.$$

Therefore

$$\sum_{i=1}^{n} f(s_i)\Delta x_i \leq \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i + M\epsilon \leq U(P, f\alpha') + M\epsilon.$$

Since this is true for arbitrary $s_i \in [a_{i-1}, a_i]$, it follows that

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\epsilon.$$

Likewise, we also obtain

$$U(P, f\alpha') \leq U(P, f, \alpha)) + M\epsilon.$$

Thus

$$|U(P, f, \alpha) - U(P, f\alpha')| < M\epsilon.$$

Exactly in the same manner, we also get

$$|L(P, f, \alpha\_L(P, f\alpha')| < M\epsilon.$$

Note that the above two inequalities hold for refinements of P as well. Now suppose $f \in \mathcal{R}(\alpha)$. we can then assume that the partition P is chosen so that

$$|U(P, f, \alpha) - L(P, f\alpha)| < M\epsilon.$$

It then follows that

$$|U(P, f\alpha') - L(P, f\alpha')| < 3M\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies $f\alpha'$ is Riemann integrable. The other way implication is similar. Moreover, the above inequalities also establish the last part of the theorem. ♠

Remark 2 The above theorems illustrate the power of Stieltjes' modification of Riemann theory. In the first case, α was a staircase function (also called a pure step function). The integral therein is reduced to a finite or infinite sum. In the latter case, α is a differentiable function and the integral reduced to the ordinary Riemann integral. Thus the R-S theory brings brings unification of the discrete case with the continuous case, so that we can treat both of them in one go. As an illustrative example, consider a thin straight wire of finite length. The moment of inertia about an axis perpendicular to the wire and through an end point is given by

$$\int_0^l x^2 dm$$

where $m(x)$ denotes the mass of the segment $[0, x]$ of the wire. If the mass is given by a density function ρ, then $m(x) = \int_0^x \rho(t)dt$ or equivalently, $dm = \rho(x)dx$ and the moment of inertia takes form

$$\int_0^l x^2 \rho(x)dx.$$

On the other hand if the mass is made of finitely many values m_i

concentrated at points x_i then the inertia takes the form

$$\sum_i x_i^2 m_i.$$

Theorem 9 Change of Variable formula *Let* $\phi : [a,b] \to [c,d]$ *be a strictly increasing differentiable function such that* $\phi(a) = c, \phi(b) = d$*. Let* α *be an increasing function on* $[c,d]$ *and* f *be a bounded function on* $[c,d]$ *such that* $f \in \mathcal{R}(\alpha)$*. Put* $\beta = \alpha \circ \phi,\ g = f \circ \phi$*. Then* $g \in \mathcal{R}(\beta)$ *and we have*

$$\int_a^b g d\beta = \int_c^d f d\alpha.$$

Proof: Since ϕ is strictly increasing, it defines a one-one correspondence of partitions of $[a,b]$ with those of $[c,d]$, given by

$$\{a = a_0 < a_1 < \cdots < a_n = b\} \leftrightarrow \{c = \phi(a) < \phi(a_1) < \cdots < \phi(a_n) = d\}.$$

Under this correspondence observe that the value of the two functions f, g are the same and also the value of function α, β are also the same. Therefore, the two upper sums lower sums are the same and hence the two upper and lower integrals are the same. The result follows. ♠

Week 8 Functions of Bounded Variation Lectures 22-24

Lecture 22 : Functions of bounded Variation

Definition 3 Let $f : [a, b] \to \mathbb{R}$ be any function. For each partition $P = \{a = a_0 < a_1 < \cdots < a_n = b\}$ of $[a, b]$, consider the *variations*

$$V(P, f) = \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})|.$$

Let

$$V_f = V_f[a, b] = sup\{V(P, f) \;:\; P \text{ is a partition of } [a, b]\}.$$

If V_f is finite we say f is of *bounded variation on* $[a, b]$. Then V_f is called the *total variation* of f on $[a, b]$. Let us denote the space of all functions of bounded variations on $[a, b]$ by $\mathcal{BV}[a, b]$.

Lemma 2 If Q is a refinement of P then $V(Q, f) \geq V(P, f)$.

Theorem 10 *(a)* $f, g \in \mathcal{BV}[a, b], \alpha, \beta \in \mathbb{R} \Longrightarrow \alpha f + \beta g \in \mathcal{BV}[a, b]$. *Indeed, we also have* $V_{\alpha f+\beta g} \leq |\alpha| V_f + |\beta| V_g$.
(b) $f \in \mathcal{BV}[a, b] \Longrightarrow f$ *is bounded on* $[a, b]$.

(c) $f, g \in \mathcal{BV}[a,b] \implies fg \in \mathcal{BV}[a,b]$. *Indeed, if* $|f| \leq K, |g| \leq L$ *then* $V_{fg} \leq LV_f + KV_g$.
(d) $f \in \mathcal{BV}[a,b]$ *and* f *is bounded away from* 0 *then* $1/f \in \mathcal{BV}[a,b]$.
(e) Given $c \in [a,b]$, $f \in \mathcal{BV}[a,b]$ *iff* $f \in \mathcal{BV}[a,c]$ *and* $f \in \mathcal{BV}[c,b]$. *Moreover, we have*

$$V_f[a,b] = V_f[a,c] + V_f[c,b].$$

(f) For any $f \in \mathcal{BV}[a,b]$ *the function* $V_f : [a,b] \to \mathbb{R}$ *defined by* $V_f(a) = 0$ *and* $V_f(x) = V_f[a,x], a < x \leq b$, *is an increasing function.*
(g) For any $f \in \mathcal{BV}[a,b]$, *the function* $D_f = V_f - f$ *is an increasing function on* $[a,b]$.
(h) Every monotonic function f *on* $[a,b]$ *is of bounded variation on* $[a,b]$.
(i) Any function $f : [a,b] \to \mathbb{R}$ *is in* $\mathcal{BV}[a,b]$ *iff it is the difference of two monotonic functions.*
(j) If f *is continuous on* $[a,b]$ *and differentiable on* (a,b) *with the derivative* f' *bounded on* (a,b), *then* $f \in \mathcal{BV}[a,b]$.
(k) Let $f \in \mathcal{BV}[a,b]$ *and continuous at* $c \in [a,b]$ *iff* $V_f : [a,b] \to \mathbb{R}$ *is continuous at* c.

Proof: (a) Indeed for every partition, we have $V(P, \alpha f + \beta g) = \alpha V(P,f) + \beta V(P,g)$. The result follows upon taking the supremum.
(b) Take $M = V_f + |f(a)|$. Then
$|f(x)| \leq |f(x) - f(a)||f(a)| \leq V(P,f) + |f(a)| \leq M$, where P is any partition in which a, x are consecutive terms.
(c) For any two points x, y we have,

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)||f(x) - g(y)| + |g(y)||f(x) - f(y)|$$

Therefore, it follows that

$$V(P,f) \leq KV_g + LV_f.$$

(d) Let $0 < m < |f(x)|$ for all $x \in [a,b]$. Then

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \left|\frac{f(x) - f(y)}{f(x)f(y)}\right|$$

It follows that $V(P, 1/f) \leq \frac{V_f}{m^2}$ for all partitions P and hence the result.
(e) Given any partition P of $[a,c]$ we can extend it to a partition Q of $[a,b]$ by including the interval $[c,b]$. Then

$$V(Q,f) = V(P,f) + |f(b) - f(c)|$$

and hence it follows that if $f \in \mathcal{BV}[a,b]$ then $f \in \mathcal{BV}[a,c]$; for a similar reason, $f \in \mathcal{BV}[c,b]$ as well. Conversely suppose $f \in \mathcal{BV}[a,c] \cap \mathcal{BV}[c,b]$. Given a partion P of $[a,b]$ we first refine it to P^* by adding the point c and then write $Q = Q_1 \cdot Q_2$ where Q_i are the resstrictions of Q to $[a,c], [c,b]$ respectively. It follows that

$$V(P,f) \leq V(Q,f) = V(Q_1,f) + V(Q_2,f) \leq V_f([a,c] + V_f[c,b].$$

Therefore $f \in \mathcal{BV}[a,b]$. In either case, the above inequality also shows that $V_f[a,b] \leq V_f[,c + V_f[c,b]$. On the other hand, since $V(P,f) \leq V(P^*,f)$ for all P it follows that

$$V_f[a,b] = sup\{V(P^*,f) \ : \ P \text{ is a partition of } [a,b]\}.$$

Since every partition P^* is of the form $P^* = Q_1 \cdot Q_2$ where Q_1, Q_2 are arbitrary partitions of $[a,c]$ and $[c,b]$ respectively, and

$$V(P^*,f) = V(Q_1,f) + V(Q_2,f)$$

it follows that

$$V_f[a,b] = V_f[a,c] + V_f[c,b].$$

(f) Follows from (e)
(g) Let $a \leq x < y \leq b$. Proving $V_f[a,x] - f(x) \leq V_f[a,y] - f(y)$ is the

same as proving $V_f|a,x] + f(y) - f(x) \leq V_f[a,y]$. For any partition P of $[a,x]$ let $P^* = P \cup \{y\}$. Then

$$V(P,f)+f(y)-f(x) \leq V(P,f)+|f(y)-f(x)| = V(P^*,f) \leq V_f[a,y].$$

Since this is true for all partitions P of $[a,x]$ we are through.
(h) May assume f is increasing. But then for every partition P we have $V(P,f) = f(b) - f(a)$ and hence $V_f = f(b) - f(a)$.
(i) If $f \in \mathcal{BV}[a,b]$, from (f) and (g), we have $f = V_f - (V_f - f)$ as a difference of two increasing functions. The converse follows from (a) and (h).
(j) This is because then f satisfies Lipschitz condition

$$|f(x) - f(y)| \leq M|x-y| \quad \text{for all } x, y \in [a,b].$$

Therefore for every partition P we have $V(P,f) \leq M(b-a)$.
(k) Observe that V_f is increasing and hence $V_f(c^{\pm})$ exist. By (h) it follows that same is true for f. We shall show that $f(c) = f(c^{\pm})$ iff $V_f(c) = V_f(c^{\pm})$ which would imply (k). So, assume that $f(c) = f(c^+)$. Given $\epsilon > 0$ we can find $\delta_1 > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $c < x < c + \delta_1, x \in [a,b]$. We can also choose a partition $P = \{c = x_0 < x_1 < \cdots < x_n = b\}$ such that

$$V_f[c,b] - \epsilon < \sum_k \Delta f_k.$$

Put $\delta = \min\{\delta_1, x_1 - c\}$. Let now $c < x < c + \delta$. Then

$$\begin{aligned}
& V_f(x) - V_f(c) \\
= \;& V_f[c,x] = V_f[c,b] - V_f[x,b] \\
< \;& \epsilon + \textstyle\sum_k \Delta f_k - V_f[x,b] \\
\leq \;& \epsilon + |f(x) - f(c)| + |f(x_1) - f(x)| + \textstyle\sum_{k \geq 2} \Delta f_k - V_f[x,b] \\
\leq \;& \epsilon + \epsilon + V_f[x,b] - V_f[x,b] = 2\epsilon.
\end{aligned}$$

This proves that $V_f(c^+) = V_f(c)$ as required.

Conversely, suppose $V_f(c^+) = V_f(c)$. Then given $\epsilon > 0$ we can find $\delta > 0$ such that for all $c < x < c + \delta$ we have $V_f(x) - V_f(c) < \epsilon$. But then given x, y such that $c < y < x < c + \delta$ it follows that

$$|f(y) - f(c)| + |f(x) - f(y)| \leq V_f([c, x] = V_f(x) - V_f(c) < \epsilon$$

which definitely implies that $|f(x) - f(y)| \leq \epsilon$. This completes the proof that $V_f(c^+) = V_f(c)$ iff $f(c^+) = f(c)$. Similar arguments will prove that $V_f(c^-) = V_f(c)$ iff $f(c^-) = f(c)$. ♠

Example 2 Not all continuous functions on a closed and bounded interval are of bounded variation. A typical examples is $f : [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \cos\left(\frac{\pi}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

For each n consider the partition

$$P = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, 1\}$$

Then $V(P, f) = \sum_{k=1}^{n} \frac{1}{k}$. As $n \to \infty$, we know this tends to ∞.

However, the function $g(x) = xf(x)$ is of bounded variation. To see this observe that g is differentiable in $[0, 1]$ and the derivative is bounded (though not continuous) and so we can apply (j) of the above theorem.

Also note that even a partial converse to (j) is not true, i.e., a differentiable function of bounded variation need not have its derivative bounded. For example $h(x) = x^{1/3}$, being increasing function, is of bounded variation on $[0, 1]$ but its derivative is not bounded.

Remark 3 We are now going extend the R-S integral with integrators α not necessarily increasing functions. In this connection, it should be noted that condition (v) of theorem 63 becomes the strongest and hence we adopt that as the definition.

Definition 4 Let $f, \alpha : [a,b] \to \mathbb{R}$ be any two functions. We say f is R-S integrable with respect to to α and write $f \in \mathcal{R}(\alpha)$ if there exists a real number η such that for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that for every refinement $Q = \{a + x_0 < x_1 < \cdots < x_n = b\}$ of P and points $t_i \in [x_{i-1}, x_i]$ we have

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \eta \right| < \epsilon.$$

We then write $\eta = \int_a^b f d\alpha$ and call it R-S integral of f with respect to α.

It should be noted that, in this general situation, several properties listed in Theorem 64 may not be valid. However, property (b) of Theorem 64 is valid and indeed becomes better.

Lemma 3 For any two functions α, β and real numbers λ, μ, if $f \in \mathcal{R}(\alpha) \cap \mathcal{R}(\beta)$, then $f \in \mathcal{R}(\lambda\alpha + \mu\beta)$. Moreover, in this case we have

$$\int_a^b f d(\lambda_\alpha + \mu\beta) = \lambda \int_a^b f d\alpha + \mu \int_a^b f d\beta.$$

Proof: This is so because for any fixed partition we have the linearity property of Δ:

$$\Delta(\lambda\alpha + \mu\beta)_i = (\lambda\alpha + \mu\beta)(x_i - x_{i-1}) = \lambda(\Delta\alpha)_i + \mu(\Delta\beta)_i$$

And hence the same is true of the R-S sums. Therefore, if $\eta = \int_a^b f d\alpha, \gamma = \int_a^b f d\beta$, then it follows that

$$\lambda\eta + \mu\gamma = \int_a^b f d(\lambda\alpha + \mu\beta).$$

♠

Theorem 11 *Let α be a function of bounded variation and let V denote its total variation function $V : [a,b] \to \mathbb{R}$ defined by $V(x) = V_\alpha[a,x]$. Let f be any bounded function. Then $f \in \mathcal{R}(\alpha)$ iff $f \in \mathcal{R}(V_\alpha)$ and $f \in \mathcal{R}(V - \alpha)$.*

Proof: The 'if' part is easy because of (a). Also, we need only prove that if $f \in \mathcal{R}(\alpha)$ then $f \in \mathcal{R}(V)$. Given $\epsilon > 0$ choose a partition P_ϵ so that for all refinements P of P_ϵ, and for all choices of $t_k, s_k \in [a_{i-1}, a_i]$, we have,

$$\left|\sum_{k=1}^{n}(f(t_k) - f(s_k))\Delta\alpha_k\right| < \epsilon, \quad V_f(b) < \sum_k \Delta\alpha_k + \epsilon.$$

We shall establish that

$$U(P,f,V) - L(P,f,V) < \epsilon K$$

for some constant K. By adding and subtracting, this task may be broken up into establishing two inequalities

$$\sum_k [M_k(f) - m_k(f)][\Delta V_k - |\Delta\alpha_k|] < \epsilon K/2; \quad \sum_k [M_k(f) - m_k(f)]|\Delta\alpha_k| < \epsilon K/2.$$

Now observe that $\Delta V_k - |\Delta\alpha_k| \geq 0$ for all k. Therefore if M is a bound for $|f|$, then

$$\begin{aligned} \textstyle\sum_k [M_k(f) - m_k(f)][\Delta V_k - |\Delta\alpha_k|] &\leq 2M \textstyle\sum_k (\Delta V_k - |\Delta\alpha_k|) \\ &\leq 2M(V_f(b) - \textstyle\sum |\Delta\alpha_k|) < 2M\epsilon. \end{aligned}$$

To prove the second inequality, let us put

$$A = \{k \ : \ \Delta\alpha_k \geq 0\}; \quad B = \{1, 2, \ldots, n\} \setminus A.$$

For $k \in A$ choose $t_k, s_k \in [a_{k-1}, a_k]$ such that

$$f(t_k) - f(s_k) > M_k(f) - m_k(f) - \epsilon;$$

and for $k \in B$ choose them so that

$$f(s_k) - f(t_k) > M_k(f) - m_k(f) - \epsilon.$$

We then have

$$\begin{array}{rl} & \sum_k [M_k(f) - m_k(f)]|\Delta\alpha_k| \\ < & \sum_{k\in A}(f(t_k) - f(s_k))|\Delta\alpha_k| + \sum_{k\in B}(f(s_k) - f(t_k))|\Delta_k| + \epsilon\sum_k |\Delta\alpha_k| \\ = & \sum_k [f(t_k) - f(s_k))\Delta\alpha_k + \epsilon V(b) = \epsilon(1 + V(b)). \end{array}$$

Putting $K = max\{2M, 1 + V(b)\}$ we are done. ♠

Corollary 1 *Let* $\alpha : [a, b] \to \mathbb{R}$ *be of bounded variation and* $f : [a, b] \to \mathbb{R}$ *be any function. If* $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *then it is so on every subinterval* $[c, d]$ *of* $[a, b]$.

Corollary 2 *Let* $f : [a, b] \to \mathbb{R}$ *be of bounded variation and* $\alpha : [a, b] \to \mathbb{R}$ *be a continuous of bounded variation. Then* $f \in \mathcal{R}(\alpha)$.

Proof: By (k) of the above theorem, we see that $V(\alpha)$ and $V(\alpha) - \alpha$ are both continuous and increasing. Hence by a previous theorem, $V(f)$ and $V(f) - f$ are both integrable with respect to $V(\alpha)$ and $V(\alpha) - \alpha$. Now we just use the additive property. ♠

Lecture 24

Example 3 :

1. Consider the double sequence,

$$s_{m,n} = \frac{m}{m+n}, \quad m, n \geq 1.$$

Compute the two iterated limits

$$\lim_m \lim_n s_{m,n}, \qquad \lim_n \lim_m s_{m,n}$$

and record your results.

2. Let $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}, n \geq 1$ and put $f(x) = \sum_n f_n(x)$. Check that f_n is continuous. Compute f and see that f is not continuous.

3. Define $g_m(x) = \lim_{n\to\infty}(\cos m!\pi x)^{2n}$ and put $g(x) = \lim_{m\to\infty} g_m(x)$. Compute g and see that g is discontinuous everywhere. Directly check that each g_m is Riemann integrable whereas g is not Riemann integrable.

4. Consider the sequence $h_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ and put $h(x) = \lim_n f_n(x)$. Check that $h \equiv 0$. On the other hand, compute $\lim_n h_n'(x)$. What do you conclude?

5. Put $\lambda_n(x) = n^2 x(1-x^2)^n$, $0 \leq x \leq 1$. Compute the $\lim_n \lambda_n(x)$. On the other hand check that

$$\int_0^1 \lambda_n(x)dx = \frac{n^2}{2n+2} \to \infty.$$

Therefore we have

$$\infty = \lim_n \left[\int_0^1 \lambda_n(x)dx\right] \neq \int_0^1 [\lim_n \lambda_n(x)]dx = 0.$$

These are all examples wherein certain nice properties of functions fail to be preserved under 'point-wise' limit of functions. And we have seen enough results to show that these properties are preserved under uniform convergence.

We know that if a sequence of continuous functions converges uniformly to a function, then the limit function is continuous. We can now ask for the converse: Suppose a sequence of continuous functions f_n converges pointwise to a function f which is also continuous. Is the convergence uniform? The answer in general is NO. But there is a situation when we can say yes as well.

Theorem 12 *Let X be a compact metric space $f_n : X \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a continuous function f. Suppose further that f_n is monotone. Then the $f_n \to f$ uniformly on X.*

Proof: Recall that f is monotone increasing means $f_n(x) \leq f_{n+1}(x)$ for all n and for all x. Likewise f_n is monotone decreasing means $f_n(x) \leq f_{n+1}(x)$ for all n and for all x. Therefore, we can define $g_n(x) = |f(x) - f_n(x)|$ to obtain a sequence of continuous functions which monotonically decreases to 0. It suffices to prove that g_n converges uniformly.

Given $\epsilon > 0$ we want to find n_0 such that $g_n(x) < \epsilon$ for all $n \geq n_0$ and for all $x \in X$. Put

$$K_n = \{x \in X \ : \ g_n(x) \geq \epsilon\}.$$

Then each K_n is a closed subset of X. Also $g_n(x) \geq g_{n+1}(x)$ it follows that $K_{n+1} \subset K_n$. On the other hand, sunce $g_n(x) \to 0$ it follows that $\cap_n K_n = \emptyset$. Since this is happening in a compact space X we conclude that $K_{n_0} = \emptyset$ for n_0. ♠

Remark 4 The compactness is crucial as illustrated by the example:

$$f_n(x) = \frac{1}{nx+1}, \quad 0 < x < 1.$$