Dieses Buch widme ich meiner Frau in Dankbarkeit.

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Table of Contents

List of Notations

Standard Spaces

1. $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n)/x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$
2. $\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n)/z_i \in \mathbb{C},\ i = 1, 2, \ldots, n\}$
3. $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},\ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
4. $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n/\|\mathbf{x}\| = 1\}$
5. $E^n = \{\mathbf{x} \in \mathbb{R}^n/\|\mathbf{x}\| \leq 1\}$
6. I^n is the standard unit cube, the Cartesian product of n copies of $[0, 1]$.
7. $\dot{I}^n$ is the (topological) boundary of I^n.
8. $\mathbb{R}P^n$ is the n-dimensional real projective space
9. $M(n, \mathbb{R})$ is the set of all $n \times n$ matrices with real entries
10. $GL(n, \mathbb{R})$ is the set of all invertible $n \times n$ matrices with real entries
11. $O(n, \mathbb{R})$ is the set of all orthogonal matrices with real entries
12. $SO(n, \mathbb{R})$ is the set of all orthogonal matrices with real entries and determinant one
13. $U(n)$ is the set of all $n \times n$ unitary matrices with complex entries
14. $SU(n)$ is the set of all $n \times n$ unitary matrices with determinant one
15. **Gr** is the category of groups
16. **AbGr** is the category of abelian groups
17. **Top** is the category of topological spaces
18. $\mathbf{Top}^2$ is the category of pairs of topological spaces
19. $\mathbf{Top}_0$ is the category of pointed topological spaces

Standard constructions:

1. $\gamma_1 * \gamma_2$ juxtaposition of paths γ_1 and γ_2
2. $[\gamma]$ the homotopy class of a loop γ (with a chosen base point)
3. $G_1 \oplus G_2$ direct sum of abelian groups or coproduct of G_1 and G_2 in **AbGr**
4. $G_1 * G_2$ Free product of groups or coproduct of G_1 and G_2 in **Gr**
5. $X \sqcup Y$ disjoint union of topological spaces or coproduct of X and Y in **Top**
6. $X \sqcup_f Y$ adjunction space
7. $G_1 \times G_2$ direct product of groups G_1 and G_2
8. $X \times Y$ product of topological spaces X and Y
9. $X \vee Y$ wedge of topological spaces X and Y
10. F_k free group on k generators or $\mathbb{Z} * \mathbb{Z} \cdots * \mathbb{Z}$ (k factors)
11. $[G, G]$ the commutator subgroup of G

Functors and related things:

1. $\pi_1(X, x_0)$ the fundamental group of X with base point x_0
2. $H_n(X)$ the $n-$th homology group of X
3. $H_n(X, A)$ the $n-$th relative homology group of the pair (X, A)
4. $Z_n(X)$ the group of singular $n-$cycles in X

Note to instructors and students

The lectures contain numerous examples and exercises all of which need not be worked through in detail. It is entirely up to the instructor to select a few for illustration in class and assign a few as home assignments. Complete solutions are provided for any exercise that is referred to in the proof of any theorem. In fact solutions to more than half of the exercises are available on line and hints (beyond what has already been indicated against the exercise) are provided for many others.

Depending on the availability of time and the background of the class the instructor may choose to omit some topics altogether. For instance if the class is not well-prepared in general topology the instructor may wish to spend more time on the material covered in the first five lectures and leave out some of the later sections of the first part or discuss them superficially. Another route is to work thorough the first part throughly and leave out some of the technical proofs in the second part. In fact some basic courses on algebraic topology cover only the theory of covering spaces and fundamental groups but this would involve discussing thoroughly the existence of a universal cover and the Galois theory of covering spaces not discussed here. The text of W. Massey may be used as a supplementary reference for these topics. Beyond these broad hints we offer no specific suggestions on what to cover/omit and leave this choice to the instructor.

The examples have been worked out in meticulous detail in order to encourage students to write out clear proofs and adhere to standard levels of mathematical rigor. Hand waving is unfortunately much too common in algebraic topology and often one finds students offering specious arguments. The material is intended for forty one hour sessions six of which are to be used for one hour tests. Some longer topics have been assigned two lectures.

Perhaps more pictures are desirable. We encourage the reader to doodle (preferably with coloured pencils) as he/she goes along drawing relevant figures and diagrams. Lovely pictures of the Klein's bottle and other things are available on the internet.

Prerequisites: This course is aimed at students who are in their second year of Master's program and who have done courses on linear algebra, real analysis, complex analysis, abstract algebra up to and including Sylow theory. Presumably a student of this course would be concurrently doing a course on multi-variable calculus leading to differential forms and Stokes' theorem. We shall freely use ideas from linear algebra and some elementary complex analysis such as properties of the exponential map and Möbius maps as a source of examples. Notions such as orthogonal matrices and the spectral theorem are ubiquitous in all of mathematics and this course is no exception. A student who is uneasy with these notions is advised to brush up these concepts before embarking upon a study of algebraic topology. We shall not use Jordan canonical forms in this course. In algebra we expect the students to be familiar with group actions, isomorphism theorems and notions such as inner automorphisms, center of a group and commutator subgroup.

Lecture I - Introduction

General topology, a language for communicating ideas of continuous geometry, provides us useful tools for studying local properties of space. Notions of compactness and connectedness though important, are quite inadequate for obtaining a reasonable understanding of the global geometry of space. For example, the sphere and the torus are not homeomorphic although they are both compact, path-connected, locally connected metric spaces.

Algebraic topology is a powerful tool in global analysis - the study involving the global geometry of space. It is difficult to define precisely at this point what global analysis is. Perhaps the few examples discussed in the following paragraphs may help in understanding this. The most basic example comes from advanced calculus in connection with Stokes' theorem where a student encounters the notion of orientability of a two dimensional surface in $\mathbb{R}^3$. A sphere is easily seen to be orientable inasmuch as it has "two sides". Small pieces of a surface obviously have "two sides" but the Möbius band "has only one side". How would one formulate a precise notion of an orientable surface and prove that the Möbius band is non-orientable? Is non-orientability an intrinsic property of the surface or does it depend on the way the surface is presented in $\mathbb{R}^3$?

Frequently one also sees an interplay between local and global analysis. The powerful algebraic techniques that we shall develop streamlines the process of piecing together local information (which is often trivial) to provide non-trivial information on the global geometry of space. A good example illustrating this "piecing of local information" is provided by the proof of the famous theorem in complex analysis asserting the impossibility of a continuous branch of the argument function on the punctured plane $\mathbb{C} - \{0\}$. Although formal use of algebraic topology can be avoided for this specific case, it is less obvious that the function $\sqrt{1-z^2}$ is holomorphic on $\mathbb{C} - [-1, 1]$. Analogous problems in several dimensions would be practically intractable without the use of algebraic topology or some other equally powerful tool in global analysis.

The first example in our list is provided by the famous Jordan curve theorem which also arose in connections with complex analysis.

Theorem 1.1 (Jordan Curve Theorem): A simple closed curve separates the plane into two disjoint open connected sets precisely one of which is bounded.

The theorem was used by Jordan in his formulation of Cauchy's theorem. Though the Jordan curve theorem no longer plays a central rôle in complex function theory it is nevertheless indispensable in many other branches such as ordinary differential equations. Let us consider the (non-trivial) problem of locating periodic solutions of systems of differential equations. In planar domains, a useful criterion is given by the following

Theorem 1.2 (Poincare Bendixon): Suppose given a planar system of differential equations

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1.1}$$

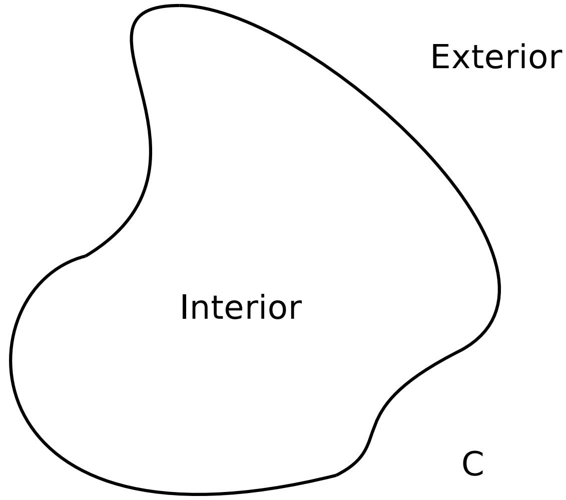


Figure 1: Simple closed curve

where $P(x, y)$ and $Q(x, y)$ are smooth functions in the plane. Assume that there is an annulus Ω not containing rest points[1] and invariant under the flow of the differential equation[2]. Then Ω must contain periodic orbits.

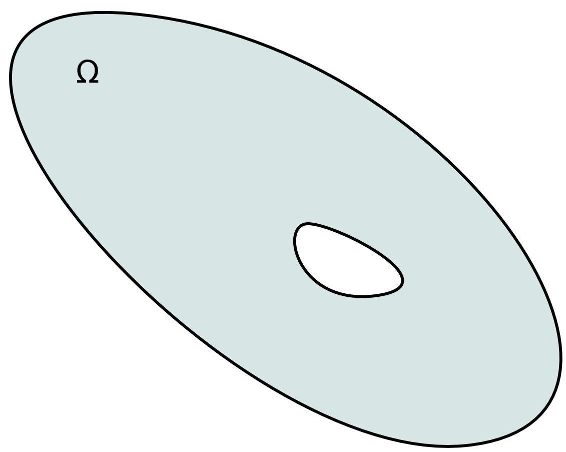


Figure 2: Invariant Annulus

The proof of this important result requires the Jordan curve theorem ([8], pp. 52-54). The analogue of theorem (1.2) is true for differential equations on the sphere but is false for differential equations on the torus. The Poincaré Bendixon theorem may be used to prove the existence of limit cycles for the Van der Pol oscillator

$$\dot{x} = -y, \quad \dot{y} = x + \epsilon(x^2 - 1)y$$

by finding an invariant annulus for the flow ([8], pp. 60-61). Another result from the theory of ordinary differential equations is the following result stated for planar systems (1.1) but holds in higher dimensions also. A proof may be given using Stokes' theorem or the Brouwer's fixed point theorem (see [8], p. 49).

[1]These are the common zeros of the pair $P(x, y)$ and $Q(x, y)$.

[2]This means a trajectory (solution curve) starting at a point of Ω stays in Ω for all times.

Theorem 1.3 Every closed trajectory of the system (1.1) contains a rest point in its interior.

Algebraic topology is a branch of geometry where properties of space are studied by assigning algebraic invariants (such as groups, rings etc.,) to space. Thus to each topological space X we attach an algebraic object such as a group $h(X)$ and to each continuous map $f : X \longrightarrow Y$ we attach a group homomorphism $h(f) : h(X) \longrightarrow h(Y)$ satisfying two basic properties:

1. If X, Y and Z are three topological spaces and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous maps, then the corresponding group homomorphisms $h(f) : h(X) \longrightarrow h(Y)$, $h(g) : h(Y) \longrightarrow h(Z)$ and $h(g \circ f) : h(X) \longrightarrow h(Z)$ must satisfy the condition
$$h(g \circ f) = h(g) \circ h(f).$$
2. The identity map $i : X \longrightarrow X$ corresponds to the identity map $h(i) : h(X) \longrightarrow h(X)$

These properties are summarized by the statement that h is a (covariant) functor from the category of topological spaces to the category of groups. We shall provide formal definitions of a category and functor elucidating them through examples as we go along.

We shall introduce two important functors - the fundamental groups and the homology groups. We also indicate how these functors help in the understanding (under restrictive conditions) of two fundamental problems in topology - the extension problem and the lifting problem. The Tietze's extension theorem which provides a solution to the extension problem in certain special but important cases, is recalled in lecture 3 where we also place it against the general background of the extension problem. The extension problem reappears again in lecture 10 in connection with the Brouwer's fixed point theorem. Certain questions in complex analysis lead us naturally to the lifting problem as elaborated in lecture 18.

The course is organized as follows. Lectures 1 through 26 constitute the first part on fundamental groups and covering spaces. The second part on singular homology is covered in lectures 28 through 40. We begin with a review of general topology in the next four lectures. We shall touch upon some of the important results on compactness, connectedness, path-connectedness and their local analogues. This is followed by a longer chapter on quotient spaces with a large supply of examples that would occur frequently in the subsequent lectures. The exercises at the end of the lectures are designed as a warm up on these notions. The universal properties of the quotient is emphasized. We shall introduce the notion of a topological group in lecture 5 and discuss some important examples.

In the next lecture we introduce one of the principal thespians of the play - the fundamental group of a topological space. The theme will be developed in the subsequent lectures. The first non-trivial result is that the fundamental group of a circle is the group of integers which in turn implies several important results such as the Brouwer's fixed point theorem and the Perron-Frobenius theorem from matrix theory. The theory of covering spaces will be developed in lectures 13-17. The theory of covering spaces is important in many areas of mathematics but we shall study it here in close connection with the theory of the fundamental group. We introduce one of the fundamental problems in topology namely, the lifting problem for which an elegant solution is available in the context of covering spaces.

Many important spaces in mathematics such as the Klein's bottle, projective spaces and Riemann surfaces (the torus being an important example) occur as orbit spaces under the action of discrete groups. Lecture 18 is devoted to many of these examples. Unfortunately limitations in space and time prevent us from discussing the existence of a universal covering for a space.

Algebraic topology is certainly not a stand alone subject and we have tried (to the extent possible) to indicate connections with other areas of mathematics.

Lecture II - Preliminaries from general topology:

We discuss in this lecture a few notions of general topology that are covered in earlier courses but are of frequent use in algebraic topology. We shall prove the existence of Lebesgue number for a covering, introduce the notion of proper maps and discuss in some detail the stereographic projection and Alexandroff's one point compactification. We shall also discuss an important example based on the fact that the sphere S^n is the one point compactifiaction of $\mathbb{R}^n$. Let us begin by recalling the basic definition of compactness and the statement of the Heine Borel theorem.

Definition 2.1: A space X is said to be compact if every open cover of X has a finite sub-cover. If X is a metric space, this is equivalent to the statement that every sequence has a convergent subsequence.

If X is a topological space and A is a subset of X we say that A is compact if it is so as a topological space with the subspace topology. This is the same as saying that every covering of A by open sets in X admits a finite subcovering. It is clear that a closed subset of a compact subset is necessarily compact. However a compact set need not be closed as can be seen by looking at X endowed the indiscrete topology, where every subset of X is compact. However, if X is a Hausdorff space then compact subsets are necessarily closed. We shall always work with Hausdorff spaces in this course. For subsets of $\mathbb{R}^n$ we have the following powerful result.

Theorem 2.1(Heine Borel): A subset of $\mathbb{R}^n$ is compact (with respect to the subspace topology) if and only if it is closed and bounded.

The theorem provides a profusion of examples of compact spaces.

1. The unit sphere $S^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$ is compact.

2. The unit square $I^2 = [0, 1] \times [0, 1]$ is compact.

3. The set of all 3×3 matrices is clearly homeomorphic to $\mathbb{R}^9$. Then the set of all 3×3 orthogonal matrices, denoted by $O(3, \mathbb{R})$ is is compact. That is to say the orthogonal group is compact. The result readily generalizes to the group of $n \times n$ orthogonal matrices.

4. Think of the set of all $n \times n$ matrices with complex entries as $\mathbb{C}^{n^2}$ which in turn may be viewed as $\mathbb{R}^{2n^2}$. The set of all $n \times n$ unitary matrices is then easily seen to be a compact space. These matrices form a group known as the unitary group $U(n)$.

5. The set of all $n \times n$ unitary matrices with determinant one is also a closed bounded subset of $\mathbb{C}^{n^2}$ and so is compact. This is the special unitary group $SU(n)$.

Theorem 2.2: Suppose that X is a compact topological space, Y is an arbitrary Hausdorff space and $f : X \longrightarrow Y$ is a continuous surjection then

1. Y is compact.

2. If A is a closed subset of X then $f(A)$ is closed.

3. It f is bijective then f is a homeomorphism.

Proof: The first assertion is proved in courses on point set topology. We remark that the Hausdorff assumption is not necessary for (i). The second follows from the first and we shall prove the third which will be of immense use in the sequel. Let g be the inverse of f and A be closed in X then $g^{-1}(A) = f(A)$ is closed in Y from which continuity of g follows.

Definition 2.2 (The Lebsesgue number for a cover): Given an open covering $\{G_\alpha\}$ of a metric space X, a Lebesgue number for the covering is a positive number ϵ such that every ball of radius ϵ is contained in some member G_α of the cover.

Theorem 2.3: Every open covering of a compact metric space has a Lebesgue number.

Proof: The student is advised to draw relevant pictures as he reads on. Suppose that a cover $\{G_\alpha\}$ has no Lebesgue number. Then for every $n \in \mathbb{N}$, $1/n$ is not a Lebesgue number and so there is a point $x_n \in X$ such that the ball of radius $1/n$ centered at x_n is not contained in any of the open sets in the covering. By compactness the sequence $\{x_n\}$ has a convergent subsequence converging to a point $p \in X$. Choose an α such that G_α contains p and there is a $\delta > 0$ such that the ball of radius δ around p is contained in G_α. Now take n large enough that $1/n < \delta/3$ and x_n is contained in the ball of radius $\delta/3$ centered at p.

Now, since the ball of radius $1/n$ with center x_n is not contained in any of the open sets in our covering, there exists $y_n \in X$ such that $y_n \notin G_\alpha$ and $d(x_n, y_n) < 1/n$. But

$$d(p, y_n) \leq d(p, x_n) + d(x_n, y_n) < 2\delta/3 < \delta.$$

So y_n is in the ball of radius δ centered at p and so $y_n \in G_\alpha$ which is a contradiction.

Definition 2.3 (Locally compact spaces): A (Hausdorff) space X is said to be locally compact if each point of X has a neighborhood whose closure is compact.

It is an exercise for the student to check that under this hypothesis each point of X has a local base of consisting of compact neighborhoods.

Examples: The reader may easily verify the following.

1. Open subsets of $\mathbb{R}^n$ are locally compact.

2. $\mathbb{Q}$ is not locally compact.

Locally compact spaces are easily realized as dense open subsets of compact spaces. One has to merely adjoin one additional point. The idea is important in many applications and is called Alexandroff's one point compactification.

One point compactification: Let X be a locally compact, non-compact Hausdorff space and $\widehat{X} = X \cup \{\infty\}$ be the one point union of X with an additional point ∞. The topology $\mathcal{T}$ consists of all the open subsets in X as well as all the subsets of the form $\{\infty\} \cup (X - K)$, where K ranges over all the compact subsets of X. The following theorem summarizes the properties of $\widehat{X}$ and the proof is left for the reader.

Theorem 2.4: (i) The collection of sets $\mathcal{T}$ is a topology on $\widehat{X}$.

(ii) The family of sets $\widehat{X} - K$, where K ranges over all compact subsets of X, forms a neighborhood base of ∞.

(iii) X with the given topology is an open dense subset of $\widehat{X}$.

(iv) The space $\widehat{X}$ is compact.

Definition 2.4 (Proper maps): A map $f : X \longrightarrow Y$ between topological spaces is said to be proper if $f^{-1}(C)$ is a compact subset of X whenever C is a compact subset of Y.

Theorem 2.5: Suppose X and Y are locally compact spaces and $f : X \longrightarrow Y$ is a continuous proper map then it extends continuously as a map $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ between their one point compactifications.

Proof: Denote the adjoined points in $\widehat{X}$ and $\widehat{Y}$ as p and q respectively and extend the given map by sending p to q. We need to show that the extension is continuous at p. Let C be any compact subset of Y so that $K = f^{-1}(C)$ is compact in X. Then $N = \widehat{X} - K$ is a neighborhood of p in $\widehat{X}$ that is mapped by $\widehat{f}$ into the preassigned neighborhood $\widehat{Y} - C$ of q. This proves the continuity of the extension.

The converse is not true as the constant map shows. However the following version in the reverse direction is easy to see,

Theorem 2.6: Suppose X and Y are locally compact Hausdorff spaces and $f : \widehat{X} \longrightarrow \widehat{Y}$ is a continuous map such that $f^{-1}(q) = \{p\}$, where p and q are as in the previous theorem, then the restriction of f to X is a proper map.

Proof: If C is a compact subset of Y then $f^{-1}(C)$ being a closed subset of $\widehat{X}$ is compact. The hypothesis says that $f^{-1}(C)$ does not contain p and hence is a compact subset of X itself.

Stereographic projection: Consider the sphere

$$S^n = \Big\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\Big\}$$

and the plane $x_{n+1} = 0$ of the equator. Let $\mathbf{n} = (0, 0, \ldots, 0, 1)$ and $\mathbf{x}$ be a general point on the equatorial plane. The line through $\mathbf{n}$ and $\mathbf{x}$ is described parametrically by $(1-t)\mathbf{n} + t\mathbf{x}$ and meets the sphere at points corresponding to the roots of the quadratic equation

$$\langle (1-t)\mathbf{n} + t\mathbf{x}, (1-t)\mathbf{n} + t\mathbf{x} \rangle \ = \ 1.$$

The root $t = 0$ corresponds to the point $\mathbf{n}$ and the second root

$$t = \frac{2(1 - \mathbf{n}\cdot\mathbf{x})}{1 + \|\mathbf{x}\|^2 - 2\mathbf{n}\cdot\mathbf{x}}$$

is continuous with respect to $\mathbf{x}$ and provides a point $F(\mathbf{x}) \in S^n - \{\mathbf{n}\}$. The map F is a bijective continuous map between the plane $x_{n+1} = 0$ and $S^n - \{\mathbf{n}\}$. Note that the origin is mapped to the south pole by F. The inverse map G is called the stereographic projection. Let us now show that G is also continuous whereby it follows that F is a homeomorphism.

Well, let $\mathbf{y}$ be a point on the sphere minus the north pole $\mathbf{n}$. The ray emanating from $\mathbf{n}$ and passing through $\mathbf{y}$ meets the plane at the point

$$G(\mathbf{y}) = \Big(\frac{y_1}{1 - y_{n+1}}, \frac{y_2}{1 - y_{n+1}}, \dots, \frac{y_n}{1 - y_{n+1}}\Big)$$

We see that G is also continuous and so the sphere minus its north pole is homeomorphic to $\mathbb{R}^n$.

It is useful to note that the stereographic projection takes points $\mathbf{y}$ close to the north pole to points $G(\mathbf{y})$ of $\mathbb{R}^n$ such that $\|G(\mathbf{y})\| \to +\infty$. We summarize the discussion as a theorem.

Theorem 2.7: The unit sphere in S^n is homeomorphic to the one point compactification of $\mathbb{R}^n$.

Theorem 2.8: Suppose that T is a linear transformation of $\mathbb{R}^n$ into itself and not the zero map, then T extends as a continuous map of S^n to itself if and only if T is non-singular.

Proof: Note that if T is non-singular, it is a proper map and so it extends continuously as a map of S^n sending the point at infinity to itself. Conversely, if T fails to be bijective then there is a sequence of points $\mathbf{x}_n$ such that $\|\mathbf{x_n}\| \to +\infty$ but $T(\mathbf{x}_n) = 0$ for every n. Thus if T were to extend continuously as a map of S^n we would be forced to map the point at infinity namely the north pole to (the point of S^n corresponding to) the origin. On the other hand since T is not the zero map, pick a vector $\mathbf{u}$ such that $T\mathbf{u} \neq 0$ and the sequence $m\mathbf{u}$ converges (as $m \to \infty$) to the point at infinity on S^n. Thus by continuity we would have $\lim T(m\mathbf{u}) = 0$, as $m \to 0$. Hence, $m\|T\mathbf{u}\| \longrightarrow 0$ which is plainly false since $T\mathbf{u} \neq 0$.

More important examples are furnished by regarding S^2 as the one point compactification of the plane $\mathbb{C}$ and using the field structure on the plane. The proof of the following is an exercise.

Theorem 2.9: Any non-constant polynomial is a proper map of $\mathbb{C}$ onto itself and so may be viewed as a continuous map of S^2 to itself fixing the point at infinity.

Exercises

1. Prove that a topological space is compact if and only if it satisfies the following condition known as the *finite intersection property.* For every family $\{F_\alpha\}$ of closed sets with $\cap_\alpha F_\alpha = \emptyset$, there is a finite sub-collection whose intersection is empty

2. Show that $f : [0,1] \longrightarrow [0,1]$ is continuous if and only if its graph is a compact subset of I^2.

3. Examine whether the exponential map from $\mathbb{C}$ onto $\mathbb{C} - \{0\}$ is proper. What about the exponential map as a map from $\mathbb{R}$ onto $(0, \infty)$?

4. (Gluing Lemma) Suppose that $\Big\{U_\alpha\Big\}_{\alpha\in\Lambda}$ is a family of open subsets of a topological space and for each $\alpha \in \Lambda$ we are given a continuous function $f_\alpha : U_\alpha \longrightarrow Y$. Assume that whenever $f_\alpha(x) = f_\beta(x)$ whenever $x \in U_\alpha \cap U_\beta$. Show that there exists a unique continuous function $f : \bigcup_{\alpha\in\Lambda} U_\alpha \longrightarrow Y$ such that $f(x) = f_\alpha(x)$ for all $x \in U_\alpha$ and for all $\alpha \in \Lambda$. Show that the result holds if all the U_α are closed sets and Λ is a finite set.

5. How would you show rigorously that the closed unit disc in the plane is homeomorphic to the closed triangular region determined by three non-collinear points? You are allowed to use results from complex analysis, provided you state them clearly.

6. Prove that any two closed triangular planar regions (as described in the previous exercise) are homeomorphic. Show that any such closed triangular region is homeomorphic to I^2.

7. Suppose that Z is a Hausdorff space and $f, g : Z \longrightarrow X$ are continuous functions then the set $\{z \in Z / f_1(z) = f_2(z)\}$ is closed in Z.

8. Show that the space obtained by rotating the circle $(x-2)^2 + y^2 = 1$ about the $y-$axis is homeomorphic to $S^1 \times S^1$.

Lecture III - More preliminaries from general topology:

In this lecture we take up the second most important notion in point set topology, namely the notion of connectedness. This topic is usually covered in good detail in point set topology courses. Again we shall merely outline the theory emphasizing examples rather than proving standard results. We begin by recalling the definition of a connected subset of a topological space ([13], p. 42).

Definition 3.1: A subset Y of a topological space X is said to be disconnected if there are non-empty subsets A and B of X such that

$$Y = A \cup B, \quad \overline{A} \cap B = \emptyset, \quad A \cap \overline{B} = \emptyset.$$

If Y is not disconnected we say that Y is connected.

Examples 3.1: (i) The intervals $[0, 1]$ and $(0, 1)$ on the real line are connected. The only connected subsets of the real line are intervals (including the empty set). Hence the only connected subsets of $\mathbb{Z}$ are singletons and the empty set.

(ii) Product of connected spaces are connected. Thus the cube $[0, 1] \times [0, 1] \times [0, 1]$ is connected.

We now state the most basic theorem on connectedness whose proof ought to be done in standard courses on general topology and will not be repeated here.

Theorem 3.1: (i) If X and Y are topological spaces and $f : X \longrightarrow Y$ is a continuous map and A is a connected subset of X then $f(A)$ is a connected subset of Y.

(ii) A topological space X is connected if and only if every continuous function $f : X \longrightarrow \mathbb{Z}$ is constant.

(iii) If $\{A_n\}$ is a sequence of connected subsets of a topological space X and $A_n \cap A_{n+1}$ is non-empty for each $n = 1, 2, 3, \ldots$ then $\cup_{n=1}^{\infty} A_n$ is connected. In particular, taking $A_2 = A_3 = \ldots$ we get the result for two connected sets.

(iv) If A_α is a family of connected subsets of a topological space such that for some connected subset B, $A_\alpha \cap B \neq \emptyset$ for each α, then $\bigcup_\alpha A_\alpha$ is also connected.

(v) Suppose that A is a connected subset of a topological space and $A \subset B \subset \overline{A}$ then B is also connected.

(vi) A space X is connected if and only if the only subsets of X that are open and closed are X and $\emptyset$.

Example 3.2: The theorem can be used to prove that the sphere

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} / x_1^2 + x_2^2 + \ldots x_{n+1}^2 = 1\}$$

is connected. Define $S^n_\pm$ to be the closed upper and lower hemispheres. Then $S^n_\pm$ are connected. The intersection of these hemispheres is S^{n-1}. One can now apply induction observing first that the circle S^1 is connected since it is the continuous image of the real line under the map

$$t \mapsto \exp(2\pi i t).$$

Example 3.3: The set $GL(n, \mathbb{R})$ of all $n \times n$ invertible matrices with real entries is disconnected as a subspace of the space of all $n \times n$ matrices with real entries (the latter may be identified with $\mathbb{R}^{n^2}$).

If $GL(n, \mathbb{R})$ were connected then so would be its image under a continuous map. Well, the determinant map $d : GL(n, \mathbb{R}) \longrightarrow \mathbb{R}$ is continuous but the image is the real line minus the origin. The same argument shows that the set of all $n \times n$ orthogonal matrices $O(n, \mathbb{R})$ is disconnected.

Definition 3.2 (Path connectedness): A space X is said to be path connected if given any two points $x, y \in X$, there is a continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Theorem 3.3: If X is path connected then it is connected.

Proof: Assume X is path connected but not connected. Then there is a non constant continuous function $g : X \longrightarrow \mathbb{Z}$ say $f(x) = m$ and $f(y) = n$ for a pair of distinct integers m and n. But there is also a continuous function $f : [0.1] \longrightarrow X$ such that $g(0) = x$ and $g(1) = y$. The composite function $f \circ g : [0, 1] \longrightarrow \mathbb{Z}$ is non constant which is a contradiction.

Corollary 3.4: A convex set in $\mathbb{R}^n$ and more generally a star shaped set in $\mathbb{R}^n$ is path connected and hence connected. In particular, the square $[0, 1] \times [0, 1]$ is path connected and hence connected.

Theorem 3.5: (i) If X and Y are topological spaces and $f : X \longrightarrow Y$ is a continuous map and A is a path connected subset of X then $f(A)$ is also a path connected subset of Y.

(ii) If $\{A_n\}$ is a sequence of path connected subsets of a topological space X and $A_n \cap A_{n+1}$ is non-empty for each $n = 1, 2, 3, \ldots$ then $\cup_{n=1}^{\infty} A_n$ is path connected. In particular, taking $A_2 = A_3 = \ldots$ we see get the result for two path connected sets.

Proof: This is usually done in point set topology courses and so the proof will not be repeated here.

Definition 3.3: A space X is said to be locally path connected if each point of X has a local base consisting of path connected neighborhoods.

Theorem 3.6: A connected, locally path-connected space is path connected. In particular, an open subset of $\mathbb{R}^n$ is path connected.

Proof: Let x and y be arbitrary points of X and let G be the set of all points of X that can be joined to x by a path. Clearly G is non-empty since it contains the point x. If we show that G is open and closed then by connectedness of X it would follow that G would equal the whole space X. In particular G contains y thereby proving that there is a path in X joining x and y. First we show that G is open. Well, let z be an arbitrary point of G and choose a path $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = z$. Choose a path connected neighborhood N of z and $w \in N$ be arbitrary. Then there is a path σ lying in N joining z and w. We now juxtapose γ and σ by defining $\eta : [0, 1] \longrightarrow X$ as

$$\eta(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2 \\ \sigma(2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

By virtue of the gluing lemma η is continuous and defines a path joining x and w. Hence w belongs to G and so $N \subset G$. We now show that G is closed as well. Let $y \notin G$ and N be a path connected neighborhood of y. Then we show that $N \subset X - G$. Well, if not, pick $z \in G \cap N$ and there is a path γ in G joining x and z and a path σ in N joining z and y. Juxtaposing we would get a path in X joining x and y which would contradict the fact that $y \notin G$. So $X - G$ is also open in X and the proof is complete.

The Tietze's extension theorem: We shall make occasional use of this in the sequel. Since we need it for the special case of metric spaces, we shall state the theorem in this context.

Theorem 3.7: Suppose that X is a metric space, A is a closed subspace of X and $f : A \longrightarrow \mathbb{R}$ is a continuous function then f extends continuously to the whole of X. Furthermore if f is bounded from above/below then the extension may be so chosen that the bound(s) are preserved.

Remarks: Note that the Tietze's extension theorem is valid for maps taking values in $\mathbb{R}^n$ or a finite product of intervals such as $[0, 1]^n$.

Exercises:

1. Prove that any continuous function $f : [-1, 1] \longrightarrow [-1, 1]$ has a fixed point, that is to say, there exists a point $x \in [-1, 1]$ such that $f(x) = x$.

2. Prove that the unit interval $[0, 1]$ is connected. Is it true that if $f : [0, 1] \longrightarrow [0, 1]$ has connected graph then f is continuous? What if connectedness is replaced by path connectedness?

3. Suppose X is a locally compact, non-compact, connected Hausdorff space, is its one point compactification connected? What happens if X is already compact and Hausdorff?

4. Show that any connected metric space with more than one point must be uncountable. Hint: Use Tietze's extension theorem and the fact that the connected sets in the real line are intervals.

5. Show that the complement of a two dimensional linear subspace in $\mathbb{R}^4$ is connected. Hint: Denoting by V be the two dimensional vector space, show that $\Sigma = \{\mathbf{x}/\|\mathbf{x}\| \ / \ \mathbf{x} \in \mathbb{R}^4 - V\}$ is connected using stereographic projection or otherwise.

6. How many connected components are there in the complement of the cone
$$x_1^2 + x_2^2 + x_3^3 - x_4^2 = 0$$
in $\mathbb{R}^4$? Hint: The complement of this cone is filled up by families of hyperboloids. Examine if there is a connected set B meeting each member of a given family.

7. A map $f : X \longrightarrow Y$ is said to be a *local homeomorphism* if for $x \in X$ there exist neighborhoods U of x and V of $f(x)$ such that $f\Big|_U : U \longrightarrow V$ is a homeomorphism. If $f : X \longrightarrow Y$ is a local homeomorphism and a proper map, then for each $y \in Y$, $f^{-1}(y)$ is a finite set. Show that the map $f : \mathbb{C} - \{1, -1\} \longrightarrow \mathbb{C}$ given by $f(z) = z^3 - 3z$ is a local homeomorphism. Is it a proper map?

Lecture IV - Further preliminaries from general topology:

We now begin with some preliminaries from general topology that is usually not covered or else is often perfunctorily treated in elementary courses on topology. Since many important examples in topology arise as quotient spaces, this lecture is completely devoted to this topic.

Quotient Spaces: Suppose that X is a topological space and $f : X \longrightarrow Y$ is a surjective mapping, let us consider the various topologies on Y with respect to which f is continuous. Certainly the function f would be continuous if Y carries the trivial topology where the only open sets are $\emptyset$ and Y. The quotient topology on Y is the strongest topology that makes f continuous. More explicitly consider the family

$$\mathcal{T} = \{A \subset Y : f^{-1}(A) \text{ is open in } X\}. \tag{4.1}$$

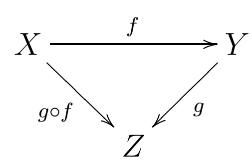
Since $\mathcal{T}$ is closed under arbitrary unions, finite intersections and contains Y and the empty set, we conclude that $\mathcal{T}$ is a topology on Y with respect to which f is continuous. It is also clear that any strictly larger topology would render f discontinuous.

Definition 4.1: (i) Given a topological space X, a set Y and a surjective map $f : X \longrightarrow Y$, the topology $\mathcal{T}$ defined by (4.1) is called the *quotient topology* on Y induced by the function f. By construction f is continuous with this topology on Y.

(ii) Given a map $f : X \longrightarrow Y$ between topological spaces X and Y, we say f is a *quotient map* if the given topology on Y agrees with the quotient topology on Y induced by f.

The quotient topology enjoys a universal property which is easy to prove but extremely useful.

Theorem 4.1 (Universal property of quotients): Suppose that X is a topological space, Y is a set, $f : X \longrightarrow Y$ is a surjective map and Y is assigned the quotient topology induced by f. Then given any topological space Z and map $g : Y \longrightarrow Z$, the map g is continuous if and only if $g \circ f : X \longrightarrow Z$ is continuous.



Proof: If g is continuous it is trivial that $g \circ f$ is continuous. Conversely suppose that $g \circ f$ is continuous. Let A be any open set in Z so that $(g \circ f)^{-1}(A)$ is open in X. Thus $f^{-1}(g^{-1}(A))$ is open in X. Invoking the definition of the quotient topology, we see that $g^{-1}(A)$ must be open in Y which means g is continuous.

Before illustrating the use of the universal property of quotients we discuss the following issue. Suppose that X and Y are topological spaces and $f : X \longrightarrow Y$ is a given continuous map, then the quotient topology on Y induced by f is weaker than the given topology on Y. When would the given topology on Y be equal to the quotient topology?

Definition 4.2: A (not necessarily continuous) map $f : X \longrightarrow Y$ between topological spaces X and Y is said to be a closed map if the image of closed sets is closed. Likewise we say f is an open mapping if the image of open sets is open.

Example 4.1: (i) Suppose X is a compact space and Y is a Hausdorff space then any surjective continuous map $f : X \longrightarrow Y$ is a closed map.

(ii) The reader may check that $\phi : \mathbb{R} \longrightarrow S^1$ given by $\phi(t) = \exp(2\pi i t)$ is an open mapping.

(iii) The map $\phi : [0, 1] \longrightarrow S^1$ given by $\phi(t) = \exp(2\pi i t)$ is closed but not open.

Theorem 4.2: Suppose that X and Y are topological spaces and $f : X \longrightarrow Y$ is a surjective continuous closed/open map then the quotient topology on Y induced by f agrees with the given topology on Y.

Proof: The quotient topology on Y induced by f is stronger than the given topology. To obtain the reverse inclusion, suppose that f is a continuous open mapping and $A \subset Y$ is open with respect to the quotient topology on Y induced by f which means $f^{-1}(A)$ is open in X whereby $f(f^{-1}(A))$ is open in Y in the given topology since f is an open mapping. But since f is surjective, $f(f^{-1}(A)) = A$ and so we conclude A is open in the given topology as well.

Let us now turn to a continuous, closed surjective map $f : X \longrightarrow Y$. Again we merely have to show that the given topology on Y is stronger than the quotient topology since the reverse inclusion is trivial. So let A be an open set in Y with respect to the quotient topology induced by f. By definition $f^{-1}(A)$ is open in X, or in other words $X - f^{-1}(A)$ is closed in X. Since f is closed, $f(X - f^{-1}(A)) = Y - A$ is closed in Y with respect to the given topology on Y. That is to say A is open with respect to the given topology on Y.

Corollary 4.3: Suppose X is a compact space and Y is a Hausdorff space then any continuous surjection from X onto Y is a quotient map.

Identification spaces: Suppose that X is a topological space and $\sim$ is an equivalence relation on X. The set of all equivalence classes is denoted by $X/\sim$ and $\eta : X \longrightarrow X/\sim$ denotes the projection map

$$\eta(x) = \overline{x}, \quad x \in X,$$

where $\overline{x}$ denotes the equivalence class of x. The space $X/\sim$ with the quotient topology induced by η is called the identification space given by the equivalence relation. An important special case deserves mention as it is of frequent occurrence. Suppose that A is a subset of a topological space then we consider the equivalence relation for which all the points of A form one equivalence class and the equivalence class of any $x \in X - A$ is a singleton. That is to say all the points of A are identified together as one point and no other identification is made. We shall refer to the resulting quotient space as **the space obtained from X by collapsing A to a singleton**.

Theorem 4.4: Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a surjective continuous map such that the given topology on Y agrees with the quotient topology on Y induced by f. Define an equivalence relation $\sim$ on X as follows:

$$x_1 \sim x_2 \text{ if and only if } f(x_1) = f(x_2), \quad x_1, x_2 \in X$$

The identification space $X/\sim$ is homeomorphic to Y via the map $\phi : X/\sim \longrightarrow Y$ given by

$$\phi(\overline{x}) = f(x).$$

Proof: It is easy to see that ϕ is well-defined, bijective and satisfies $\phi \circ \eta = f$. Since f is continuous and η is a quotient map we see by the universal property that ϕ is continuous. Since f is a quotient map and η is continuous we may invoke the universal property again but this time to $\phi^{-1} \circ f = \eta$ to conclude that ϕ^{-1} is continuous as well.

The real projective spaces $\mathbb{R}P^n$: The projective space $\mathbb{R}P^n$ is the identification space obtained from the sphere S^n by the equivalence relation $\sim$ given by

$$\mathbf{x} \sim \mathbf{y} \text{ if and only if } \mathbf{x} = -\mathbf{y}, \quad \mathbf{x}, \mathbf{y} \in S^n. \tag{4.2}$$

That is to say, each pair of antipodal points are identified.

Theorem 4.5: (i) The projective spaces are compact and connected.

(ii) The projective space $\mathbb{R}P^n$ is homeomorphic to the identification space $(\mathbb{R}^{n+1} - \{0\})/\sim$ where

$$\mathbf{x} \sim \mathbf{y} \text{ if and only if for some } \lambda \in \mathbb{R},\ \mathbf{x} = \lambda \mathbf{y}.$$

(iii) The projective space $\mathbb{R}P^n$ is homeomorphic to the identification space $E^n/\sim$ where $\mathbf{x} \sim \mathbf{y}$ if and only if either $\mathbf{x} = \mathbf{y}$ or else $\mathbf{x}, \mathbf{y} \in S^{n-1}$ and $\mathbf{x} = -\mathbf{y}$.

Proof: (i) The sphere S^n is compact and connected and $\mathbb{R}P^n$ is the continuous image of S^n under the projection map η.

(ii) Let $\eta : S^n \longrightarrow \mathbb{R}P^n$ and $p : \mathbb{R}^{n+1} - \{0\} \longrightarrow (\mathbb{R}^{n+1} - \{0\})/\sim$ be the projection maps. We have a continuous map $\phi : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}P^n$ given by the prescription

$$\phi(x) = \overline{x}, \quad x \in \mathbb{R}^{n+1} - \{0\},$$

where $\overline{x}$ is the equivalence class of x in $S^{n+1}/\sim$. Denoting by $[x]$ the equivalence class of x in $(\mathbb{R}^{n+1} - \{0\})/\sim$, the associated map $\overline{\phi} : (\mathbb{R}^{n+1} - \{0\})/\sim\ \longrightarrow \mathbb{R}P^n$ given by

$$\overline{\phi}([x]) = \overline{x}.$$

It is readily checked that $\overline{\phi}$ is bijective and $\overline{\phi} \circ \eta = \phi$. The universal property now gives us the continuity of $\overline{\phi}$. Consider now the map

$$\psi : S^n \longrightarrow (\mathbb{R}^{n+1} - \{0\})/\sim$$

given by $\psi(\mathbf{x}) = [\mathbf{x}]$ which is evidently continuous map. There is a unique map

$$\overline{\psi} : \mathbb{R}P^n \longrightarrow (\mathbb{R}^{n+1} - \{0\})/\sim$$

such that $\overline{\psi} \circ \eta = \psi$. By the universal property of the quotient we see that $\overline{\psi}$ is continuous. It is left as an exercise to check that $\overline{\psi}$ and $\overline{\phi}$ are inverses of each other. Proof of (iii) is left as an exercise.

We shall see later that the spaces are Hausdorff as well. The space $\mathbb{R}P^1$ is a familiar space and the proof of the following result will be left for the reader.

Theorem 4.6: The space $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

The Möbius band and the Klein's bottle: We describe the Möbius band and Klein's bottle as quotient spaces of I^2 via identifications which are described as follows. Each point in the interior of I^2 forms an equivalence class in itself. That is to say a point in the interior of I^2 is not identified with any other point. Points on the boundary are identified according to the following scheme:

1. Möbius band: On the part of the boundary $(\{0\} \times [0,1]) \cup (\{1\} \times [0,1])$, the pair of points $(0,y)$ and $(1, 1-y)$ are identified for each y with $0 \leq y \leq 1$. Points on the remaining part of the boundary namely
$$(0,1) \times \{0\} \cup (0,1) \times \{1\} \tag{4.3}$$
are left as they are. That is to say the equivalence class of each of the points (4.3) is a singleton.

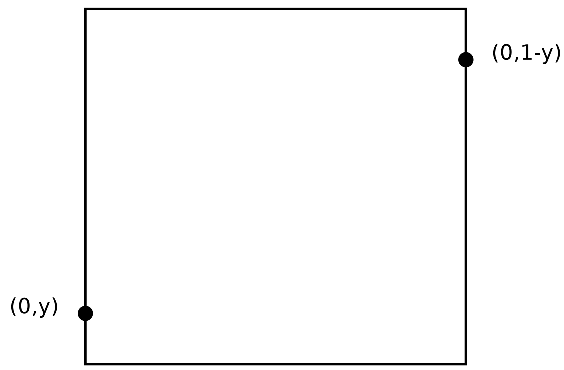


Figure 3: Möbius Band

2. Klein's bottle: As in the case of the Möbius band, for each $0 \leq y \leq 1$, the pair of points $(0,y)$ and $(1, 1-y)$ are identified. However also for each $0 \leq x \leq 1$, the pair of points $(x,0)$ and $(x,1)$ are identified.

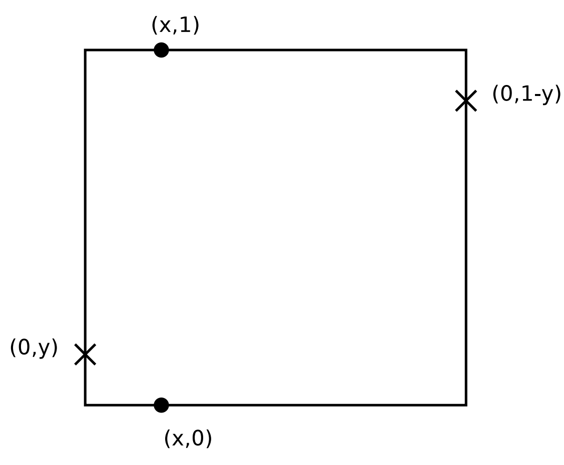


Figure 4: Klein's Bottle

The torus: This is obtained by identifying the opposite sides of the square I^2 according to the following scheme. For each $x \in [0,1]$, the pair of points $(x,0)$ and $(x,1)$ are identified. Likewise for each $y \in [0,1]$ the pair of points $(0,y)$ and $(1,y)$ are identified. One first obtains a cylinder $S^1 \times [0,1]$ which is then "bent around" and the circular ends are glued together. One obtains a space which looks like the crust of a dough-nut (or medu vada).

Example: The torus defined above is homeomorphic to $S^1 \times S^1$. To see this, let T denote the torus and $\eta : I^2 \longrightarrow T$ be the quotient map. Define the map $f : I^2 \longrightarrow S^1 \times S^1$ as

$$f(x,y) = (\exp(2\pi i x), \exp(2\pi i y)).$$

There is a unique bijection $\overline{f} : T \longrightarrow S^1 \times S^1$ such that $\overline{f} \circ \eta = f$. The universal property shows that $\overline{f}$ is continuous and since T is compact and $S^1 \times S^1$ is Hausdorff, the map $\overline{f}$ is a closed map and so a homeomorphism.

The wedge: The *wedge* of two topological spaces X and Y, denote by $X \vee Y$, is the following subspace of $X \times Y$

$$(X \times \{y_0\}) \cup (\{x_0\} \times Y),$$

where (x_0, y_0) is a chosen point of $X \times Y$.

Theorem 4.7: The quotient space $(S^1 \times S^1)/(S^1 \vee S^1)$ is homeomorphic to the sphere S^2.

Proof: It is an exercise that the space obtained by collapsing the boundary of I^2 to a singleton is homeomorphic to S^2. Let η_1 denote the quotient map $I^2 \longrightarrow S^2$ which collapses the boundary to a singleton and likewise let $\eta_2 : S^1 \times S^1 \longrightarrow (S^1 \times S^1)/(S^1 \vee S^1)$ be the quotient map. The map $\phi : S^1 \times S^1 \longrightarrow S^2$ given by the prescription

$$\phi(\exp(2\pi i x), \exp(2\pi i y)) = \eta_1(x,y)$$

is well-defined and surjective. Since η_1 is continuous it follows that ϕ is continuous (why?). There is a unique bijective map $\overline{\phi} : (S^1 \times S^1)/(S^1 \vee S^1) \longrightarrow S^2$ such that $\overline{\phi} \circ \eta_2 = \phi$, from which follows that $\overline{\phi}$ is continuous and a closed map since the domain is compact and the codomain is Hausdorff. Hence $\overline{\phi}$ is a homeomorphism between $(S^1 \times S^1)/(S^1 \vee S^1)$ and S^2.

Surfaces: The sphere S^2, torus, Klein's bottle and projective plane are the four basic examples of a class of spaces called *surfaces.* We shall not formally define a surface but provide one more example namely, the double torus. Roughly the double torus is obtained by taking two copies of the torus and cutting out a little disc form each of them so as to obtain a pair of tori each with a boundary. One then glues these boundaries together to obtain a double torus. Analytically the double torus is the identification space obtained by identifying pairs of opposite sides of an octagon according to the following scheme. Obviously the process can be generalized and one can obtain for instance a triple torus by identifying pairs of opposite sides of a twelve sided polygon. The classification of surfaces forms an important chapter in topology and we refer to the book of [11].

Hausdorff Quotients: The quotient of a Hausdorff space need not be Hausdorff. Since quotient spaces occur in abundance we need easily verifiable sufficient conditions for a quotient space to be Hausdorff. We provide here one such condition which suffices for most applications [16]. Let X be a

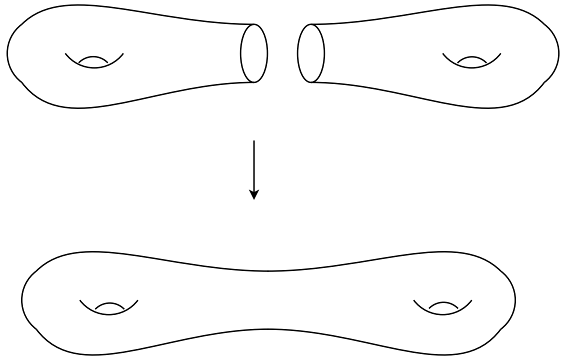


Figure 5: Double torus as a connected sum

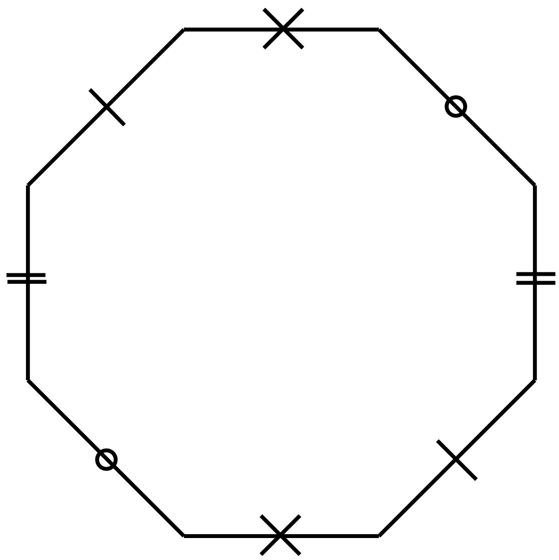


Figure 6: Double torus

space on which an equivalence relation $\sim$ has been defined. Note that a relation on X is a subset Γ of the cartesian product $X \times X$ on which we have the product topology. Thus,

$$\Gamma = \{(x, y) \in X \times X \ / \ x \sim y\}$$

Definition 4.3: The relation $\sim$ is said to be closed if Γ is a closed subset of $X \times X$.

We shall employ the two projection maps

$$\begin{array}{cc} p : X \times X \longrightarrow X, & q : X \times X \longrightarrow X, \\ (x, y) \mapsto x, & (x, y) \mapsto y, \end{array}$$

and denote by η the quotient map $\eta : X \longrightarrow X/\sim$.

Theorem 4.8: Let X be a compact Hausdorff space and $\sim$ be a closed relation on X. Then,

(a) The map η is a closed map.

(b) The quotient space $X/\sim$ is Hausdorff.

Proof: Let C be a closed subset of X. Since $p^{-1}(C)$ is closed in $X \times X$ we note that $p^{-1}(C) \cap \Gamma$ is closed in $X \times X$ and hence is compact. Thus $q(p^{-1}(C) \cap \Gamma)$ is compact and so is closed in X. Now,

$$\begin{aligned} q(p^{-1}(C) \cap \Gamma) &= \{y \in X \;/\; (x, y) \in p^{-1}(C) \cap \Gamma \text{ for some } x \in X\} \\ &= \{y \in X \;/\; y \sim x \text{ for some } x \in C\} \\ &= \eta^{-1}(\eta(C)) \end{aligned}$$

showing that $\eta(C)$ is closed. This proves (a) and in particular we note that singleton sets in $X/\sim$ are closed since they are images of singletons. Turning to the proof of (b), for an arbitrary pair of distinct elements $\overline{x}$ and $\overline{y}$ in $X/\sim$, the sets $\eta^{-1}(\overline{x})$ and $\eta^{-1}(\overline{y})$ are a pair of disjoint closed sets in X. Since X is normal there exist disjoint open sets U and V in X such that

$$\eta^{-1}(\overline{x}) \subset U \text{ and } \eta^{-1}(\overline{y}) \subset V.$$

The sets $\eta(X - U)$ and $\eta(X - V)$ are closed in $X/\sim$ by (a). We leave it to the reader to verify that the complements

$$(X/\sim) - \;\; \eta(X - U) \text{ and } (X/\sim) - \;\; \eta(X - V)$$

are disjoint sets. Now $\eta^{-1}(\overline{x}) \subset U$ implies $\overline{x} \notin \eta(X - U)$ whereby $\overline{x} \in (X/\sim) - \;\; \eta(X - U)$. Likewise $\overline{y} \in (X/\sim) - \;\; \eta(X - V)$ and the proof is complete.

Corollary 4.9: The projective spaces $\mathbb{R}P^n$ are Hausdorff.

Proof: The relation $\sim$ on S^n given by (4.2) defines a closed subset of $S^n \times S^n$.

Exercises

1. What happens if we omit the surjectivity hypothesis on the function $f : X \longrightarrow Y$ in the definition of quotient topology on Y induced by f ?
2. Show that the space obtained from the unit ball $\{\mathbf{x} \in \mathbb{R}^n / \|\mathbf{x}\| \leq 1\}$ by collapsing its boundary to a singleton, is homeomorphic to the sphere S^n.
3. Show that $\mathbb{R}P^1 \cong S^1$ by considering the map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^2$.
4. Try to show that S^2 is not homeomorphic to $\mathbb{R}P^2$. Would the Jordan curve theorem help?
5. Show that the boundary of the Möbius band is homeomorphic to S^1.
6. Does a Möbius band result upon cutting the projective plane $\mathbb{R}P^n$ along a closed curve on it?

Lecture V - Topological Groups

A topological group is a topological space which is also a group such that the group operations (multiplication and inversion) are continuous. They arise naturally as continuous groups of symmetries of topological spaces. A case in point is the group $SO(3,\mathbb{R})$ of rotations of $\mathbb{R}^3$ about the origin which is a group of symmetries of the sphere S^2. Many familiar examples of topological spaces are in fact topological groups. The most basic example of-course is the real line with the group structure given by addition. Other obvious examples are $\mathbb{R}^n$ under addition, the multiplicative group of unit complex numbers S^1 and the multiplicative group $\mathbb{C}^*$.

In the previous lectures we have seen that the group $SO(n,\mathbb{R})$ of orthogonal matrices with determinant one and the group $U(n)$ of unitary matrices are compact. In this lecture we initiate a systematic study of topological groups and take a closer look at some of the matrix groups such as $SO(n,\mathbb{R})$ and the unitary groups $U(n)$.

Definition 5.1: A topological group is a group which is also a topological space such that the singleton set containing the identity element is closed and the group operation

$$G \times G \longrightarrow G$$
$$(g_1, g_2) \mapsto g_1 g_2$$

and the inversion $j : G \longrightarrow G$ given by $j(g) = g^{-1}$ are continuous, where $G \times G$ is given the product topology.

We leave it to the reader to prove that a topological group is a Hausdorff space. It is immediate that the following maps of a topological group G are continuous:

1. Given $h \in G$ the maps $L_h : G \longrightarrow G$ and $R_h : G \longrightarrow G$ given by $L_h(g) = hg$ and $R_h(g) = gh$. These are the left and right translations by h.

2. The inner-automorphism given by $g \mapsto hgh^{-1}$ which is a homeomorphism.

Note that the determinant map is a continuous group homomorphism from $GL_n(\mathbb{R}) \longrightarrow R - \{0\}$. The image is surjective from which it follows that $GL_n(\mathbb{R})$ is disconnected as a topological space.

Theorem 5.1: The connected component of the identity in a topological group is a subgroup.

Proof: Let G_0 be the connected component of G containing the identity and $h, k \in G_0$ be arbitrary. The set $h^{-1}G_0$ is connected and contains the identity and so $G_0 \cup h^{-1}G_0$ is also connected. Since G_0 is a component, we have $G_0 \cup h^{-1}G_0 = G_0$ which implies $h^{-1}G_0 \subset G_0$. In particular $h^{-1}k$ belongs to G_0 from which we conclude that G_0 is a subgroup.

Interesting properties of topological groups arise in connection with quotients:

Theorem 5.2: Suppose that G is a topological group and K is a subgroup and the coset space G/K is given the quotient topology then

1. If K and G/K are connected then G is connected.

2. If K and G/K are compact then G is compact.

Proof: If G is connected then so is G/K since the quotient map $\eta : G \longrightarrow G/K$ is a continuous surjection. To prove the converse suppose that K and G/K are connected and $f : G \longrightarrow \{0,1\}$ be an arbitrary continuous map. We have to show that f is constant. The restriction of f to K must be constant and since each coset gK is connected, f must be constant on gK as well taking value $f(g)$. Thus we have a well defined map $\tilde{f} : G/K \longrightarrow \{0,1\}$ such that $\tilde{f} \circ \eta = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is continuous and so must be constant since G/K is connected. Hence f is also constant and we conclude that G is connected. $\square$

Since we shall not need (2), we shall omit the proof. A proof is available in [12], p. 109.

Theorem 5.3: The groups $SO(n, \mathbb{R})$ are connected when $n \geq 2$.

Proof: It is clear that $SO(2, \mathbb{R})$ is connected (why?). Turning to the case $n \geq 3$, we consider the action of $SO(n, \mathbb{R})$ on the standard unit sphere S^{n-1} in $\mathbb{R}^n$ given by

$$(A, \mathbf{v}) \mapsto A\mathbf{v},$$

where $A \in SO(n, \mathbb{R})$ and $\mathbf{v} \in S^{n-1}$. It is an exercise for the student to check that this group action is transitive and that the stabilizer of the unit vector $\hat{\mathbf{e}}_n$ is the subgroup K consisting of all those matrices in $SO(n, \mathbb{R})$ whose last column is $\hat{\mathbf{e}}_n$. The subgroup K is homeomorphic to $SO(n-1, \mathbb{R})$ and so, by induction hypothesis, is connected. By exercise 3, the quotient space $SO(n, \mathbb{R})$ is homeomorphic to S^{n-1} which is connected. So the theorem can be applied with $G = SO(n, \mathbb{R})$, $H = SO(n-1, \mathbb{R})$ and G/H is the sphere S^{n-1} with $n \geq 2$. $\square$

Theorem 5.4: If G is a connected topological group and H is a subgroup which contains a neighborhood of the identity then $H = G$. In particular, an open subgroup of G equals G.

Proof: Let U be the open neighborhood of the identity that is contained in H and $h \in H$ be arbitrary. Since multiplication by h is a homeomorphism, the set $Uh = \{uh/u \in U\}$ is also open and also contained in H. Hence the set

$$L = \bigcup_{h \in H} Uh$$

is open and contained in H. Since U contains the identity element, $H \subset L$ and we conclude that H is open. Our job will be over if we can show that H is closed as well. Let $x \in \overline{H}$ be arbitrary. Since the neighborhood Ux of x contains a point $y \in H$, there exists $u \in U$ such that $y = ux$ which, in view of the fact that $U \subset H$, implies $x \in H$. Hence $\overline{H} = H$. $\square$

Theorem 5.5: Suppose G is a connected topological group and H is a discrete normal subgroup of G then H is contained in the center of G.

Proof: Since H is discrete, the identity element is not a limit point of H and so there is a neighborhood U of the identity such that $U \cap H = \{1\}$. We may assume U has the property that if u_1, u_2 are in U then the product $u_1^{-1}u_2$ is in U. This follows from the continuity of the group operation and a detailed verification is left as an exercise. It is easy to see that if h_1 and h_2 are two distinct elements of H then

$$Uh_1 \cap Uh_2 = \emptyset.$$

Fix $h \in H$ and consider now the set K given by

$$K = \{g \in G \;/\; gh = hg\}$$

We shall show that the subgroup K contains a neighborhood of the identity. Pick a neighborhood V of the identity such that $V = V^{-1}$ and $(hVh^{-1}V) \cap H = \{1\}$. Then for any $g \in V$, we have on the one hand

$$hgh^{-1}g^{-1} \in hVh^{-1}V$$

and on the other hand $hgh^{-1}g^{-1} \in H$ since H is normal. Hence $hgh^{-1}g^{-1} \in (hVh^{-1}V) \cap H = \{1\}$ which shows that g belongs to K and K contains a neighborhood of the unit element. We may now invoke the previous theorem. $\square$

Remark: The result is false if the hypothesis of normality of H is dropped. For example consider a cube in $\mathbb{R}^3$ with center at the origin and H be the subgroup of $G = SO(3, \mathbb{R})$ that map the cube to itself. Then H is the symmetric group on four letters (proof?). Clearly H is not in the center of G.

Exercises

1. Show that in a topological group, the connected component of the identity is a normal subgroup.

2. Show that the action of the group $SO(n, \mathbb{R})$ on the sphere S^{n-1} given by matrix multiplication is transitive. You need to employ the Gram-Schmidt theorem to complete a given unit vector to an orthonormal basis.

3. Suppose a group G acts transitively on a set S and x, y are a pair of points in S and $y = gx$. Then the subgroups stab x and stab y are conjugates and $g^{-1}(\text{stab } y)g = \text{stab } x$.

   (i) Show that the map $\overline{\phi} : G/\text{stab } x \longrightarrow S$ given by $\overline{\phi}(\overline{g}) = gx$ is well-defined, bijective and $\overline{\phi} \circ \eta = \phi$.

   (ii) Suppose that S is a topological space, G is a topological group and the action $G \times S \longrightarrow S$ is continuous. Show that the map $\overline{\phi}$ is continuous.

   (iii) Deduce that if G is compact and S is Hausdorff then $G/\text{stab } x$ and S are homeomorphic.

4. Examine whether the map $\phi : SU(n) \times S^1 \longrightarrow U(n)$ given by $\phi(A, z) = zA$ is a homeomorphism.

5. Show that the group of all unitary matrices $U(n)$ is compact and connected. Regarding $U(n-1)$ as a subgroup of $U(n)$ in a natural way, recognize the quotient space as a familiar space.

6. Show that the subgroups $SU(n)$ consisting of matrices in $U(n)$ with determinant one are connected for every n.

7. Suppose G is a topological group and H is a normal subgroup, prove that G/H is Hausdorff if and only if H is closed.

Lecture VI (Test - I)

1. Prove that the intervals (a, b) and $[a, b)$ are non-homeomorphic subsets of $\mathbb{R}$. Prove that if A and B are homeomorphic subsets of $\mathbb{R}$, then A is open in $\mathbb{R}$ if and only if B is open in $\mathbb{R}$. Is an injective continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ a homeomorphism onto its image?

2. Using Tietze's extension theorem or otherwise construct a continuous map from $\mathbb{R}$ into $\mathbb{R}$ such that the image of $\mathbb{Z}$ is not closed in $\mathbb{R}$.

3. If K is a compact subset of a topological group G and C is a closed subset of G, is it true that KC is closed in G? What if K and C are merely closed subsets of G?

4. Removing three points from $\mathbb{R}P^2$ we get an open set G and a continuous map $f : G \longrightarrow \mathbb{R}P^2$ given by $f([x_1, x_2, x_3]) = [x_2x_3, x_3x_1, x_1x_2]$. Which three points need to be removed? Prove the continuity of f.

5. Let $C = \{(\mathbf{v}_1, \mathbf{v}_2) \in S^2 \times S^2 \;/\; \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0\}$. Is C connected? Is C homeomorphic to $SO(3, \mathbb{R})$?

6. Prove that $\mathbb{R}P^1$ is homeomorphic to S^1.

Solutions to Test - I

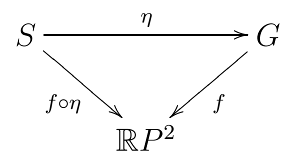
1. Suppose that $(0,1)$ and $(0,1]$ are homeomorphic subsets of $\mathbb{R}$ and ϕ is a homeomorphism then $\phi(t) = 1$ for some $t \in (0,1)$. Then, $(0,t) \cup (t,1)$ is homeomorphic to $(0,1)$ by restricting the map ϕ and this is a contradiction. Next suppose that A and B are homeomorphic subsets of $\mathbb{R}$ and A is open in $\mathbb{R}$. To show that B is open, let $q \in B$ and $q = \phi(p)$ for some $p \in A$. Since A is open we can find $\delta > 0$ such that $I = (p-\delta, p+\delta) \subset A$. The image $\phi(I)$ is then an interval containing q and this interval cannot be compact since I is not compact. This interval cannot be of the form $(a,b]$ or $[a,b)$ by what we have proved. This $\phi(I)$ is an open interval in $\mathbb{R}$ containing q and so q is an interior point of B. Let f be an injective continuous map from $\mathbb{R}$ onto $B \subset \mathbb{R}$. Then f is strictly increasing or strictly decreasing. It is a basic fact proved in real analysis courses that under these circumstances the inverse map $f^{-1} : B \longrightarrow \mathbb{R}$ is continuous and so f must be a homeomorphism onto its range.

2. Enumerate the rationals in $[0,1]$ as $q_1, q_2, q_3, \ldots$. An arbitrary bijection $\phi : \mathbb{Z} \longrightarrow \mathbb{Q} \cap [0,1]$ is continuous since $\mathbb{Z}$ carries the discrete topology. By Tietze's extension theorem ϕ extends continuously as a map (still denoted by ϕ) from $\mathbb{R} \longrightarrow \mathbb{R}$. The image of the closed set $\mathbb{Z}$ is not closed. A more direct example is the function $(\sin x)/x$.

3. To show that KC is closed, let g be a limit point of G and $\{k_n c_n\}$ be a sequence in KC converging to $g \in G$, where $\{k_n\}$ and $\{c_n\}$ are sequences in K and C respectively. Passing on to a subsequence if necessary, we may assume that $\{k_n\}$ converges to say k. Then $\{k_n^{-1}\}$ converges to k^{-1} and so
$$c_n = k_n^{-1}(k_n c_n) \longrightarrow k^{-1} g$$
But C being closed, $k^{-1}g \in C$ and so $g = k(k^{-1}g) \in KC$. The result is false if K and C are merely closed. Take
$$C = \{2n + \frac{1}{n} \;/\; n = 1, 2, 3, \ldots\}, \quad K = \{-2n + \frac{1}{n} \;/\; n = 1, 2, 3, \ldots\}.$$

4. For the function to be well-defined one must clearly remove the points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$. Let S be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes and η is the restriction of the quotient map $\mathbb{R}^3 - \{(0,0,0)\} \longrightarrow \mathbb{R}P^2$. Consider
$$g : S \longrightarrow \mathbb{R}P^2$$
given by $g(x_1, x_2, x_3) = [x_1x_2, x_2x_3, x_3x_1]$. The map $g = f \circ \eta$ is continuous and by the universal

property of quotients (see the commutative diagram)



we conclude that f is continuous.

5. Let us consider the map $F : SO(3, \mathbb{R}) \longrightarrow C$ given by

$$F(A) = (A\mathbf{e}_1, A\mathbf{e}_2).$$

Then F is a continuous surjection so that C is connected. In fact F is bijective (why?) and so F is a homeomorphism.

6. Let $\sigma : S^1 \longrightarrow S^1$ denote the map $\sigma(z) = z^2$ and η denote the standard quotient map $S^1 \longrightarrow \mathbb{R}P^1$. Define $\overline{\sigma} : \mathbb{R}P^1 \longrightarrow S^1$ by the prescription

$$\overline{\sigma}(\overline{z}) = \sigma(z)$$

It is easily checked (draw relevant diagram) that $\overline{\sigma} \circ \eta = \sigma$ and the universal property of quotients reveals that $\overline{\sigma}$ is a continuous bijection. Since the domain of $\overline{\sigma}$ is compact and the codomain is Hausdorff it follows that $\overline{\sigma}$ is a homeomorphism.

Lecture VII - Paths, homotopies and the fundamental group

In this lecture we shall introduce the most basic object in algebraic topology, the fundamental group. For this purpose we shall define the notion of homotopy of paths in a topological space X and show that this is an equivalence relation. We then fix a point $x_0 \in X$ in the topological space and look at the set of all equivalence classes of loops starting and ending at x_0. This set is then endowed with a binary operation that turns it into a group known as the fundamental group $\pi_1(X, x_0)$. Besides being the most basic object in algebraic topology, it is of paramount importance in low dimensional topology. A detailed study of this group will occupy the rest of part I of this course. However in this lecture we shall focus only on the most elementary results.

All spaces considered here are path connected Hausdorff spaces.

Definition 7.1 (homotopy of paths): Two paths $\gamma_0,\ \gamma_1$ in X with parameter interval $[0, 1]$ such that

$\gamma_0(0) = \gamma_1(0),\ \gamma_0(1) = \gamma_1(1)$ (that is with the same end points) are said to be homotopic if there exists a continuous map $F : [0, 1] \times [0, 1] \to X$ such that

$$\begin{aligned} F(0,t) &= \gamma_0(t) \\ F(1,t) &= \gamma_1(t) \\ F(s,0) &= \gamma_0(0) = \gamma_1(0) \\ F(s,1) &= \gamma_0(1) = \gamma_1(1) \end{aligned}$$

The definition says that the path $\gamma_0(t)$ can be continuously deformed into $\gamma_1(t)$ and F is the continuous

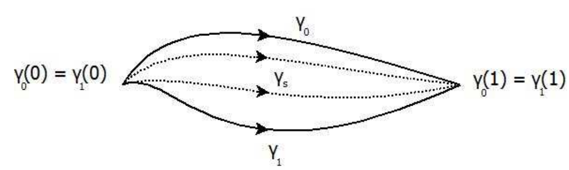


Figure 7: Homotopy of paths

function that does the deformation. The deformation takes place in unit time parametrized by s. For $s \in [0, 1]$, the function $\gamma_s\ :\ t \to F(s, t)$ is the intermediate path. Finally, the conditions

$$F(s, 0) = \gamma_0(0) \quad \text{and} \quad F(s, 1) = \gamma_0(1)$$

imply that the ends $\gamma_0(0),\ \gamma_0(1)$ do not move during the deformation. We shall now show that homotopy is an equivalence relation.

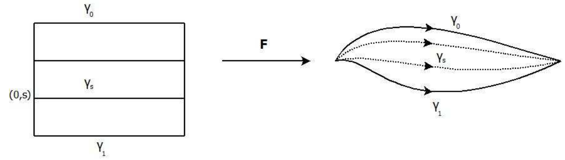


Figure 8: Homotopy of paths

Theorem 7.1: If $\gamma_1,\ \gamma_2,\ \gamma_3$ are three paths in X such that

$$\gamma_1(0) = \gamma_2(0) = \gamma_3(0) \ \text{ and } \ \gamma_1(1) = \gamma_2(1) = \gamma_3(1),$$

γ_1 and γ_2 are homotopic; γ_2 and γ_3 are homotopic then γ_1 and γ_3 are homotopic.

Proof: It is clear that the homotopy is reflexive, and symmetry is left for student to verify. To prove transitivity let

$$F:\ [0,1]\times[0,1] \to X \ \text{ and } \ G:\ [0,1]\times[0,1] \to X$$

be homotopies between the pairs γ_1, γ_2 and γ_2, γ_3 respectively. Define $H : [0,1]\times[0,1] \to X$ by the prescription:

$$H(s,t) = \begin{cases} F(2s,t) & 0 \le s \le 1/2 \\ G(2s-1) & 1/2 \le s \le 1 \end{cases}$$

Note that by gluing lemma H is continuous. We need to check the conditions at endpoints.

$$H(s,0) = \begin{cases} F(2s,0) = \gamma_1(0) = \gamma_3(0), & 0 \le s \le 1/2 \\ G(2s-1,0) = \gamma_2(0) = \gamma_3(0), & 1/2 \le s \le 1 \end{cases}$$

Likewise one verifies easily $H(s,1) = \gamma_1(1) = \gamma_3(1)$ for all $s \in [0,1]$. Finally we see that $H(0,t) = F(0,t) = \gamma_1(t)$ and $H(1,t) = G(1,t) = \gamma_3(t)$, which proves the result.

Notation: The equivalence class of γ will be denoted by $[\gamma]$ and called the homotopy class of the path γ. When γ_1, γ_2 are homotopic we write $\gamma \sim \gamma_2$.

Theorem 7.2 (Reparametrization theorem): Let X be a topological space. Suppose that $\phi : [0,1] \longrightarrow [0,1]$ is a continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$. Then for any given path γ in X, we have a homotopy

$$\gamma \sim \gamma \circ \phi$$

Proof: We must remark that we are not assuming anything about ϕ besides continuity and the fact that it fixes 0 and 1. In particular ϕ need not be monotone. The idea of proof is simple. The convexity of the unit square $[0,1]\times[0,1]$ is used to tweak the graph of ϕ onto the graph of the identity map of $[0,1]$. Thus we define a continuous map $F : [0,1]\times[0,1] \longrightarrow X$ by the prescription

$$F(s,t) = \gamma(s\phi(t) + (1-s)t)$$

Now $F(0,t) = \gamma(t)$, $F(1,t) = \gamma\circ\phi(t)$. For verifying that the end points are fixed during deformation,

$$\begin{aligned} F(s,0) &= \gamma(s\phi(0)) = \gamma(0) \\ F(s,1) &= \gamma(s\phi(1) + (1-s)) = \gamma(1), \quad 0 \le s \le 1. \end{aligned}$$

Juxtaposition of paths: Suppose that γ_1, γ_2 are two paths such that $\gamma_1(1) = \gamma_2(0)$, that is to say, the end point of γ_1 is the initial point of γ_2. The paths γ_1 and γ_2 can be juxtaposed to produce a path from $\gamma_1(0)$ to $\gamma_2(1)$ called called the juxtaposition γ_1 and γ_2,, denoted by $\gamma_1 * \gamma_2$ and defined as :

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Lemma 7.3: If γ_1' and γ_1'' are two homotopic paths starting at $\gamma_0(1)$ then

$$\gamma_0 * \gamma_1' \sim \gamma_0 * \gamma_1''$$

Proof: Let $F : [0,1] \times [0,1] \to X$ be a homotopy between γ_1' and γ_1'' so that $F(0,t) = \gamma_0(1)$, $F(1,t) = \gamma_1'(1) = \gamma_1''(1)$. The homotopy we seek is the map $H(s,t)$ given by

$$H(s,t) = \begin{cases} \gamma_0(2t), & 0 \leq t \leq 1/2 \\ F(s, 2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

It can be checked that the definition is meaningful along $[0,1] \times \{\frac{1}{2}\}$ and the continuity of H follows by the gluing lemma. The reader may complete the proof by verifying that

$$H(0,t) = \gamma_0 * \gamma_1'; \quad H(1,t) = \gamma_0 * \gamma_1''. \qquad \square$$

Corollary 7.4: If γ_1', γ_1'' are homotopic paths starting at $\gamma_0(1)$ then $[\gamma_0 * \gamma_1'] = [\gamma_0 * \gamma_1'']$. Likewise if γ_1 is a path in X and γ_0', γ_0'' are homotopic paths whose terminal points are at $\gamma_1(0)$ then $\gamma_0' * \gamma_1 \sim \gamma_0'' * \gamma_1$.

Definition 7.2: If γ_1, γ_2 are two paths in X such that initial point of γ_1 is the terminal point of γ_2, then we define the product of the homotopy classes of paths as

$$[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2.] \tag{7.1}$$

The Inverse Path and the constant path: Suppose $\gamma : [0,1] \to X$ is a path then the inverse path $\gamma^{-1}(t)$ is the path traced in the reversed direction namely the map $\gamma^{-1} : [0,1] \to X$ given by

$$\gamma^{-1}(t) = \gamma(1-t).$$

The initial point of γ is the terminal point of γ^{-1} and vice versa.

The constant path at x_0 is the path $\varepsilon_{x_0} : [0,1] \longrightarrow X$ given by

$$\varepsilon_{x_0}(t) = x_0 \quad \text{for all } t \in [0,1].$$

The following lemma summarizes the main properties of the constant and the inverse paths in terms of the homotopy classes of paths. Theorem (7.6) spells out the associativity of multiplication of homotopy classes of paths. The reader would see analogies with the defining properties of a group.

Lemma 7.5:

(i) $\gamma * \gamma^{-1} \sim \varepsilon_{\gamma(0)}$. Thus $[\gamma] \cdot [\gamma^{-1}] = [\varepsilon_{\gamma(0)}]$.

(ii) $\gamma * \varepsilon_{\gamma(1)} \sim \gamma$. Thus $[\gamma][\varepsilon_{\gamma(1)}] = [\gamma]$

(iii) $\varepsilon_{\gamma(0)} * \gamma \sim \gamma$. Thus $[\varepsilon_{\gamma(0)}][\gamma] = [\gamma]$.

Proofs: One uses the reparametrization theorem to prove (ii) and (iii). Proof of (i) is more involved and we indicate two different methods by which this can be achieved. On the boundary $\dot{I}^2$ of the unit square I^2 we define a map $\phi : \dot{I}^2 \longrightarrow [0,1]$ as follows.

$$\phi(0,t) = 0, \quad \phi(s,0) = 0, \quad \phi(s,1) = 0$$

Along the part $(1,t)$ of the boundary,

$$\phi(1,t) = \begin{cases} 2t & 0 \leq t \leq 1/2 \\ 2-2t & 1/2 \leq t \leq 1 \end{cases}$$

By Tietze's extension theorem ϕ extends continuously to I^2 taking values in $[0,1]$. Consider now the map $H : I^2 \longrightarrow X$ given by

$$H(s,t) = \gamma \circ \phi(s,t).$$

It is readily checked that H establishes a homotopy between $\gamma * \gamma^{-1}$ and the constant path $\varepsilon_{\gamma(0)}$. □

Theorem 7.6: Suppose γ_1, γ_2, γ_3 are three paths in X such that $\gamma_1(1) = \gamma_2(0)$; $\gamma_2(1) = \gamma_3(0)$ then

$$(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$$

Hence

$$([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$$

Proof: By direct calculation we get on the one hand

$$(\gamma_1 * \gamma_2) * \gamma_3 = \begin{cases} \gamma_1(4t) & 0 \leq t \leq 1/4 \\ \gamma_2(4t-1) & 1/4 \leq t \leq 1/2 \\ \gamma_3(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

On the other hand, for $\gamma_1 * (\gamma_2 * \gamma_3)$ we find

$$\gamma_1 * (\gamma_2 * \gamma_3) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(4t-2) & 1/2 \leq t \leq 3/4 \\ \gamma_3(4t-3) & 3/4 \leq t \leq 1. \end{cases}$$

These two are homotopic by the reparametrization theorem. To see this define $\phi : [0,1] \to [0,1]$ by

$$\phi(t) = \begin{cases} 2t & 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & \frac{1}{2} \leq t \leq 1. \end{cases}$$

one verifies that $\gamma \circ \phi = (\gamma_1 * \gamma_2) * \gamma_3$ where $\gamma = \gamma_1 * (\gamma_2 * \gamma_3)$. By theorem (7.2) the result follows.

We are now ready to define the fundamental group.

Definition 7.3 (The fundamental group $\pi_1(X, x_0)$): Let X be a path connected topological space and x_0 be a point of X. We define $\pi_1(X, x_0)$ to be the set of all homotopy classes of paths beginning and ending at the given point x_0 namely homotopy classes $[\gamma]$ where $\gamma : [0, 1] \to X$ is continuous and $\gamma(0) = \gamma(1) = x_0$:

$$\pi_1(X, x_0) = \{ [\gamma] / \gamma : [0, 1] \to X \text{ continuous and } \gamma(0) = \gamma(1) = x_0\}.$$

Terminology: Paths in X starting and ending at x_0 will be referred to as *loops* based at x_0. The distinguished point $x_0 \in X$ is called the *base point* of X.

Note that if γ_1, γ_2 are two loops based at x_0, their juxtaposition $\gamma_1 * \gamma_2$ is defined whereby both the products $[\gamma_1][\gamma_2]$ and $[\gamma_2][\gamma_1]$ are defined. Also for $[\gamma] \in \pi_1(X, x_0)$, $[\gamma^{-1}]$ also belongs to $\pi_1(X, x_0)$. $[\varepsilon_{x_0}] \in \pi_1(X, x_0)$ and lemma (7.5) and theorem (7.6) imply that $\pi_1(X, x_0)$ is a group with unit element $[\varepsilon_{x_0}]$. This group is written multiplicatively and the unit element $[\varepsilon_{x_0}]$ will be denoted by 1 when there is no danger of confusion. Summarizing,

Theorem 7.7: The set $\pi_1(X, x_0)$ of homotopy classes of loops in X based at x_0 is a group with respect to the binary operation defined by (7.1). The unit element of the group is the homotopy class of the constant loop at the base point x_0 and the inverse of $[\gamma]$ is the homotopy class of the loop γ^{-1}.

Definition: The group $\pi_1(X, x_0)$ is called the fundamental group of the space X based at x_0. This group can be non-abelian although we need to do some work to produce an example. Indeed we need to do some work to produce such an example for which $\pi_1(X, x_0)$ is non-trivial. All we shall do in the rest of this lecture is to show that it is trivial in case X is a convex subset of $\mathbb{R}^n$. First we shall see what happens when the base point is changed.

Theorem 7.8: Let X be a path connected topological space and x_1, x_2 be two arbitrary points of X. Then $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$ are isomorphic.

Proof: Let σ be a path joining x_1 and x_2. Observe that if γ is a loop at the x_1 then $\sigma * \gamma * \sigma^{-1}$ is a loop at x_2 thereby enabling us to define a map

$$\begin{aligned} h_{[\sigma]} : \quad & \pi_1(X, x_1) \longrightarrow \pi_1(X, x_2) \\ & [\gamma] \ \mapsto [\sigma * \gamma * \sigma^{-1}]. \end{aligned}$$

Corollary (7.4) shows that the function is well defined and lemma (7.5) shows that it is a group

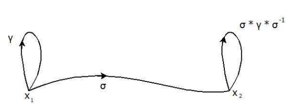


Figure 9: Change of base point

homomorphism. Let Γ be a loop at x_2. Then $\sigma^{-1} * \Gamma * \sigma$ is a loop at x_1 and $h_{[\sigma]}([\sigma^{-1} * \Gamma * \sigma]) = [\Gamma]$ showing that $h_{[\sigma]}$ is surjective. The map

$$h_{[\sigma^{-1}]} : [\Gamma] \longrightarrow [\sigma^{-1} * \Gamma * \sigma]$$

is the inverse of $h_{[\sigma]}$. □

Remarks: The isomorphism h depends on the homotopy class of the path σ joining x_1 and x_2 justifying the notation $h_{[\sigma]}$. The reason for the elaborate notation is that it will reappear in lecture 11. The next theorem tells us what happens when we choose various paths from x_1 to x_2.

Theorem 7.9: Suppose γ_0', γ_0'' are two paths joining x_1 and x_2 and h', h'' are the corresponding group isomorphisms from $\pi_1(X, x_1) \longrightarrow \pi_1(X, x_2)$ given by the previous theorem. Then there exists an inner automorphism

$$\sigma : \pi_1(X, x_2) \longrightarrow \pi_1(X, x_2)$$

such that $h' = \sigma \circ h''$. In fact σ is the inner automorphism determined by $[\gamma_0'][\gamma_0'']^{-1}$.

If $\pi_1(X, x_0)$ is abelian then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are naturally isomorphic. That is the isomorphism $h_{[\sigma]}$ is canonical in this case.

Proof: Using lemma (7.5) we begin by writing

$$h'[\gamma] = [\gamma_0' * \gamma * (\gamma_0')^{-1}] = [\gamma_0' * (\gamma_0'')^{-1} * \gamma_0'' * \gamma * (\gamma_0'')^{-1} * \gamma_0'' * \gamma_0^{-1}].$$

By definition (7.2), the right hand side equals

$$[\gamma_0'][\gamma_0'']^{-1}h''([\gamma])[\gamma_0''][\gamma_0']^{-1} = (\sigma \circ h'' \circ \sigma^{-1})[\gamma]$$ □

Definition: A path connected space X is said to be simply connected if $\pi_1(X, x_0) = \{1\}$, $x_0 \in X$.

Definition 7.4 (Convex and star-shaped domains): (i) A subset X of $\mathbb{R}^n$ is said to be convex if for every pair of points a and b in X, the line segment $ta + (1-t)b$, $0 \leq t \leq 1$ lies entirely in X.

(ii) A subset X of $\mathbb{R}^n$ is said to be star shaped with respect to a point x_0 if for every $a \in X$, the line segment $ta + (1-t)x_0$, $0 \leq t \leq 1$ lies entirely in X.

So a convex domain is star shaped with respect to any of its points.

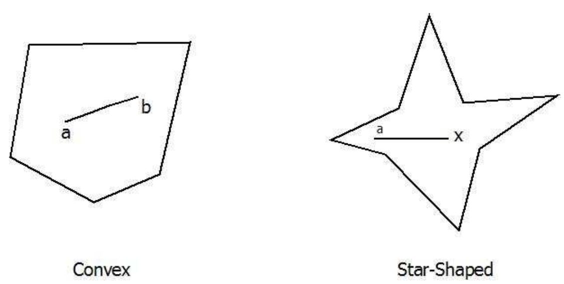


Figure 10: Convex and star-shaped domains

Theorem 7.10: If X is star shaped then $\pi_1(X, x_0) = \{1\}$. In particular the fundamental groups of the unit disc and $\mathbb{R}^n$ are both equal to the trivial group.

Proof: By the previous result it is immaterial which point x_0 is chosen as the base point. Assume that X is star shaped with respect to x_0. Let $\gamma : [0,1] \longrightarrow X$ be a loop in X based at x_0. We shall prove $[\gamma] = [\varepsilon_{x_0}] = 1$ by constructing a homotopy F between γ and the constant loop ε_{x_0}, namely

$$F(s,t) = (1-s)\gamma(t) + sx_0.$$

This makes sense because X is star shaped with respect to x_0. Turning to the boundary conditions,

$$\begin{aligned} F(0,t) &= \gamma(t), \quad F(1,t) = \varepsilon_{x_0} \\ F(s,0) &= (1-s)\gamma(0) + sx_0 = (1-s)x_0 + sx_0 = x_0 \\ F(s,1) &= (1-s)\gamma(1) + sx_0 = (1-s)x_0 + sx_0 = x_0. \end{aligned}$$

Exercises:

1. Explicitly construct a homotopy between the loop $\gamma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$ on the sphere S^2 and the constant loop based at $(1,0,0)$. Note that an explicit formula is being demanded here.

2. Show that a loop in X based at a point $x_0 \in X$ may be regarded as a continuous map $f : S^1 \longrightarrow X$ such that $f(1) = x_0$. Show that if f is homotopic to the constant loop ε_{x_0} then f extends as a continuous map from the closed unit disc to X.

3. Show that if γ is a path starting at x_0 and γ^{-1} is the inverse path then prove by imitating the proof of the reparametrization theorem (that is by taking convex combination of two functions) that $\gamma * \gamma^{-1}$ is homotopic to the constant loop ϵ_{x_0}.

4. Prove theorems (7.2) and theorem (7.6) using Tietze's extension theorem.

5. Suppose $\phi : [0,1] \longrightarrow [0,1]$ is a continuous function such that $\phi(0) = \phi(1) = 0$ and γ is a closed loop in X based at $x_0 \in X$. Is it true that $\gamma \circ \phi$ is homotopic to the constant loop ε_{x_0}?

6. Show that the group isomorphism in theorem (7.8) is natural namely, if $f : X \longrightarrow Y$ is continuous and $x_1, x_2 \in X$ then
$$h_{[f\circ\sigma]} \circ f'_* = h_{[\sigma]} \circ f''_*$$
where, $y_1 = f(x_1)$, $y_2 = f(x_2)$ and σ is a path joining x_1 and x_2. The maps f'_* and f''_* are the maps induced by f on the fundamental groups. This information is better described by saying that the following diagram *commutes:*

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{f'_*} & \pi_1(Y,y_1) \\ {\scriptstyle h_{[\sigma]}}\downarrow & & {\scriptstyle h_{[f\circ\sigma]}}\downarrow \\ \pi_1(X,x_2) & \xrightarrow{f''_*} & \pi_1(Y,y_2) \end{array}$$

Lecture VIII - Categories and Functors

Note that one often works with several types of mathematical objects such as groups, abelian groups, vector spaces and topological spaces. Thus one talks of the family of all groups or the family of all topological spaces. These entities are huge and do not qualify to be sets. We shall call them families or classes and their individual members as objects. Between two objects of a family say between two topological spaces X and Y one is interested in the class of all continuous functions. Instead if we take two objects G and H from the class of all groups we are interested in the set of all group homomorphisms from G into H. Abstracting from these examples we say that a category consists of a family of objects and for each pair of objects X and Y we are given a family of maps $X \to Y$ called the set of morphisms $\mathrm{Mor}(X, Y)$ subject to the following properties:

(i) To each pair $\mathrm{Mor}(X, Y)$ and $\mathrm{Mor}(Y, Z)$ there is a map

$$\mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \longrightarrow \mathrm{Mor}(X, Z)$$
$$(f, g) \mapsto g \circ f$$

such that for $f \in \mathrm{Mor}(X, Y), g \in \mathrm{Mor}(Y, Z)$ and $h \in \mathrm{Mor}(Z, W)$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(ii) To each object X there is a unique element $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ such that for any $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Z, X)$

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

Example 8.1: We see that the family of all groups **Gr** forms a category where $\mathrm{Mor}(G, H)$ consists of the set of all group homomorphisms from G to H.

(ii) Likewise we can look at the family **AbGr** of all abelian groups and as before $\mathrm{Mor}(G, H)$ consists of all group homomorphisms from G to H.

(iii) The class of all topological spaces **Top** forms a category if we take as morphisms between X and Y the set of all continuous functions from X to Y.

Definition 8.1 (Covariant functor): Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, a covariant functor is a rule that assigns to each object $A \in \mathcal{C}_1$ an object $h(A) \in \mathcal{C}_2$ and to each morphism $f \in \mathrm{Mor}(A, B)$, where A, B are objects in $\mathcal{C}_1$, a unique morphism $h(f) \in \mathrm{Mor}(h(A), h(B))$ such that the following hold:

(i) Given objects A, B and C in $\mathcal{C}_1$ and a pair of morphisms $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$,

$$h(g \circ f) = h(g) \circ h(f)$$

(ii) $h(\mathrm{id}_A) = \mathrm{id}_{h(A)}$

Definition 8.2 (Contravariant functor): A contravariant functor between the two given categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is a rule that assigns to each object $A \in \mathcal{C}_1$ an object $h(A) \in \mathcal{C}_2$ and to each morphism $f \in \mathrm{Mor}(A, B)$, where A, B are objects in $\mathcal{C}_1$, a unique morphism $h(f) \in \mathrm{Mor}(h(B), h(A))$ such that the following conditions hold:

(i) Given objects A, B and C in $\mathcal{C}_1$ and a pair of morphisms $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$,

$$h(g \circ f) = h(f) \circ h(g)$$

(ii) $h(\mathrm{id}_A) = \mathrm{id}_{h(A)}$

Example 8.2: To each group G we assign its commutator subgroup $[G, G]$. A group homomorphism $f : G \longrightarrow H$ maps the commutator subgroup into the commutator subgroup $[H, H]$ so that the restriction

$$f\Big|_{[G,G]} : [G, G] \longrightarrow [H, H]$$

is a meaningful group homomorphism enabling us to assign to the morphism f its restriction to $[G, G]$. The conditions of definition 8.1 are readily verified.

Example 8.3: Between the categories **Gr** and **AbGr** we define a map as follows. For $G \in$ **Gr** we denote by A_G its abelianization namely the quotient group:

$$A_G = G/[G, G].$$

The quotient is an abelian group and so belongs to **AbGr**. For example if we take $G = S_n$ the symmetric group on n letters then its abelianization is the cyclic group of order two (why?). If G and H are two groups and $f : G \longrightarrow H$ is a group homomorphism then

$$\eta_H \circ f : G \longrightarrow H/[H, H]$$

is a group homomorphism into an abelian group where η_H is the quotient map $H \longrightarrow H/[H, H]$. The kernel of $\eta_H \circ f$ must contain all the commutators and so defines a group homomorphism

$$\begin{aligned} \tilde{f} : G/[G, G] &\longrightarrow H/[H, H] \\ \overline{x} &\mapsto \overline{f(x)}, \end{aligned}$$

where the bar over x denotes the residue class of x in the quotient. Thus to each object G of **Gr** we have assigned a unique object of **AbGr** namely the abelianization $G/[G, G]$ and to each morphism $f \in \mathrm{Mor}(G, H)$ we have associated a unique morphism $\tilde{f}$. The following properties are quite clear:

(i) If $f \in \mathrm{Mor}(G, H)$ and $g \in \mathrm{Mor}(H, K)$ then

$$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$$

(ii) For any group G,

$$\widetilde{\mathrm{id}_G} = \mathrm{id}_{G/[G,G]}$$

This is an example of a covariant functor from one category to another.

Example 8.4: Here we give an example of a contra-variant functor. The family of all real vector spaces, denoted by **Vect** is a category and for a pair of real vector spaces V and W, the set $\text{Mor}(V, W)$ consists of all linear transformations from V to W. We define a functor from **Vect** to itself by assigning to each V its dual V^* and to each $T \in \text{Mor}(V, W)$ the adjoint map T^*. Again,

$$(\text{id}_V)^* = \text{id}_{V^*}$$

But if U, V and W are three vector spaces and $T \in \text{Mor}(U, V)$ and $S \in \text{Mor}(V, W)$ are two linear maps then

$$(S \circ T)^* = T^* \circ S^*$$

Let us look at an example of a functor from the category of topological spaces to the category **Rng** of commutative rings. We shall always assume that every ring that we shall deal with, has a unit element.

Example 8.5: Let X be a topological space and $C(X)$ be the set of all continuous functions from X to the real line (with its usual topology). Then $C(X)$ is a commutative ring with unity. Suppose that $f : X \longrightarrow Y$ is a continuous map between topological spaces then we define f^* to be the map

$$f^* : C(Y) \longrightarrow C(X)$$
$$\phi \mapsto \phi \circ f$$

It is obvious to see that f^* is a ring homomorphism and $\text{id}_X^* = \text{id}_{C(X)}$. Further, $(g \circ f)^* = f^* \circ g^*$ for $f \in \text{Mor}(X, Y)$ and $g \in \text{Mor}(Y, Z)$. We thus have a contravariant functor **Top** $\longrightarrow$ **Rng** sending the object $X \in$ **Top** to the object $C(X) \in$ **Rng** and assigning to $f \in \text{Mor}(X, Y)$ the ring homomorphism $f^* \in \text{Mor}(C(Y), C(X))$.

Category of pairs: Given topological spaces X and Y one is often not interested in the class of all continuous maps $f : X \longrightarrow Y$ but a restricted class of continuous functions satisfying some "side conditions" such as mapping a given subset A of X into a given subset B of Y.

Definition 8.3: The category $\mathbf{Top^2}$ of pairs has as its objects the class of all pairs of topological spaces (X, A) where X is a topological space and $A \subset X$. Given two pairs (X, A) and (Y, B) the set of morphisms between them is the class of all continuous functions $f : X \longrightarrow Y$ such that $f(A) \subset B$.

Exercises:

1. Recast the notion of homotopy of paths in terms of morphisms of the category $\mathbf{Top^2}$.
2. Define a binary operation on $\mathbb{Z} \times \mathbb{Z}$ as follows

   $$(a, b) \cdot (c, d) = (a + c, b + (-1)^a d)$$

   Show that this defines a group operation on $\mathbb{Z} \times \mathbb{Z}$ and this group is called the semi-direct product of $\mathbb{Z}$ with itself. The standard notation for this is $\mathbb{Z} \ltimes \mathbb{Z}$. Compute the inverse of (a, b), compute the conjugate of (a, b) by (c, d) and the commutator of two elements. Determine the commutator subgroup and the the abelianization of $\mathbb{Z} \ltimes \mathbb{Z}$.

3. A morphism $\phi \in \mathrm{Mor}(X, Y)$ in a category is said to be an equivalence if there exists $\psi \in \mathrm{Mor}(Y, X)$ such that $\phi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \phi = \mathrm{id}_X$. In a category whose objects are sets and morphisms are maps, show that if $g \circ f$ is an equivalence for $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Y, Z)$ then g is surjective and f is injective.

4. We say a category $\mathcal{C}$ admits finite products if for every pair of objects U, V in $\mathcal{C}$ there exists an object W and a pair of morphisms $p : W \longrightarrow U$, $q : W \longrightarrow V$ such that the following property holds. For every pair of morphisms $f : Z \longrightarrow U$, $g : Z \longrightarrow V$ there exists a unique morphism $f \times g \in \mathrm{Mor}(Z, W)$ such that

$$p \circ (f \times g) = f, \quad q \circ (f \times g) = g.$$

Show that the categories **Top**, **Gr** and **AbGr** admit finite products and in fact the usual product of topological spaces/groups serve the purpose with p and q being the two projection maps.

5. Discuss arbitrary products in a category generalizing the preceding exercise and discuss the existence of arbitrary products in the categories **Top**, **Gr** and **AbGr**.

6. We say a category $\mathcal{C}$ admits finite coproducts if for every pair of objects U, V in $\mathcal{C}$ there exists an object W and a pair of morphisms $p : U \longrightarrow W$, $q : V \longrightarrow W$ such that the following property holds. For every pair of morphisms $f : U \longrightarrow Z$, $g : V \longrightarrow Z$ there exists a unique morphism $f \oplus g \in \mathrm{Mor}(W, Z)$ such that

$$(f \oplus g) \circ p = f, \quad (f \oplus g) \circ q = g.$$

Show that the category **AbGr** admits finite coproducts and in fact the usual product of groups serves the purpose where the maps p and q are the canonical injections:

$$\begin{aligned} p : G \longrightarrow G \times H, &\quad q : H \longrightarrow G \times H \\ p(g) = (g, 1), &\quad q(h) = (1, h) \end{aligned}$$

What happens when this (naive construction) is tried out in the category **Gr** instead of **AbGr**? In the context of abelian groups the coproduct is referred to as the direct sum.

7. Discuss the coproduct of an arbitrary family of objects in the category **AbGr**. It is referred to as the direct sum of the family.

8. Suppose that X and Y are two topological spaces, form their disjoint union $X \sqcup Y$ which is the set theoretic union of their homeomorphic copies $X \times \{1\}$ and $Y \times \{2\}$. A subset G of $X \sqcup Y$ is declared open if $G \cap (X \times \{1\})$ and $G \cap (Y \times \{2\})$ are both open. Check that this defines a topology on $X \sqcup Y$ and the maps

$$\begin{aligned} p : X \longrightarrow X \sqcup Y, &\quad q : Y \longrightarrow X \sqcup Y \\ p(x) = (x, 1), &\quad q(y) = (y, 2) \end{aligned}$$

are both continuous. Show that the category **Top** admits finite coproducts.

Lecture IX - Functorial Property of the Fundamental Group

We now turn to the most basic functor in algebraic topology namely, the π_1 functor. Recall that the fundamental group of a space involves a base point and according to theorem (7.8) the fundamental group of a path connected space is unique upto isomorphism. However, this isomorphism is not canonical as theorem 7.9 shows and isomorphism classes of groups do not form a category. To get around this difficulty and to obtain a well-defined functor, we introduce the category of pointed topological spaces.

Definition 9.1 (The category of pointed topological spaces): This category will be denoted by $\mathbf{Top_0}$ and its objects consists of all pairs (X, x_0) where X is a topological space and x_0 is a point of X. Given two pairs of pointed spaces (X, x_0) and (Y, y_0), the morphisms between them consists of all continuous functions $f : X \longrightarrow Y$ such that $f(x_0) = y_0$.

Suppose that X, Y are path connected spaces and $f : X \longrightarrow Y$ is a continuous map such that $f(x_0) = y_0$ then f clearly defines a morphism, denoted by same letter, between pointed spaces

$$f : (X, x_0) \longrightarrow (Y, y_0).$$

The map $f_* : \pi_1(X.x_0) \longrightarrow \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$ where γ in X based at x_0, is well defined since $f \circ \gamma$ is a loop in Y based at y_0 Therefore f_* is well defined because if γ_1, γ_2 are homotopic loops in X based at x_0 and F is the homotopy then $f \circ F$ is a homotopy between $f \circ \gamma_1$ and $f \circ \gamma_2$ in Y. It is immediately checked that $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$ thereby giving a group homomorphism:

$$f_*([\gamma_1][\gamma_2]) = f_*([\gamma_1])f_*([\gamma_2]).$$

The group homomorphism f_* is called the map induced by f on the fundamental groups. In other words we obtain a functor π_1 from $\mathbf{Top}_0$ to $\mathbf{Gr}$.

Lemma 9.1: Suppose that $(X, x_0), (Y, y_0)$ and (Z, z_0) are pointed topological spaces. Let $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ be continuous maps of pairs, that is continuous maps satisfying $f(x_0) = y_0$; $g(y_0) = z_0$, then the induced homomorphisms on the respective fundamental groups satisfies

$$(g \circ f)_* = g_* \circ f_*.$$

If $\text{id}_x : X \longrightarrow X$ is the identity map then $(\text{id}_x)_* = \text{id}_{\pi_1(X,x_0)}$. That is to say, the identity map on X induces the identity homomorphism on $\pi_1(X, x_0)$.

Proof: The second part is obvious. To prove the first part, for any loop γ in X based at x_0,

$$(g \circ f) \circ \gamma = g \circ (f \circ \gamma)$$

so we get upon passing to equivalence classes,

$$(g \circ f)_*[\gamma] = g_*[f \circ \gamma] = g_*(f_*([\gamma]))$$

In particular if $f : X \longrightarrow Y$ is a homeomorphism then $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism of groups. □

Definition 9.2 (Retraction): Given a topological space X, a subset $A \subseteq X$ is said to be retract of X if there exits a continuous function $r : X \longrightarrow A$ such that $r(a) = a$ for all $a \in A$.

It is immediate that a retract of a Hausdorff space must be closed. The condition that A be a retract of X is quite a strong condition. For example if X is compact and connected then so must A. Thus $\{0, 1\}$ cannot be a retract of $[0, 1]$. The boundary $\dot{I}^2$ of I^2 is not a retract of I^2 but this is highly non-trivial.

Example 9.1:

(i) $S^1 \times \{1\}$ is a retract of $S^1 \times S^1$. A retraction is given by $r(z, w) = (z, 1)$.

(ii) $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ is not a retract of $S^1 \times S^1$ as we shall see later.

(iii) S^1 is a retract of $\mathbb{R}^2 - \{(0, 0)\}$ and the retraction is given by the map $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$.

(iv) Suppose A is a retract of X then every continuous map $f : A \longrightarrow Y$ extends continuously to a map $\tilde{f} : X \longrightarrow Y$.

We shall show later (lectures 12-13) that $\pi_1(S^1, 1) = \mathbb{Z}$ is non-trivial but we present it here as a theorem for immediate use in the next lecture on the Brouwer's fixed point theorem.

Theorem 9.2: $\pi_1(S^1, 1) = \mathbb{Z}$ and the generator is given by the homotopy class of the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1.$$

Lemma 9.3: Suppose $r : X \longrightarrow A$ is a retraction, $j : A \longrightarrow X$ is the inclusion, then for $a \in A$

$$r_* : \pi_1(X, a) \longrightarrow \pi_1(A, a)$$

is surjective and

$$j_* : \pi_1(A, a) \longrightarrow \pi_1(X, a)$$

is injective.

Proof: Since $r \circ j = \mathrm{id}_A$ we see that $r_* \circ j_* = \mathrm{id}_{\pi_1(A,a)}$. Hence r_* is surjective and j_* is injective. □

Corollary 9.4 (No retraction theorem): S^1 is not a retract of $E^2 = \{\mathbf{x} \in \mathbb{R}^2 / \|\mathbf{x}\| \leq 1\}$

Proof: Suppose we have a retraction $r : E^2 \longrightarrow S^1$ then the induced map

$$r_* : \pi_1(E^2, 1) \longrightarrow \pi_1(S^1, 1)$$

would be surjective which means we have a surjective group homomorphism

$$r_* : \{1\} \longrightarrow \mathbb{Z}$$

which is impossible. □

Corollary 9.5 (Brouwer's fixed point theorem): Every continuous function $f : E^2 \longrightarrow E^2$ has a fixed point where $E^2 = \{\mathbf{x} \in \mathbb{R}^2 / \|\mathbf{x}\| \leq 1\}$.

Proof: Will be done in the next lecture.

Fundamental group of a Product: The fundamental group functor has the pleasant property that it respects products. The following theorem summarizes the matter for finite products.

Theorem 9.6: Suppose that X and Y are two topological spaces and $x_0 \in X$ and $y_0 \in Y$. Then

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof: Let p_1 and p_2 be the usual projection maps $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$ respectively and γ be a loop in $X \times Y$ based at (x_0, y_0). Then $p_1 \circ \gamma$ and $p_2 \circ \gamma$ are loops in X and Y based at x_0 and y_0 respectively. The map

$$\begin{aligned} \phi : \pi_1(X \times Y, (x_0, y_0)) &\longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ [\gamma] &\mapsto ([p_1 \circ \gamma], [p_2 \circ \gamma]) \end{aligned}$$

is well-defined and easily seen to be a surjective group homomorphism. Injectivity is also easy to check. Well, suppose that $[\gamma]$ is in the kernel of ϕ then $p_1 \circ \gamma$ and $p_2 \circ \gamma$ are homotopic to the constant loops ε_{x_0} and ε_{y_0} respectively via homotopies F_1 and F_2. That is to say there exists continuous maps $F_1 : I^2 \longrightarrow X$ and $F_2 : I^2 \longrightarrow Y$ such that

$$F_1(0, t) = p_1 \circ \gamma, F_1(1, t) = \varepsilon_{x_0}, F_2(0, t) = p_2 \circ \gamma, F_2(1, t) = \varepsilon_{y_0}.$$

and $F_1(s, 0) = F_1(s, 1) = x_0$, $F_2(s, 0) = F_2(s, 1) = y_0$ for all $s \in [0, 1]$. Putting these together we get a continuous map $F_1 \times F_2 : I^2 \longrightarrow X \times Y$ namely

$$(s, t) \mapsto (F_1(s, t), F_2(s, t))$$

which is a homotopy between γ and the constant loop at (x_0, y_0) proving that the kernel is trivial.

Corollary 9.7: $\pi_1(S^1 \times S^1, (1, 1)) = \mathbb{Z} \times \mathbb{Z}$

Exercises

1. Show that the sphere S^2 retracts onto one of its longitudes. If X is the space obtained from S^2 by taking its union with a diameter, there is a surjective group homomorphism $\pi_1(X) \longrightarrow \mathbb{Z}$.

2. Prove that A is a retract of X if and only if every space Y, every continuous map $f : A \longrightarrow Y$ has a continuous extension $\tilde{f} : X \longrightarrow Y$.

3. Show that the fundamental group respects arbitrary products.

4. Construct a retraction from $\{(x, y) : x \text{ or } y \text{ is an integer}\}$ onto the boundary of I^2.

5. Show that every homeomorphism of E^2 onto itself must map the boundary to the boundary.

6. Given that there exists a functor T from the category **Top** to the category **AbGr** such that $T(X)$ is the trivial group for every convex subset X of a Euclidean space and $T(S^n)$ is a non-trivial group, prove that S^n is not a retract of the closed unit ball in $\mathbb{R}^{n+1}$.

Lecture X - Brouwer's Theorem and its Applications.

In this lecture we shall prove the Brouwer's fixed point theorem and deduce some of its consequences such as the Perron-Frobenius' theorem. The one dimensional Brouwer's theorem follows from the intermediate value property as is indicated in the exercises of lecture 3. We also include a proof of the fact that the spheres S^n have trivial fundamental group when $n \geq 2$. This result has been included here to demonstrate why the fundamental group is insufficient to prove the Brouwer's fixed point theorem in dimension three or higher.

We begin by defining the fixed point property for a space. Here we require the fixed point property to hold for all continuous functions of the space into itself. Note that in analysis the spaces considered are somewhat special and so are the maps whose fixed point property are sought. A classic example of such a restricted fixed point theorem is the Banach's fixed point theorem.

Definition 10.1: A space X is said to have the fixed point property if every continuous map $f : X \longrightarrow X$ has a fixed point namely, there exists $p \in X$ such that $f(p) = p$.

Theorem 10.1: The fixed point property is a topological property. That is, if X and Y are homeomorphic and X has the fixed point property then so does Y.

Proof: Suppose that X has the fixed point property and $h : X \longrightarrow Y$ is a homeomorphism. Let $g : Y \longrightarrow Y$ be an arbitrary continuous map. Applying the fixed point property to the map $f = h^{-1} \circ g \circ h$ we get a point $p \in X$ such that $f(p) = p$. The fixed point of g is seen to be $h(p)$.

Examples 10.1: (i) The closed unit interval $[0, 1]$ has the fixed point property (exercise 1, lecture 3).

(ii) A non-trivial topological group does not have the fixed point property.

(iii) The space $\mathbb{R}P^{2n}$ has the fixed point property but we are not yet ready to prove this.

(iv) The open unit disc $U = \{z \in \mathbb{C}/|z| < 1\}$ does not have the fixed point property. For if a is a non-zero complex number with $|a| < 1$ then the map $f : U \longrightarrow U$ given by

$$f(z) = \frac{z - a}{1 - \overline{a}z}$$

has no fixed points in U. The reader must first check that f maps the open unit disc to itself and examine if it has any fixed points.

Theorem 10.2 (Brouwer's fixed point theorem): Every continuous function $f : E^2 \longrightarrow E^2$ has a fixed point where $E^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$.

Proof: We assume the contrary, that is to say a continuous function f of the closed unit disc into itself exists which has no fixed points. We produce a retraction from E^2 onto S^1 which would be a contradiction. The ray emanating from $f(x) \in E^2$ and passing through $x \in E^2$ namely

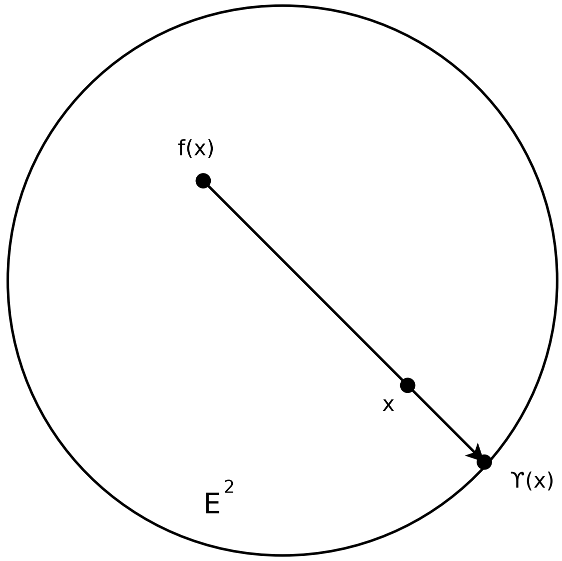


Figure 11: E^2 is not a retract of S^1

$$tx + (1-t)f(x), \quad t \geq 0,$$

meets the circle S^1 at a point denoted by $r(x) = t_0 x + (1-t_0)f(x)$ where, t_0 is a root of the quadratic

$$\langle tx + (1-t)f(x),\ tx + (1-t)f(x)\rangle = 1. \tag{10.1}$$

We recast this quadratic as

$$t^2(|f(\mathbf{x}) - \mathbf{x}|^2) - 2tf(\mathbf{x}) \cdot (f(\mathbf{x}) - \mathbf{x}) - (1 - |f(\mathbf{x})|^2) = 0. \tag{10.2}$$

Since the coefficient of t^2 is never zero, the roots are continuous functions of $\mathbf{x}$ and they are real. Moreover the roots differ in sign or one of the roots is zero. Take t_0 to be the larger root for constructing $r(x)$. From (10.1) we see that r maps E^2 to S^1. Note that if $|x| = 1$ then $t = 1$ satisfies the quadratic and so must be the larger root. Hence we conclude $r(x) = x$ if $|x| = 1$ and we get a retraction of E^2 onto S^1 which is a contradiction. □

Remark: Note that the proof merely used the fact that π_1 functor is trivial on discs and nontrivial on circles. Any functor with this property may be used to prove the Brouwer's fixed point theorem.

Theorem 10.3 (Perron-Frobenius): A 3×3 matrix with strictly positive real entries has a positive eigen-value. The corresponding eigen-vector has non-negative entries.

Proof: Let A be a 3×3 matrix with strictly positive real entries and S be the part of the sphere

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \quad x \geq 0, y \geq 0, z \geq 0\}$$

Then S is homeomorphic to the closed unit disc in the plane (why?) and so has the fixed point property. If $\mathbf{v}$ is any unit vector with non-negative entries then the entries of $A\mathbf{v}$ are non-negative and at-least one of the entries must be positive. Hence the map $f : S \longrightarrow S$ given by $f(\mathbf{v}) = A\mathbf{v}/\|A\mathbf{v}\|$ is continuous. By Brouwer's fixed point theorem, f has a fixed point $\mathbf{v}_0$ which means $A\mathbf{v}_0/\|A\mathbf{v}_0\| = \mathbf{v}_0$ from which we infer that $\|A\mathbf{v}_0\|$ is an eigen-value of A and this must be positive.

Fundamental groups of spheres: We close this lecture with a proof of the fact that $\pi_1(S^n) = \{1\}$ when $n \geq 2$. The student ought to try and figure out intuitively why is this so.

Theorem 10.4: If U and V are simply connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is path connected then X is simply connected.

Proof: Let us choose a base point $x_0 \in U \cap V$ and γ be an arbitrary loop in X based at x_0. The open cover $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ of $[0,1]$ has a Lebesgue number ϵ. Choose a partition

$$\{t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1\}.$$

such that the length of each sub-interval is less than ϵ. Then γ maps each $[t_j, t_{j+1}]$ into U or V. If γ maps two adjacent intervals into U or into V then drop the abutting point of the two intervals thereby coarsening the partition. Thus may arrange it such that for each $j = 1, 2, \ldots, n-1$, the point $\gamma(t_j)$ lies in $U \cap V$. We now choose a path σ_j joining x_0 and $\gamma(t_j)$ such that the image of σ_j lies entirely in $U \cap V$. This is possible since $U \cap V$ is path connected and $x_0 \in U \cap V$. Also let γ_j denote the restriction of γ to the sub-interval $[t_{j-1}, t_j]$ $(j = 1, 2, \ldots, n)$. We may reparametrize γ_j (retaining the name) so that its domain is $[0,1]$. Now

$$\gamma \sim \gamma_1 * \sigma_1^{-1} * \sigma_1 * \gamma_2 * \sigma_2^{-1} * \sigma_2 * \gamma_3 * \cdots * \sigma_{n-1}^{-1} * \sigma_{n-1} * \gamma_n$$

Now each of the loops $\gamma_1 * \sigma_1^{-1}$, $\sigma_1 * \gamma_2 * \sigma_2^{-1}$,..., $\sigma_{n-1} * \gamma_n$ based at x_0 lies in one of the simply connected open sets U or V and so each of them is homotopic to the constant loop via a homotopy F_j. These homotopies F_j may be juxtaposed to provide a homotopy between

$$\gamma_1 * \sigma_1^{-1} * \sigma_1 * \gamma_2 * \sigma_2^{-1} * \sigma_2 * \gamma_3 * \cdots * \sigma_{n-1}^{-1} * \sigma_{n-1} * \gamma_n$$

and the constant loop. The proof is complete.

Theorem 10.5: For $n \geq 2$, the sphere S^n is simply connected.

Proof: Let U be the sphere minus the north pole and V be the sphere minus the south pole. Using the stereo-graphic projections, we see that U and V are simply connected open subsets of S^n and it is easily verified that $U \cap V$ is path connected. The result follows from the previous theorem.

Exercises

1. Suppose that a space X has the fixed point property, is it necessary that it be connected? Does it have to be path-connected?

2. Explain why a non-trivial topological group cannot have the fixed point property.

3. Prove the Brouwer's fixed point theorem for the closed unit ball in $\mathbb{R}^n$ given that that there exists a functor T from the category **Top** to the category **AbGr** such that $T(X)$ is the trivial group for every convex subset X of a Euclidean space and $T(S^{n-1})$ is a non-trivial group.
4. Show that the Brouwer's fixed point theorem implies the no retraction theorem.
5. Explain how the homotopies F_j in the proof of theorem 10.4 can be juxtaposed.
6. Show that the circle S^1 is not a retract of the sphere S^2.

Lecture XI - Homotopies of maps. Deformation retracts.

We generalize the notion of homotopy of paths to homotopy of a pair of continuous maps between topological spaces. This would be particularly useful in the second part of the course. It also leads to a powerful notion of deformation retracts which is often useful in deciding whether two spaces have the same fundamental group. Homotopy of maps is a useful coarsening of the notion of homeomorphism of two spaces leading to the notion of homotopy equivalence of spaces. Over the decades homotopy has proved to be the most important notion in topology, susceptible to considerable generalization with wide applicability.

Definition 11.1 (Homotopies of maps): (i) Given continuous maps $f, g : X \longrightarrow Y$ between topological spaces we say that f and g are homotopic if there exists a continuous map $F : X \times [0,1] \longrightarrow Y$ such that

$$F(x,0) = f(x), \quad F(x,1) = g(x), \quad \text{for all } x \in X \tag{11.1}$$

We shall occasionally use the notation $f \sim g$ to indicate that f and g are homotopic. One can formulate a notion for pairs of spaces:

(ii) Two continuous maps $f, g : (X, A) \longrightarrow (Y, B)$ between pairs of topological spaces are said to be homotopic if there exists $F : (X \times I, A \times I) \longrightarrow (Y, B)$ such that in addition to (11.1) the following condition holds:

$$F(a,t) \in B, \quad \text{for all } a \in A, t \in [0,1]. \tag{11.2}$$

Condition (11.2) is a boundary condition which states that the intermediate functions

$$F_t : x \mapsto F(x,t)$$

all map A into B. Note that when $A = \{x_0\}$ and $B = \{y_0\}$, the condition says that all the intermediate maps F_t are base point preserving. We leave it to the reader to prove the following two simple results.

Theorem 11.1: Homotopy is an equivalence relation.

Theorem 11.2: Suppose that f and g are homotopic maps of pairs (X, x_0) and (Y, y_0) then the induced group homomorphisms f_* and g_* from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$ are equal.

Now suppose that f and g are homotopic maps from X to Y such that for a base point $x_0 \in X$, $f(x_0) = g(x_0) = y_0$ say, but the intermediate maps do not respect these base points. Then it is not necessary that $f_* = g_*$ as maps from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. The following theorem addresses this issue.

Theorem 11.3: Suppose that F is a homotopy between maps $f, g : X \longrightarrow Y$ and for a point $x_0 \in X$, $f(x_0) = g(x_0) = y_0$. Then the group homomorphisms f_* and g_* are conjugate by the inner-automorphism generated by the loop

$$\sigma : t \mapsto F(x_0, t) \tag{11.3}$$

Proof: The idea of proof is simple. Observe that (11.3) is the image of the base point x_0 under the deformation suggesting the use of theorem (7.8). If we fix an intermediate time $s \in [0, 1]$ then the curve σ_s given by $\sigma_s(t) = t \mapsto \sigma(st)$ starts at y_0 and we could use it to construct a loop at y_0 namely

$$\sigma_s * F(\gamma(\ .\ ), s) * \sigma_s^{-1}$$

In detail, for each loop $\gamma(t) \in X$ based at x_0, the homotopy $\phi : [0, 1] \times [0, 1] \longrightarrow Y$ given by

$$\begin{aligned} \phi(s,t) &= \sigma(3st) \ \text{ if } 0 \leq t \leq 1/3 \\ &= F(\gamma(3t-1), s) \ \text{ if } 1/3 \leq t \leq 2/3 \\ &= \sigma(3s - 3st) \ \text{ if } 2/3 \leq t \leq 1. \end{aligned}$$

establishes the equality of $f_*[\gamma]$ and $[\sigma](g_*[\gamma])[\sigma^{-1}]$.

Corollary 11.4: Suppose that F is a homotopy between maps $f, g : X \longrightarrow Y$ and for a point $x_0 \in X$, $f(x_0) = g(x_0) = y_0$. If $\pi_1(Y, y_0)$ is abelian then the group homomorphisms f_* and g_* are equal.

If we drop the hypothesis $f(x_0) = g(x_0)$ in theorem 11.3 the proof still goes through but since σ is no longer a loop we merely get that the induced maps f_* and g_* differ by a composition through the isomorphism $h_{[\sigma]}$ encountered in theorem (7.8). We record the result as a theorem and the reader may rework the proof of theorem 11.3 to fit it in the present context.

Theorem 11.5: Suppose that F is a homotopy between maps $f, g : X \longrightarrow Y$ then for $x_0 \in X$, the induced maps $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ and $g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, g(x_0))$ satisfy the relation

$$h_{[\sigma]} \circ f_* = g_* \tag{11.4}$$

where $h_{[\sigma]}$ is the isomorphism

$$h_{[\sigma]} : [\gamma] \mapsto [\sigma * \gamma * \sigma^{-1}] \tag{11.5}$$

we have encountered earlier with σ being the curve $F(x_0, t)$ joining $f(x_0)$ and $g(x_0)$.

Definition 11.2 (Homotopy equivalence): (i) A map $f : X \longrightarrow Y$ is said to be a homotopy equivalence if there exists a map $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are respectively homotopic to the identity maps id_Y and id_X respectively. Under this circumstance we say that the spaces X and Y are homotopically equivalent or have the same homotopy type.

(ii) A space that is homotopy equivalent to a point is said to be contractible. This is equivalent to the statement that the identity map on X is homotopic to a constant map.

The student may check that if X and Y are homotopy equivalent and Y and Z are homotopically equivalent then X and Z are homotopy equivalent.

Theorem 11.6: If $f : X \longrightarrow Y$ is a homotopy equivalence then the groups $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$ are isomorphic.

Proof: There exists $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are respectively homotopic to id_Y and id_X. By theorem 11.5 $f_* \circ g_*$ differs from the identity map on $\pi_1(Y, (f \circ g)(y_0))$ by a composition with the isomorphism $h_{[\sigma]}$ where σ is a path joining $f(g(y_0))$ and y_0. In particular $f_* \circ g_*$ is bijective and so f_* is surjective and g_* is injective. Likewise, working with $g \circ f$ one concludes that g_* is surjective and f_* is injective. Hence f_* is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$.

Deformation retract: A subspace A of X is said to be a deformation retract if there exists a continuous map $r : X \longrightarrow A$ such that $r \circ j = \mathrm{id}_A$ and $j \circ r \sim \mathrm{id}_X$ where j denotes the inclusion of A into X. In particular, X and A have the same homotopy type.

Theorem 11.7: Suppose that A is a deformation retract of X via a map $r : X \longrightarrow A$. Then for $x_0 \in A$, the maps $r_* : \pi_1(X, x_0) \longrightarrow \pi_1(A, x_0)$ and $i_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ are isomorphisms.

Proof: Let $r : X \longrightarrow A$ be a retraction such that $j \circ r \sim \mathrm{id}_X$. By (the proof of) theorem 11.6, r_* is injective. But the composition $r \circ j = \mathrm{id}_A$ shows that r_* is surjective. Hence r_* establishes an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(A, x_0)$.

Example 11.1: The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n - \{0\}$. A retraction $r : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$ is given by the formula $r(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$. The homotopy between $j \circ r$ and the identity map on $\mathbb{R}^n - \{0\}$ is provided by the convex combination

$$F(\mathbf{x}, t) = t\mathbf{x} + (1-t)\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{11.6}$$

The student must however check that $F(\mathbf{x}, t)$ omits the zero vector. From this we get the following important result.

Theorem 11.8: (i) The fundamental group of the punctured plane is the additive group $\mathbb{Z}$ and the homotopy class of the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1 \tag{11.7}$$

provides a generator for the group.

(ii) The fundamental group of $\mathbb{R}^n - \{0\}$ is the trivial group.

Example 11.2: Let X be the union of the sphere S^2 and one of its diameters. Then X is homotopy equivalent to the space $S^2 \vee S^1$. While it is easy to construct the map $f : X \longrightarrow S^2 \vee S^1$, the map g in the opposite direction is not easy to write down. Exercise 6 shows how to get around the difficulty.

Example 11.3: Let L be the line $\{(0, 0, x_3)/x_3 \in \mathbb{R}\}$ in $\mathbb{R}^3$ and C be the circle

$$(x_1 - 1)^2 + x_2^2 = 1/4, \quad x_3 = 0.$$

We show that the torus is a deformation retract of the space $X = \mathbb{R}^3 - (L \cup C)$. The idea is simple but some details ought to be examined. Let us begin with the punctured half plane

$$H_0' = \{(x_1, 0, x_3)/x_1 > 0\} - \{(1, 0, 0)\}$$

which clearly deformation retracts to the circle C_0 given by

$$C_0 : (x_1 - 1)^2 + x_3^2 = 1/4, \quad x_2 = 0.$$

The homotopy $F : H_0' \times [0,1] \longrightarrow H_0'$ is simply given by the convex combination:

$$F(\mathbf{x}, t) = (1-t)\mathbf{x} + t\Big(\mathbf{e}_1 + \frac{\mathbf{x} - \mathbf{e}_1}{\|\mathbf{x} - \mathbf{e}_1\|}\Big), \quad \mathbf{e}_1 = (1, 0, 0).$$

The idea is to rotate the picture about the x_3-axis. It is expedient to use spherical polar coordinates given by

$$x_1 = \rho \cos\theta \sin\phi,\ x_2 = \rho \sin\theta \sin\phi,\ x_3 = \rho\cos\phi, \quad 0 < \phi < \pi,\ \theta \in \mathbb{R}.$$

Let H_θ' be the half plane bounded by the x_3-axis making angle θ with H_0' and R_θ denote the rotation about the x_3-axis mapping H_θ' onto H_0' namely,

$$R_\theta(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) = (\rho\sin\phi, 0, \rho\cos\phi)$$

The homotopy we are looking for is then the map $G : X \times [0,1] \longrightarrow X$ given by

$$G(\mathbf{x}, t) = R_\theta^{-1} \circ F(R_\theta(\mathbf{x}), t). \tag{11.8}$$

It is easy to see using the properties of rotations, that

(i) G is well defined, that is the image of G avoids the circle C

(ii) Satisfies the requisite boundary conditions at $t = 0$ and $t = 1$.

However, the continuity of G is not automatic since the θ appearing in the definition of G depends also on $\mathbf{x}$ and we know that θ cannot be defined as a continuous function of $\mathbf{x}$ on X. One can either write a formula (which is easy) and see that θ occurs in (11.8) only as $\cos\theta$ and $\sin\theta$ which are continuous functions on X or better still use the property of quotients. We leave the amusing details to the reader.

Corollary 11.9: The fundamental group of the complement of $L \cup C$ in $\mathbb{R}^3$ is $\mathbb{Z} \times \mathbb{Z}$.

Exercises:

1. Check that the map ϕ constructed in the proof of theorem 11.3 is continuous and is indeed a homotopy. Work out the proof of theorem 11.5.

2. Show that the boundary ∂M of the Möbius band M is not a deformation retract of M by taking a base point x_0 on the boundary and computing explicitly the group homomorphism

$$i_* : \pi_1(\partial M, x_0) \longrightarrow \pi_1(M, x_0).$$

3. Show that the boundary of the Möbius band is not even a retract of the Möbius band.

4. Fill in the details on the continuity of the map G in the example preceding corollary 11.9.

5. Show that the space $\mathbb{R}^3 - \{(x, y, z)/x^2 + y^2 = 1,\ z = 0\}$ deformation retracts to a sphere with a diameter attached to it.

6. Let X be the union of S^2 and one of its diameters D, $Y = S^2 \vee S^1$ and Z be the union of S^2 with a punctured half disc contained in a half with edge along D. Show that X and Y are both deformation retracts of Z and so they have the same homotopy type.

Lectures XII - XIII The fundamental group of the circle.

We have already stated the fact that the fundamental group of the circle is the group of integers and derived some important consequences form it. The importance of this result is attested by the fact that the Brouwer's fixed point theorem for a disc follows immediately from it. In this lecture will provide a detailed proof that $\pi_1(S^1, 1) = \mathbb{Z}$. Some of the ideas of the proof would appear again later in a general context of covering spaces. Though this result is a special one from the theory of covering spaces it is worthwhile looking at this important special case without reference to the general theory but rather as a preview to it. This topic will be covered in two lectures but the numbering will be as that of lecture 12. We begin with an algebraic lemma [14] (p. [//]).

Lemma 12.1: Suppose S is a set on which two binary operations $*$ and $*'$ are defined such that

(a) Both $*$ and $*'$ have a common two sided unit.

(b) The binary operations $*$ and $*'$ are mutually distributive. That is,

$$(f_1 * g_1) *' (f_2 * g_2) = (f_1 *' f_2) * (g_1 *' g_2), \quad f_1,\ f_2,\ g_1,\ g_2 \in S.$$

Then,

(i) both $*$ and $*'$ are associative and commutative.

(ii) $f * g = f *' g$ for all $f, g \in S$.

Proof: Denoting the common two sided identity by 1,

$$(f * g) = (f *' 1) * (1 *' g) = (f * 1) *' (1 * g) = f *' g$$

which proves (ii). Next we prove commutativity:

$$g * f = (1 *' g) * (f *' 1) = (1 * f) *' (g * 1) = f *' g = f * g.$$

Finally, using (ii) we prove associativity:

$$(f * g) * h = (f * g) *' (1 * h) = (f *' 1) * (g *' h) = f * (g *' h) = f * (g * h)$$

Corollary 12.2: If X is a topological group with unit element e then $\pi_1(X, e)$ is abelian. Moreover, if γ_1, γ_2 are two loops based at e define the binary operation $\circ$ on $\pi_1(X, e)$ by[3]

$$[\gamma_1] \circ [\gamma_2] = [\gamma_1(t) \cdot \gamma_2(t)]$$

where $\gamma_1(t) \cdot \gamma_2(t)$ denotes the group multiplication in X. Then

$$[\gamma_1] \circ [\gamma_2] = [\gamma_1][\gamma_2],$$

the right hand side being the product in $\pi_1(X, e)$. In other words, $\gamma_1(t) \cdot \gamma_2(t) \sim \gamma_1 * \gamma_2$.

Proof: Let $\bar{e}$ denote the homotopy class of the constant loop based at e. We first show that the operation $\circ$ is well defined. If $\gamma_1' \sim \gamma_1''$ and $\gamma_2' \sim \gamma_2''$ via the respective homotopies $F, G : I \times I \longrightarrow X$, it is easily checked that the map $F \cdot G : [0, 1] \times [0, 1] \longrightarrow X$ given by

$$F \cdot G(s, t) = F(s, t) \cdot G(s, t),$$

the product on the right denoting with group multiplication in X, is a homotopy between $\gamma_1'(t)\gamma_2'(t)$ and $\gamma_1''(t)\gamma_2''(t)$. We conclude that $\circ$ is a well defined binary operation on $\pi_1(X, e)$ with a two sided unit $\bar{e}$. Clearly, $\bar{e}$ is a common two sided unit element for both binary operations on $\pi_1(X, e)$. To invoke the lemma we show that the two binary operations are mutually distributive. Let $\gamma_1', \gamma_2'\gamma_1'', \gamma_2''$ be loops based at e

$$(\ [\gamma_1'][\gamma_1'']\ ) \circ (\ [\gamma_2'][\gamma_2'']\ ) = [\ (\gamma_1' * \gamma_1'')(t) \cdot (\gamma_2' * \gamma_2'')(t)\ ]$$

We first verify through direct calculation that $(\gamma_1' * \gamma_1'') \cdot (\gamma_2' * \gamma_2'') = (\gamma_1' \cdot \gamma_2') * (\gamma_1'' \cdot \gamma_2'')$. Well,

$$\begin{aligned}
(\gamma_1' * \gamma_1'')(t) \cdot (\gamma_2' * \gamma_2'')(t) &= \gamma_1'(2t)\gamma_2'(2t), \qquad \text{if } 0 \le t \le \frac{1}{2} \\
&= \gamma_1''(2t-1)\gamma_2''(2t-1), \text{ if } \frac{1}{2} \le t \le 1. \\
\therefore \quad [(\gamma_1' * \gamma_1'')(t) \cdot (\gamma_2' * \gamma_2'')(t)] &= [\gamma_1'(t) \cdot \gamma_2'(t)][\gamma_1''(t) \cdot \gamma_2''(t)]
\end{aligned}$$

So finally

$$([\gamma_1'][\gamma_1'']) \circ ([\gamma_2'][\gamma_2'']) = [\gamma_1'(t)\gamma_2'(t)][\gamma_1''(t)\gamma_2''(t)] = ([\gamma_1'] \circ [\gamma_2'])([\gamma_1''] \circ [\gamma_2''])$$

Thus lemma (12.1) is applicable for the binary operations $*$ and $\circ$ and the proof is complete.

Theorem 12.3: $\pi_1(S^1, 1) = \mathbb{Z}$ and the group is generated by homotopy class of the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \le t \le 1$$

Proof: The proof is broken into several steps. We shall employ the exponential map ex: $\mathbb{R} \longrightarrow S^1$ given by

$$\text{ex}(t) = e^{2\pi i t}. \qquad (12.1)$$

The function ex maps $(\frac{-1}{2}, \frac{1}{2})$ homeomorphically onto $S^1 - \{-1\}$ and we denote its inverse by

$$\text{lg} : S^1 - \{-1\} \longrightarrow (\frac{-1}{2}, \frac{1}{2}) \qquad (12.2)$$

which is also a homeomorphism.

[3] To avoid introducing more notation we are being notationally imprecise. The expression $\gamma_1(t) \cdot \gamma_2(t)$ inside the brackets refers to the map $t \mapsto \gamma_1(t) \cdot \gamma_2(t)$.

Lemma 12.4 (The lifting lemma): Let X be a compact subset of $\mathbb{R}^n$ that is star shaped with respect to origin. Let $f : X \longrightarrow S^1$ be a continuous function such that $f(0) = \text{ex}(t_0)$ for some $t_0 \in \mathbb{R}$. Then, there exists a continuous function $\tilde{f} : X \longrightarrow \mathbb{R}$ such that

$$\text{ex}\tilde{f}(x) = f(x), \quad \tilde{f}(0) = t_0 \tag{12.3}$$

Moreover the function $\tilde{f}$ satisfying (12.3) is unique and is called the lift of f with respect to ex.

Proof: Invoking the uniform continuity of f with $\varepsilon = 2$, there exists $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < 2$$

which in turn implies that $f(x) \neq -f(y)$. Now choose $n \in \mathbb{N}$ such that $n^{-1}\|x\| < \delta$ for all $x \in X$ which is possible since X is compact. This n is now fixed for the rest of the discussion. For $x \in X$ and $j = 0, 1, \ldots, n-1$

$$\Big\|\frac{j}{n}x - \frac{(j+1)}{n}x\Big\| < \delta,$$

whereby,

$$f(\frac{j+1}{n}x) \neq -f(\frac{j}{n}x).$$

From this we conclude that the function given by

$$\text{lg}\Big(\frac{f(\frac{j+1}{n}x)}{f(\frac{j}{n}x)}\Big), \quad x \in X$$

is continuous with respect to x. We now claim that

$$\tilde{f}(x) = t_0 + \sum_{j=0}^{n-1} \text{lg}\Big(\frac{f(\frac{j+1}{n}x)}{f(\frac{j}{n}x)}\Big)$$

is the required continuous function. Observe that $\text{lg}(1) = 0$, $\tilde{f}(0) = t_0$ and

$$\text{ex}\tilde{f}(x) = (\text{ex}(t_0)) \cdot \frac{f(\frac{1}{n}x)}{f(0)} \cdot \frac{f(\frac{2}{n}x)}{f(\frac{1}{n}x)} \cdots \frac{f(\frac{n}{n}x)}{f(\frac{n-1}{n}x)} = f(x).$$

Turning to the proof of uniqueness of the lift $\tilde{f}$, suppose $\tilde{f}_1, \tilde{f}_2 : X \longrightarrow \mathbb{R}$ are two continuous functions such that $\tilde{f}_1(0) = \tilde{f}_2(0) = t_0$ and $\text{ex}\tilde{f}_1(x) = \text{ex}\tilde{f}_2(x) = f(x)$. Then $\text{ex}(\tilde{f}_1(x) - \tilde{f}_2(x)) = 1$, which implies $\tilde{f}_1(x) - \tilde{f}_2(x) \in \mathbb{Z}$ (see note below). Since both functions are continuous, agree at the origin and X is connected, we conclude that

$$\tilde{f}_1(x) \equiv \tilde{f}_2(x).$$

□

Note: The properties of the exponential function used here must be established using power series expansions. Specifically prove using power series the following:

(i) $\text{ex}(z_1 + z_2) = \text{ex}(z_1) \cdot \text{ex}(z_2)$

(ii) There exists a unique positive real root of $\cos(x) = 0$ in $[0, 2]$ (via the real power series for the cosine function) and we denote this root by $\pi/2$.

(iii) $\cos(2\pi + x) = \cos x$, $\sin(2\pi + x) = \sin x$ (using addition formula for sin and cos following (i))

(iv) If $\cos x = \cos y$, $\sin x = \sin y$ then there exists $k \in \mathbb{Z}$ such that $x - y = 2\pi i k$.

Definition 12.1: Let $\gamma : [0,1] \longrightarrow S^1$ be a loop based at 1. By the lifting lemma there exists unique lift $\tilde{\gamma} : [0,1] \longrightarrow \mathbb{R}$ such that $\tilde{\gamma}(0) = 0$, $\text{ex}\tilde{\gamma}(1) = 1$. Thus, $\tilde{\gamma}(1) \in \mathbb{Z}$ and we call this integer the degree of the loop γ.

Lemma 12.5: If γ_1 and γ_2 are two homotopic loops based at 1. then $\text{deg}\gamma_1 = \text{deg}\gamma_2$. Thus the map $\phi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ given by $[\gamma] \mapsto \deg \gamma$ is well-defined.

Proof: Let $F : I \times I \longrightarrow S^1$ be the homotopy between γ_1 and γ_2. Since $I \times I$ is star shaped with respect to $(0,0)$ and $F(0,0) = 1 = \text{ex}(0)$, the lifting lemma gives a unique lift $\tilde{F} : I \times I \longrightarrow \mathbb{R}$ with $\tilde{F}(0,0) = 0$. The image $F(s,0)$ is a connected subset of $\mathbb{R}$ as s runs from 0 to 1 and $\exp\tilde{F}(s,0) = F(s,0) = 1$ for all $s \in [0,1]$. So $\tilde{F}(s,0)$ is integer valued and hence constant. From $\tilde{F}(0,0) = 0$ we conclude that $\tilde{F}(s,0) = 0$ for all $s \in [0,1]$. In particular the lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ both start at the origin and so

$$\deg \gamma_1 = \tilde{\gamma}_1(1), \quad \deg \gamma_2 = \tilde{\gamma}_2(1).$$

Our job will be over if we show that $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. Well, $\tilde{F}$ must map the connected set $\{(s,1)/0 \leq s \leq 1\}$ onto a connected subset J of $\mathbb{R}$ and since

$$\text{ex}\tilde{F}(s,1) = F(s,1) = 1,$$

this connected subset J must be a subset of $\mathbb{Z}$ and hence reduces to a singleton which means

$$\tilde{F}(s,1) = \tilde{F}(0,1), \text{ for all } s \in [0,1]$$

Setting $s = 0$ and 1 we see that

$$\tilde{\gamma}_2(1) = \tilde{F}(1,1) = \tilde{F}(0,1) = \tilde{\gamma}_1(1),$$

thereby completing the proof that the map $\phi : [\gamma] \mapsto \text{deg}\gamma$ is well defined.

Lemma 12.6: The map ϕ defined in lemma (12.5) is a group isomorphism.

Proof: Suppose γ_1 and γ_2 are two loops at 1 with lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ starting at origin. Then the path $\tilde{\gamma}$ given by $\tilde{\gamma}(t) = \tilde{\gamma}_1(t) + \tilde{\gamma}_2(t)$ also starts at the origin and satisfies

$$\text{ex}\ \tilde{\gamma}(t) = \text{ex}\ \tilde{\gamma}_1(t) \cdot \text{ex}\ \tilde{\gamma}_2(t) = \gamma_1(t) \cdot \gamma_2(t).$$

Hence $\tilde{\gamma}$ is the unique lift of $\gamma_1(t) \cdot \gamma_2(t)$ whereby,

$$\deg(\gamma_1(t)\gamma_2(t)) = \tilde{\gamma}(1) = \tilde{\gamma}_1(1) + \tilde{\gamma}_2(1) = \deg \gamma_1 + \deg \gamma_2.$$

Thus $\phi([\gamma_1 \cdot \gamma_2]) = \phi([\gamma_1]) + \phi([\gamma_2])$. From corollary (12.2), $[\gamma_1 \cdot \gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_1][\gamma_2]$ whence $\phi([\gamma_1][\gamma_2]) = \phi([\gamma_1]) + \phi([\gamma_2])$ which means that ϕ is a group homomorphism.

Surjectivity of ϕ is easy to see. Let $n \in \mathbb{Z}$ be arbitrary and $\tilde{\gamma}(t) = nt$. Then $\tilde{\gamma}$ is the unique lift of $\gamma(t) = \text{ex}\tilde{\gamma}(t)$ starting at the origin so that $\phi([\gamma]) = \tilde{\gamma}(1) = n$. We now show that the group homomorphism ϕ is injective. Suppose γ_1, γ_2 are two loops at 1 in S^1 such that $\deg \gamma_1 = \deg \gamma_2$. Then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$, where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the lifts of γ_1 and γ_2 starting at the origin. Since $\mathbb{R}$ is convex and the two curves $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ have common end points, they are homotopic. That is to say, there exists a continuous function $\tilde{F} : I \times I \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} \tilde{F}(0,t) &= \tilde{\gamma_1}(t), \quad \tilde{F}(1,t) = \tilde{\gamma_2}(t); \text{ for all } t \in [0,1] \\ \tilde{F}(s,0) &= 0, \quad \tilde{F}(s,1) = \tilde{\gamma_1}(1) = \tilde{\gamma_2}(1), \text{ for all } s \in [0,1]. \end{aligned}$$

The function $F : [0,1] \times [0,1]$ given by

$$F(s,t) = \mathrm{ex}\tilde{F}(s,t)$$

is then a homotopy between γ_1 and γ_2 and we have shown that $\deg \gamma_1 = \deg \gamma_2$ implies $[\gamma_1] = [\gamma_2]$. This suffices for a proof.

Corollary 12.7 (Generators for $\pi_1(S^1, 1)$): (1) The generators for $\pi_1(S^1, 1)$ are given by the loops

$$\eta : t \mapsto \exp(\pm 2\pi i t) \tag{12.4}$$

(2) The loops (12.4) also generate the group $\pi_1(\mathbb{C} - \{0\}, 1)$.

Proof: The lifts of these starting at the origin are ± 1 so that these loops have degrees ± 1 respectively. The second conclusion follows from the fact that a deformation retraction induces an isomorphism of fundamental groups. □

Definition 12.2 (Degree of a map): Suppose that $f : S^1 \longrightarrow S^1$ is a continuous map such that $f(1) = 1$, the degree of f is defined to be the degree of the loop

$$f \circ \eta : t \mapsto f(\exp(\pm 2\pi i t)), \quad 0 \le t \le 1. \tag{12.5}$$

Theorem 12.8: For a continuous map $f : S^1 \longrightarrow S^1$ with $f(1) = 1$, the degree satisfies the equation

$$f_*[\eta] = (\deg f)[\eta] \tag{12.6}$$

where the group operation on $\pi_1(S^1, 1)$ is viewed additively.

Proof: Since $[\eta]$ generates $\pi_1(S^1, 1)$, writing the group operation additively, we have

$$f_*[\eta] = c[\eta] \tag{12.6}$$

We have to show that $c = \deg f$. By definition, $f_*[\eta] = [f \circ \eta]$ which is mapped to $\deg f$ by the isomorphism ϕ of lemma (12.5). But this isomorphism maps $[\eta]$ to 1 and hence applying ϕ to (12.6) we get the result. □

Theorem 12.9 (The Borsuk Ulam Theorem): Suppose $f : S^n \longrightarrow \mathbb{R}^n$ is a continuous map. Then there exists a pair of antipodal points $x, -x$ such that $f(x) = f(-x)$

Proof for the case $n = 2$: We follow the elegant proof given in [17] (p. 109). We first show that any continuous function $g : E^2 \longrightarrow S^1$ maps a pair of antipodal points on the *boundary* of E^2 to the same point. That is there exists $z \in E^2$ such that $|z| = 1$ and $g(z) = g(-z)$. Since E^2 is a compact convex set, by lemma (12.4) we see that any continuous map $g : E^2 \longrightarrow S^1$ has a continuous lift $\tilde{g} : E^2 \longrightarrow \mathbb{R}$. Since the real valued map

$$\theta \mapsto \tilde{g}(e^{2\pi i\theta}) - \tilde{g}(e^{-2\pi i\theta}), \quad 0 \leq \theta \leq 1,$$

changes sign we see that there is a pair of antipodal points $z, -z \in S^1$ such that $\tilde{g}(z) = \tilde{g}(-z)$ and hence $g(z) = g(-z)$. Turning now to a continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, assume $f(\mathbf{x}) \neq f(-\mathbf{x})$ for every $\mathbf{x} \in S^2$. We construct the continuous function $g : E^2 \longrightarrow S^1$

$$g(z) = h(z)/|h(z)|$$

where

$$h(x_1, x_2) = f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) - f(-x_1, -x_2, -\sqrt{1 - x_1^2 - x_2^2}), \quad (x_1, x_2) \in E^2.$$

Since $|h(z)| = |h(-z)|$, we infer that there is no $z \in E^2$ satisfying $|z| = 1$ and $g(z) = g(-z)$ resulting in a contradiction.

Corollary 12.10: S^2 is not homeomorphic to any subset of $\mathbb{R}^2$

Proof: The Borsuk Ulam theorem shows that a continuous function $S^2 \longrightarrow \mathbb{R}^2$ cannot be injective.

Theorem 12.11 (Fundamental theorem of algebra): Every non-constant polynomial with complex coefficients has a complex root.

Proof: If the polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has no zeros, then in particular, $p(1) \neq 0$. For $t \neq 0$, we define

$$p(z/t)t^n = \Big(z^n + a_1 z^{n-1} t + \cdots + a_n t^n\Big).$$

The right hand side makes sense even when $t = 0$ and we denote the right hand side by $g(z, t)$. Observe that $g(z, 0) = z^n$ and $g(z, 1) = p(z)$. However we need a homotopy of maps of S^1 preserving the base point 1. To this end we modify it consider instead the map $F : S^1 \times [0, 1] \longrightarrow S^1$ given by

$$F(z, t) = \frac{g(z, t)}{|g(z, t)|} \frac{|g(1, t)|}{g(1, t)}. \tag{12.7}$$

Clearly $g(z, 0) \neq 0$ for any $z \in S^1$ and if $0 < t \leq 1$ then again $g(z, t) = p(z/t)t^n \neq 0$. Thus (12.7) is a base point preserving homotopy between the function $f : S^1 \longrightarrow S^1$ given by

$$f(z) = \frac{p(z)}{|p(z)|} \frac{|p(1)|}{p(1)} \tag{12.8}$$

and the map $z \mapsto z^n$. We conclude that degree of f is n. However we have a base point preserving homotopy between (12.8) and the constant map namely, $G : S^1 \times [0, 1] \longrightarrow S^1$ given by

$$G(z, s) = \frac{p(sz)}{|p(sz)|} \frac{|p(s)|}{p(s)}.$$

We now conclude that degree of (12.8) is zero and we have a contradiction.

Exercises:

1. Formulate and prove the Borsuk Ulam theorem for continuous maps from S^1 to the real line.

2. Use the Borsuk Ulam theorem to prove that a pair of homogeneous polynomials of odd degree in three real variables have a common non-trivial zero.

3. For the following three maps $f : S^1 \longrightarrow S^1$ compute the induced map $f_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$. All three maps preserve the base point 1.

   (i) $f(z) = z^n$

   (ii) $f(z) = \bar{z}$.

   (iii) $f(z) = \frac{z^2 - z + \frac{3}{2}}{|z^2 - z + \frac{3}{2}|}$. Hint: Is $(z^2 - z)t + 3/2 = 0$ for any $z \in S^1$ and $0 \leq t \leq 1$?

4. Let X be the union of the sphere S^2 and one of its diameters. Use exercise 1 of lecture 8 to determine a generator for $\pi_1(X, x_0)$, where x_0 is a point on the sphere.

5. Determine the generators of the group $\pi_1(S^1 \times S^1, (1, 1))$. Determine the generators for the fundamental group of the space X of example 11.3.

6. Compute $f_* : \pi_1(\mathbb{C} - \{0\}, 1) \longrightarrow \pi_1(\mathbb{C} - \{0\}, 1)$ for the function $f(z) = z^k$.

Lecture XIV (Test - II)

1. Suppose X is a metric space and A is a retract of X. Show that A is closed in X. Is the space homeomorphic to the letter Y a deformation retract of a space homeomorphic to E^2?

2. Show that if X has the fixed point property and A is a retract of X then A also has the fixed point property.

3. Show that S^1 is not homeomorphic to any subset of $\mathbb{R}$. Can S^2 be homeomorphic to a subset of $\mathbb{R}^2$?

4. Determine $\pi_1(\mathbb{R}P^2 - \{p\})$ where p is any point of $\mathbb{R}P^2$. Show that $\pi_1(\mathbb{R}P^2 - \{p\})$ deformation retracts to a space homeomorphic to S^1. For this purpose you need the following fact:

   If $\eta : X \longrightarrow Y$ is a quotient map and Z is locally compact and Hausdorff then $\eta \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$ is also a quotient map.

5. For the map $f : S^1 \longrightarrow S^1 \times S^1$ given by $f(z) = (z^p, z^q)$, where p and q are positive integers, find the induced group homomorphism $f_* : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$.

Solutions to Test - II

1. Let $r : X \longrightarrow A$ be a retraction. To show that A is closed assume that $\{a_n\}$ is a sequence of points of A converging to a. We have to show that $a \in A$. Since $r(a_n) = a_n$ we have, by continuity, $r(a) = a$ and since $r(a) \in A$ we conclude that $a \in A$.

   As a concrete representation we take the letter Y to be the union of the three radii of S^1 terminating at the cube roots of unity namely

   $$J_0 = \{t \;/\; 0 \leq t \leq 1\}, \quad J_1 = \{t\omega \;/\; 0 \leq t \leq 1\}, \quad J_2 = \{t\omega^2 \;/\; 0 \leq t \leq 1\}$$

   and A be the sector between J_0 and J_1. We construct the retraction piece by piece beginning with A. It is convenient to use Tieze's extension theorem. First we define the obvious map $\lambda : [-1, 1] \longrightarrow J_0 \cup J_1$ given by

   $$\lambda(t) = \begin{cases} -t\omega & t \leq 0 \\ t & t \geq 0 \end{cases}$$

   Next, we define $g(t\omega) = -t$ and $g(t) = t$ for $0 \leq t \leq 1$. The function g is continuous on the closed subset $J_0 \cup J_1$ of A and so by Tietze's theorem extends continuously from $A \longrightarrow [-1, 1]$. Let $r = \lambda \circ g$ and we chech that $r(z) = z$ for all $z \in J_0 \cup J_1$. Similarly one can define r on the other two sectors and we have the desired retraction.

2. Suppose $r : \; X \longrightarrow A$ is a retraction and X has the fixed point property. Let $g : A \longrightarrow A$ be continuous. Then $i \circ g \circ r : X \longrightarrow X$ is continuous where $i : A \longrightarrow X$ is the inclusion map. By the fixed point property of X, there exists $p \in X$ such that

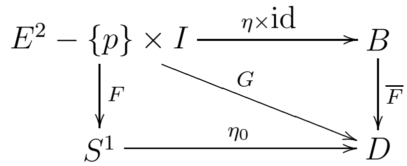
   $$i \circ g \circ r(p) = p.$$

   But $r(p) \in A$ and so $g(r(p)) \in A$ and consequently $p = i \circ g \circ r(p) \in A$. As a result $r(p) = p$ and the displayed equation simplifies to $g(p) = p$.

3. If S^1 is homeomorphic to a subset B of $\mathbb{R}$ then B is compact and connected and hence $B = [a, b]$ for some $a, b \in \mathbb{R}$ and $a < b$. However removal of a point from (a, b) disconnects $[a, b]$ but S^1 minus a point remains connected and hence it is impossible for S^1 and B to be homeomorphic. The second part was a result discussed in the lecture.

4. Recall (theorem (4.5) (iii)) that $\mathbb{R}P^2$ is obtained from E^2 by identifying pairs of antipodal points on the boundary. We take the point p to be the center of the disc E^2 and denote $X = \mathbb{R}P^2 - \{p\}$. Let $F : (E^2 - \{p\}) \times I \longrightarrow S^1$ be the deformation retraction onto the boundary S^1. Let $\eta : E^2 \longrightarrow \mathbb{R}P^2$ and $\eta_0 : S^1 \longrightarrow \mathbb{R}P^1 = S^1$ be the standard quotient maps. We also denote by η the restriction $\eta : E^2 - \{p\} \longrightarrow \mathbb{R}P^2 - \{p\}$ which is also a quotient map (why?).

We now have the commutative diagram where $G = \eta_0 \circ F$ and $\overline{F}$ is the unique map given by $\overline{F}(\overline{x}, t) = \eta_0 \circ F(x, t)$:



Since G is continuous and $\eta \times \mathrm{id}$ is a quotient map we have the continuity of $\overline{F}$ and $\overline{F}$ is the deformation retraction from X onto S^1.

5. Let g denote the generator for $\pi_1(S^1, 1)$ namely $g = [\gamma]$ where γ is the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1.$$

Then, by corollary (12.2) the map $S^1 \longrightarrow S^1$ given by $z \mapsto z^k$ induces on the fundamental group the homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by

$$x \mapsto kx,$$

where the group operation on $\mathbb{Z}$ is written additively whereas when we regard $\pi_1(S^1, 1)$ as an abstract group we shall denote the operation multiplicatively. Now,

$$f_*(g) = [f \circ \gamma] = \phi^{-1}(g^p, g^q),$$

where ϕ denotes the canonical isomorphism of theorem (9.6). The group $\pi_1(S^1 \times S^1)$ is identified as $\mathbb{Z} \times \mathbb{Z}$ via the isomorphism ϕ and we see that the map f_* regarded as a map $\mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ is given by

$$m \mapsto (mp, mq).$$

Lecture XV - Covering Projections

The theory of covering projections sets a common stage for the development of diverse branches of mathematics. In this course we develop the theory of covering projections only to the extent that is relevant for the computation of the fundamental group. It may be useful for the student to review the proof that $\pi_1(S^1) = \mathbb{Z}$. In fact one of the paradigms for a covering projection is the map

$$t \mapsto \exp(2\pi i t)$$

wrapping the real line onto the circle.

Definition 15.1: A covering projection is a triple $(\tilde{X}, X, p)$ where $\tilde{X}$, X are connected topological spaces and a continuous map $p : \tilde{X} \longrightarrow X$ satisfying the following properties:

(i) The map p is surjective

(ii) Each $x \in X$ has a neighborhood U such that the inverse image $p^{-1}(U)$ is a disjoint union of a collection open subsets $\{U_\alpha\}$ of $\tilde{X}$.

(iii) Each U_α is mapped onto U homeomorphically by p.

The neighborhood U described in the definition above is called an *evenly covered neighborhood of x,* the open sets U_α are referred to as *sheets* lying above U and for $x \in X$, the subset $p^{-1}(x)$ of $\tilde{X}$ is called the *fiber* over x. This terminology will be used frequently. We shall also say that $\tilde{X}$ is a covering space of X when it is fairly clear what the map p is.

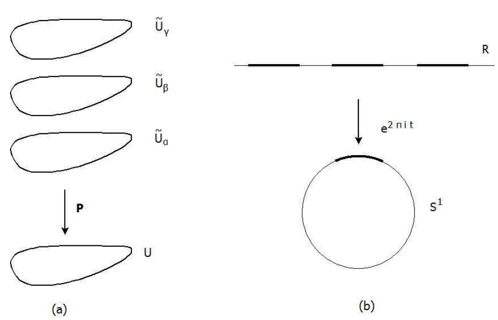


Figure 12: Covering projection

Remark: It is NOT sufficient that each U_α be homeomorphic to U but the homeomorphism must be given by the restricting p to U_α.

Examples 15.1: We now present four examples to illustrate the concept of a covering projection.

1. As indicated in the beginning the most basic example is the map $\mathrm{ex} : \mathbb{R} \longrightarrow S^1$ given by

$$\mathrm{ex}(t) = \exp(2\pi i t)$$

For each point z on the circle take an arc U centered at z and of length say $\pi/2$. The reader may check that the inverse image of U under ex is a disjoint union of open intervals on the line.

2. Consider the map $T : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ given by $T(z) = z^2$. If we pick a point $z \in \mathbb{C} - \{0\}$ and a small disc U centered at z not containing a pair of antipodal points then $T^{-1}(U)$ is a disjoint union of two open sets each of which is mapped bijectively onto U by T.

3. Consider the map $p : \mathbb{C} - \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} - \{\pm 2\}$ given by

$$p(z) = z^3 - 3z$$

The equation $p(z) = w$ has three distinct roots for each $w \in \mathbb{C} - \{\pm 2\}$ and the roots are continuous functions of w. For a sufficiently small neighborhood U of w, $p^{-1}(U)$ is a disjoint union of three open sets each of which is mapped onto U homeomorphically onto U by the open mapping theorem. Several examples of this type related to complex analysis are discussed in [6].

4. Consider the quotient map $\eta : S^n \longrightarrow \mathbb{R}P^n$. We show that η is a covering projection. Let U_1 be an open subset of S^n not containing a pair of anti-podal points and

$$U_2 = \{-x / x \in U_1\}.$$

Then, $\eta(U_1) = \eta(U_2)$. Denoting these images by U, we see that $\eta^{-1}(U) = U_1 \cup U_2$ which is an open set in S^n and so U is open in $\mathbb{R}P^n$. Second, η maps each of U_1 and U_2 bijectively onto U. To see that η maps each of U_1 and U_2 homeomorphically onto U, we merely have to show that η is an open mapping. So let V_1 be an open subset of U_1 and $V_2 = \{-x/x \in V_1\}$. Then

$$\eta^{-1}(\eta(V_1)) = V_1 \cup V_2$$

is open in S^n so that $\eta(V_1)$ is an open subset of $\mathbb{R}P^n$. Thus we have shown that η restricted to each U_j is an open mapping and that suffices for a proof.

We now summarize the most basic properties of covering projections.

Theorem 15.1: Suppose that $p : \tilde{X} \longrightarrow X$ is a covering projection. Then

(i) The map p is a local homeomorphism (see exercise 7, lecture 3).

(ii) The function p is an open mapping.

(iii) The fibers $p^{-1}(x)$ are discrete for each $x \in X$.

Proof: Let $\tilde{x} \in \tilde{X}$ be arbitrary and $x = p(\tilde{x})$. Choose an evenly covered neighborhood U of x and $\tilde{U}$ be a sheet lying over U and containing $\tilde{x}$. Then $\tilde{U}$ is an open set in $\tilde{X}$ containing $\tilde{x}$ that is mapped by p homeomorphically onto U. Thus p is a local homeomorphism and we have proved (i). Let $\tilde{G}$ be an arbitrary open set in $\tilde{X}$. Then $\tilde{G}$ can be covered by open sets $\tilde{U}$ such that p maps each $\tilde{U}$ homeomorphically onto an evenly covered open subset U of X (why?). Then $G = p(\tilde{G})$ is the union of such evenly covered neighborhoods U implying that G is an open set in X. Thus p is an open mapping. Finally to prove (iii) suppose that $\tilde{z} \in \tilde{X}$ is a limit point of $p^{-1}(x)$. Pick an arbitrary evenly covered neighborhood U of $z = p(\tilde{z})$ and a sheet $\tilde{U}$ lying over U containing $\tilde{z}$. In particular the restriction of p to the sheet $\tilde{U}$ is injective. But since $\tilde{z}$ is a limit point of $p^{-1}(x)$, this sheet must contain infinitely many points of $p^{-1}(x)$ which means p restricted to $\tilde{U}$ cannot be injective which is a contradiction.

The lifting problem: Suppose that $p : E \longrightarrow B$ is a surjective continuous map between topological spaces and $f : T \longrightarrow B$ is a given continuous map, a lift of f is by definition a continuous map $\tilde{f} : T \longrightarrow E$ such that

$$p \circ \tilde{f} = f$$

The lifting problem involves giving sufficient conditions for the existence of the lift $\tilde{f}$. The main point here is of course the continuity of the lift. The significance of the problem can be understood from complex analysis.

Example 15.2: Consider the exponential map $\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$ and an open set $\Omega \subset \mathbb{C}$ and the inclusion map

$$j : \Omega \longrightarrow \mathbb{C} - \{0\}.$$

To say that the inclusion map j has a lift with respect to the exponential map means the existence of a continuous $\tilde{j} : \Omega \longrightarrow \mathbb{C}$ such that

$$\exp(\tilde{j}(z)) = z, \quad \text{for all } z \in \Omega.$$

In other words the existence of a lift of j is equivalent to the existence of a continuous branch of the logarithm on Ω. We know from complex variable theory that such a continuous branch need not exist in general such as for instance the case $\Omega = \mathbb{C} - \{0\}$.

In place of the exponential map we could consider the map $S : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ given by $S(z) = z^2$. The problem of lifting the inclusion map of a domain $\Omega \subset \mathbb{C} - \{0\}$ is then equivalent to the existence of a continuous branch of the square root function on Ω. We also know from complex analysis that if the lift exists it need not be unique. Well, if a domain $\Omega \subset \mathbb{C} - \{0\}$ admits a continuous branch of the square root then it admits two branches. If it admits a continuous branch of the logarithm then it admits infinitely many any two of which differ by an integer multiple of $2\pi i$. On a connected domain, the branch however is uniquely specified by specifying a value at a point of the domain. The following theorem generalizes this in the context of covering spaces.

Theorem 15.2 (uniqueness of lifts): Suppose $p : \tilde{X} \longrightarrow X$ is a covering projection, T is a connected topological space and $f_1 : T \longrightarrow \tilde{X}$ and $f_2 : T \longrightarrow \tilde{X}$ are two lifts of a given continuous map $f : T \longrightarrow X$ such that $f_1(t_0) = f_2(t_0)$ for some $t_0 \in T$. Then the two lifts agree on T namely, $f_1(t) = f_2(t)$ for all $t \in T$.

Proof: Let G be the subset given by $G = \{t \in T / f_1(t) = f_2(t)\}$. The set G is non-empty since $t_0 \in G$. We shall show that G is both open and closed in T from which the result would follow since T is connected. For $t \in G$ pick an evenly covered neighborhood U of

$$x = p(f_1(t)) = p(f_2(t)).$$

and $\tilde{U}$ be the sheet lying over U and containing $f_1(t) = f_2(t)$. The set

$$N = f_1^{-1}(\tilde{U}) \cap f_2^{-1}(\tilde{U})$$

is open and contains t. If $z \in N$ then $f_1(z)$ and $f_2(z)$ both belong to $\tilde{U}$ and $p(f_1(z)) = p(f_2(z)) = f(z)$. But p restricted to $\tilde{U}$ is injective and so $f_1(z) = f_2(z)$ for all $z \in N$ and we conclude that $N \subset G$. The proof that G is closed is left as an exercise. The student may assume that the spaces involved are Hausdorff (see exercise 7 of lecture 2).

Exercises:

1. Explain why the map $\phi : \mathbb{C} - \{0, 1/2\} \longrightarrow \mathbb{C} - \{-1/4\}$ given by $\phi(z) = z(z-1)$ is not a covering projection?

2. Show that the map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^k$ is a covering projection for every $k \in \mathbb{N}$.

3. Suppose $p : \tilde{X} \longrightarrow X$ is a covering projection and E is a closed subset of X. Is the map

$$p : \tilde{X} - p^{-1}(E) \longrightarrow X - E$$

   a covering projection?

4. Find a discrete subset E of $\mathbb{C}$ such that $\sin : \mathbb{C} - E \longrightarrow \mathbb{C} - \{-1, 1\}$ is a covering projection.

5. Suppose that $p : \tilde{X} \longrightarrow X$ and $q : \tilde{Y} \longrightarrow Y$ are covering projections then the product map $(p, q) : \tilde{X} \times \tilde{Y} \longrightarrow X \times Y$ given by

$$(p, q)(z, w) = (p(z), q(w)), \quad z \in \tilde{X}, w \in \tilde{Y},$$

   is a covering projection. In particular the plane $\mathbb{R}^2$ is a covering space of the torus $S^1 \times S^1$.

6. Let Y be the infinite grid

$$Y = \{(x, y) \in \mathbb{R}^2 / x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$$

   is a covering projection of the figure eight loop. Draw the figure eight loop on the torus.

7. Show that the set G in theorem (15.2) is closed without using the Hausdorff assumption on T.

Lecture XVI - Lifting of paths and homotopies

In the last lecture we discussed the lifting problem and proved that the lift if it exists is uniquely determined by its value at one point. In this lecture we shall prove the important result that covering projections enjoy the path lifting and covering homotopy properties. This theorem is fundamental in the the theory of covering projections and will be used in the next lecture to define an action of the fundamental group on the fibers.

Theorem 16.1 (path lifting lemma): Let $p : \tilde{X} \longrightarrow X$ be a covering projection and $\gamma : [0,1] \longrightarrow X$ be a path such that for some $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$,

$$\gamma(0) = x_0 = p(\tilde{x}_0). \tag{16.1}$$

Then there exists a unique path $\tilde{\gamma} : [0,1] \longrightarrow \tilde{X}$ such that

$$p \circ \tilde{\gamma} = \gamma, \quad \tilde{\gamma}(0) = \tilde{x}_0 \tag{16.2}$$

Thus each path in X lifts to a unique path in $\tilde{X}$ with a prescribed initial point in $p^{-1}(\gamma(0))$.

Proof: Let $\mathcal{O}$ be the open cover of X by evenly covered open sets and $\gamma^{-1}(\mathcal{O})$ be the family

$$\gamma^{-1}(\mathcal{O}) = \{\gamma^{-1}(G)/G \in \mathcal{O}\}$$

of open sets covering $[0,1]$. There is a Lebesgue number η for this cover and we choose n to be a natural number such that $1/n < \eta$. Consider the partition

$$\Big\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\Big\}.$$

For each $j = 1, 2, \dots, n$, the piece $\gamma([\frac{j-1}{n}, \frac{j}{n}])$ lies in an evenly covered open set in X. In particular if γ_0 denotes the restriction of γ to $[0, 1/n]$ then the image of γ_0 lies in an open set $G_0 \in \mathcal{O}$. The conditions

(16.1)-(16.2) say that there is a sheet $\tilde{G}_0$ lying over G_0 and containing the point $\tilde{x}_0$. Let p_0 denote the restriction of p to the sheet $\tilde{G}_0$ and q_0^{-1} be its inverse. On the sub-interval $[0, 1/n]$, we define

$$\tilde{\gamma}_0 = q_0 \circ \gamma_0$$

thereby obtaining an initial piece of the desired lift $\tilde{\gamma}$. We shall construct the lift $\tilde{\gamma}$ piece by piece defining it on each subinterval of the partition of $[0, 1]$. In what follows γ_j denotes the restriction of γ to the sub-interval $[\frac{j}{n}, \frac{j+1}{n}]$. Assume inductively that

$$\tilde{\gamma}_j : [\frac{j}{n}, \frac{j+1}{n}] \longrightarrow \tilde{X}$$

has been defined such that

$$\begin{aligned} p \circ \tilde{\gamma}_j &= \gamma_j \\ \tilde{\gamma}_j(j/n) &= \tilde{\gamma}_{j-1}(j/n), \qquad \text{in case } j \geq 1. \\ \tilde{\gamma}_0(0) &= \tilde{x}_0 \end{aligned}$$

For the inductive step we set up the notations for the endpoints of the lift $\tilde{\gamma}_j$ namely, let

$$\gamma_j\Big(\frac{j+1}{n}\Big) = x_{j+1}, \quad \tilde{\gamma}_j\Big(\frac{j+1}{n}\Big) = \tilde{x}_{j+1}, \quad p(\tilde{x}_{j+1}) = x_{j+1}.$$

Let $G_{j+1} \in \mathcal{O}$ be an evenly covered neighborhood containing x_{j+1} such that γ maps $[\frac{j+1}{n}, \frac{j+2}{n}]$ into G_{j+1} and $\tilde{G}_{j+1}$ be the sheet lying over G_{j+1} containing the point $\tilde{x}_{j+1}$. The restriction of p to $\tilde{G}_{j+1}$ is a homeomorphism with inverse q_{j+1} say, so that $q_{j+1}(x_{j+1}) = \tilde{x}_{j+1}$. We set

$$\tilde{\gamma}_{j+1} = q_{j+1} \circ \gamma_{j+1}$$

Then $\tilde{\gamma}_{j+1}$ is continuous, $p \circ \tilde{\gamma}_{j+1} = \gamma_{j+1}$ and

$$\tilde{\gamma}_{j+1}\Big(\frac{j+1}{n}\Big) = q_{j+1}(x_{j+1}) = \tilde{x}_{j+1} = \tilde{\gamma}_j\Big(\frac{j+1}{n}\Big)$$

By gluing lemma, the pieces $\tilde{\gamma}_j$ may be glued together to yield a continuous function $\tilde{\gamma} : [0, 1] \longrightarrow \tilde{X}$ such that

$$p \circ \tilde{\gamma} = \gamma, \quad \tilde{\gamma}(0) = \tilde{x}_0.$$

The proof is complete. The uniqueness has been already proved in general.

Lifting of homotopies: We now examine what happens when we lift homotopic paths with the lifts having the same initial points.

Theorem 16.2 (Covering homotopy property): Let $p : \tilde{X} \longrightarrow X$ be a covering projection and $\tilde{x}_0 \in \tilde{X}, x_0 \in X$ be chosen base points such that $p(\tilde{x}_0) = x_0$. Let γ_1, γ_2 be two curves in X starting at x_0 and having the same terminal points and $F : [0, 1] \times [0, 1] \longrightarrow X$ be a homotopy between γ_1 and γ_2. There is a unique lift $\tilde{F} : [0, 1] \times [0, 1] \longrightarrow \tilde{X}$ of F such that $\tilde{F}(0, 0) = \tilde{x}_0$. In particular the unique lifts of γ_1 and γ_2 starting at $\tilde{x}_0$ have the same terminal points.

Proof: The idea behind the proof is simple and parallels the proof of the previous theorem except that the book-keeping gets a bit more involved. Consider a covering $\mathcal{O}$ of X by evenly covered open neighborhoods and choose a Lebesgue number ϵ for the covering

$$\{F^{-1}(U)/U \in \mathcal{O}\}. \tag{16.3}$$

Choose n so large that any square in $[0,1] \times [0,1]$ of side $1/n$ is contained in one of the sets $F^{-1}(U)$ in (16.3). Partition $[0,1] \times [0,1]$ using the grid points

$$\Big\{\Big(\frac{j}{n}, \frac{k}{n}\Big)/0 \leq j \leq n,\ 0 \leq k \leq n\Big\}$$

and $S_{j,k}$ be the square with vertices

$$\Big(\frac{j}{n}, \frac{k}{n}\Big),\ \Big(\frac{j+1}{n}, \frac{k}{n}\Big),\ \Big(\frac{j+1}{n}, \frac{k+1}{n}\Big),\ \Big(\frac{j}{n}, \frac{k+1}{n}\Big).$$

Let $U_{0,0}$ be an evenly covered neighborhood in X such that $F(S_{0,0}) \subset U_{0,0}$ and $\tilde{U}_{0,0}$ be the sheet in

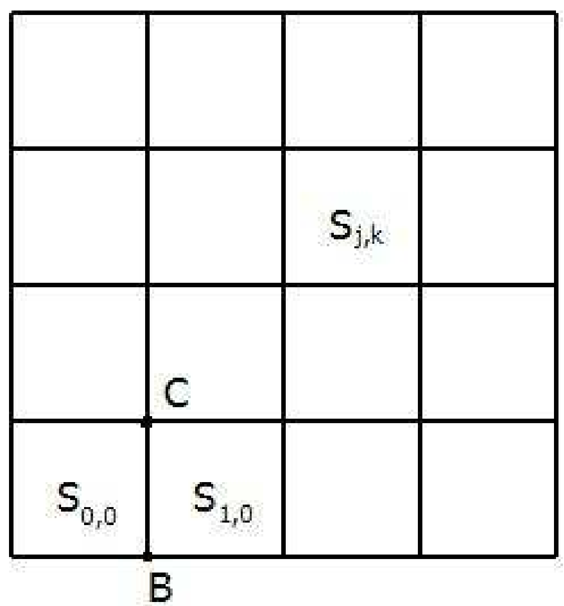


Figure 13: Homotopy lifting property

$\tilde{X}$ lying above $U_{0,0}$. Denoting by $p_{0,0}$ and $F_{0,0}$ the restrictions of p and F to $\tilde{U}_{0,0}$ and $S_{0,0}$ respectively, define

$$\tilde{F}_{0,0} = p_{0,0}^{-1} \circ F.$$

Thus $\tilde{F}_{0,0} : S_{0,0} \longrightarrow \tilde{X}$ is continuous, takes the value $\tilde{x}_0$ at the origin and is a part of the lift $\tilde{F}$ under construction. As in the previous theorem we shall construct the lift $\tilde{F}$ piece by piece and we now turn to the adjacent square $S_{1,0}$ which is mapped by F to an evenly covered neighborhood $U_{1,0}$ in the cover $\mathcal{O}$. In particular (referring to the figure) $F(B) \in U_{1,0}$. Choose a sheet $\tilde{U}_{1,0}$ lying above $U_{1,0}$ containing $\tilde{F}(B)$ and the restriction

$$p_{1,0} = p\Big|_{\tilde{U}_{1,0}}$$

maps $\tilde{U}_{1,0}$ homeomorphically onto $U_{1,0}$. Now we define the next piece of the lift $\tilde{F}_{1,0}$ as

$$\tilde{F}_{1,0} = p_{1,0}^{-1} \circ F$$

which is continuous on the square $S_{1,0}$ and

$$p \circ \tilde{F}_{1,0} = F\Big|_{S_{1,0}}$$

In order to glue together the pieces $\tilde{F}_{0,0}$ and $\tilde{F}_{1,0}$ we must ensure that they agree all along the common edge BC of the adjacent squares $S_{0,0}$ and $S_{1,0}$. Their restrictions along BC where $t = 0$ and $0 \leq s \leq 1/n$ agree at B namely

$$\tilde{F}_{0,0}(0, \frac{1}{n}) = \tilde{F}_{1,0}(0, \frac{1}{n})$$

and are both lifts of the map

$$s \mapsto F(s, \frac{1}{n}), \quad 0 \leq s \leq \frac{1}{n}$$

which implies, by uniqueness of lifts,

$$\tilde{F}_{0,0}(s, \frac{1}{n}) = \tilde{F}_{1,0}(s, \frac{1}{n}), \quad 0 \leq s \leq \frac{1}{n},$$

as desired. It is now clear how the construction ought to proceed and we get a lift $\tilde{F} : [0,1] \times [0,1] \longrightarrow \tilde{X}$ of F.

We now have to check that $\tilde{F}$ is indeed a homotopy of paths with fixed endpoints. Well,

$$p \circ \tilde{F}(s, 0) = F(s, 0) = x_0, \text{ for all } s \in [0,1]$$

so that the connected set

$$\{\tilde{F}(s, 0)/0 \leq s \leq 1\}$$

is contained in the discrete set $p^{-1}(x_0)$ and so must reduce to a singleton. Likewise $\tilde{F}(s, 1)$ is constant as s varies over $[0,1]$. Also $p \circ \tilde{F}(0,t) = F(0,t) = \gamma_1(t)$ and $p \circ \tilde{F}(1,t) = F(1,t) = \gamma_2(t)$ showing that $\tilde{F}$ is the desired homotopy between the lifts of γ_1 and γ_2 starting at $\tilde{x}_0$. □

Theorem 16.3: Given a covering projection $p : \tilde{X} \longrightarrow X$, for any $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ the induced group homomorphism

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$$

is injective.

Proof: Let $\tilde{\gamma}$ be a loop in $\tilde{X}$ based at $\tilde{x}_0$ that represents an element of ker p_*. This means the loop $\gamma = p \circ \tilde{\gamma}$ is homotopic to the constant loop in X based at x_0. But the constant loop ε_{x_0} at x_0 lifts as the constant loop $\varepsilon \tilde{x}_0$ at $\tilde{x}_0 \in \tilde{X}$. By the covering homotopy theorem we conclude that $\tilde{\gamma}$ and the constant loop $\varepsilon_{\tilde{x}_0}$ are homotopic. That is to say $[\tilde{\gamma}]$ is the trivial element in $\pi_1(\tilde{X}, \tilde{x}_0)$. □

Remark: The above theorem enables us to identify $\pi_1(\tilde{X}, \tilde{x}_0)$ as a subgroup of $\pi_1(X, x_0)$.

We shall now discuss another important consequence of the path lifting property.

Theorem 16.4: Given a covering projection $p : \tilde{X} \longrightarrow X$ where X and $\tilde{X}$ are path-connected, for any points $x_1, x_2 \in X$ the fibers $p^{-1}(x_1)$ and $p^{-1}(x_2)$ have the same cardinality.

Proof: We shall construct injective maps from $p^{-1}(x_1)$ into $p^{-1}(x_2)$ and vice versa. Fix a path γ in X joining x_1 and x_2. Pick $\tilde{x}_1 \in p^{-1}(x_1)$ and let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{x}_1$ and define a map $T : p^{-1}(x_1) \longrightarrow p^{-1}(x_2)$ by the prescription

$$T : \tilde{x}_1 \mapsto \tilde{\gamma}(1).$$

Likewise let $S : p^{-1}(x_2) \longrightarrow p^{-1}(x_1)$ be the map in the reverse direction constructed using the path γ^{-1}. Since the inverse path $\tilde{\gamma}^{-1}$ is the unique lift of γ^{-1} starting at $\tilde{\gamma}(1)$, we see that

$$S(\tilde{\gamma}(1)) = \tilde{\gamma}(0) = \tilde{x}_1,$$

whereby we conclude $S \circ T$ is the identity map on $p^{-1}(x_1)$. By symmetry $T \circ S$ is the identity map on $p^{-1}(x_2)$ as desired. □

Exercises

1. Use the general results of this section to give an efficient and transparent proof that $\pi_1(S^1, 1) = \mathbb{Z}$. First show that for any loop γ based at 1, the map $\pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ given by $[\gamma] \mapsto \tilde{\gamma}(1)$ is well defined by theorem 16.1, is a group homomorphism using uniqueness of lifts. Show that surjectivity follows from uniqueness of lifts and injectivity follows from theorem 16.1.

2. Let X be a topological spaces and $a, b \in X$. A simple chain connecting a and b is a finite sequence $U_1, U_2, \dots, U_n$ of open sets such that $a \in U_1$, $b \in U_n$ and for $1 \leq i < j \leq n$, $U_i \cap U_j \neq \emptyset$ implies $j = i + 1$. Show that if X is a connected metric space and $\mathcal{U}$ is an open covering of X then any

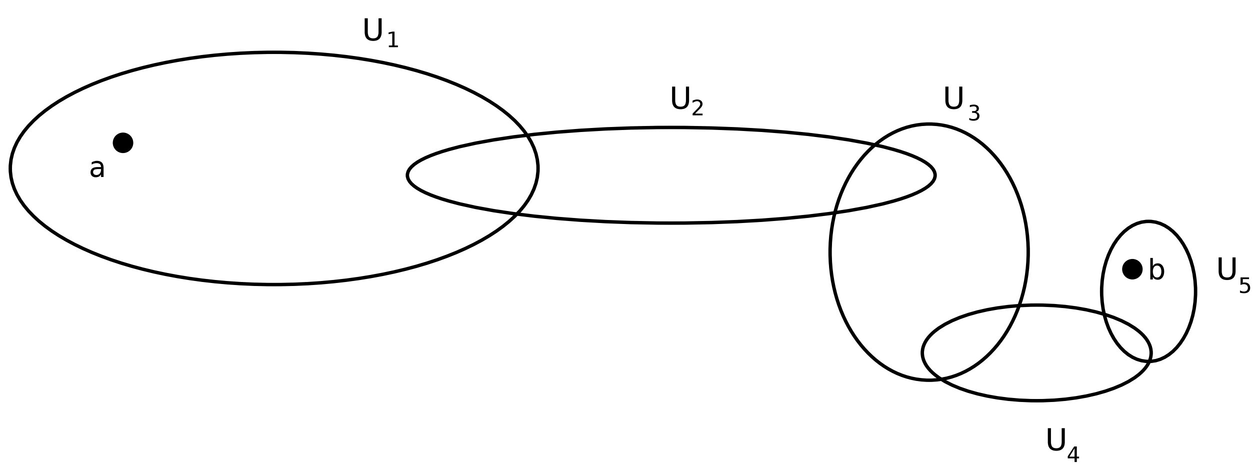


Figure 14: Chain connectedness

two points $a, b \in X$ can be connected by a simple chain. This property is referred to as chain connectedness. Is $\mathbb{Q}$ chain connected?

3. Use the above exercise to show that if X is a chain-connected space and $p : \tilde{X} \longrightarrow X$ is a covering projection then for any pair of points $x, y \in X$ the fibers $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality. The point here is that X need not be path connected and the idea of using a path joining x and y as was done in the proof of theorem 14.4 is no longer available.

4. A toral knot is a group homomorphism $\kappa : S^1 \longrightarrow S^1 \times S^1$ given by $z \mapsto (z^m, z^n)$ where $m, n \in \mathbb{N}$. Regarding the toral knot as a loop on the torus determine its lifts with respect to the covering projection $\mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1$.

5. For the group homomorphism κ of the previous exercise describe the induced map κ_*.

Lecture XVII - Action of $\pi_1(X, x_0)$ on the fibers $p^{-1}(x_0)$

Given a covering projection $p : \tilde{X} \longrightarrow X$, the lifting lemma would imply that the fundamental group of the base space X acts naturally on the fibers $p^{-1}(x_0)$ ($x_0 \in X$). We define this action and examine its basic properties such as its transitivity. The action provides a great deal of information about the fundamental group $\pi_1(X, x_0)$ and this is the primary application of the theory of covering spaces in this course.

Definition 17.1: Let $p : \tilde{X} \longrightarrow X$ be a covering projection and $x_0 \in X$ be a given point. For a loop γ in X based at x_0, define the right-action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ as follows. For $\tilde{x}_1 \in p^{-1}(x_0)$,

$$\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}(1), \tag{17.1}$$

where $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_1$.

Theorem 17.1: The prescription (17.1) defines a right action of the fundamental group $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

Proof: We first show that the action is well-defined. That is to say if γ_1 and γ_2 are homotopic loops based at x_0 then for $\tilde{x} \in p^{-1}(x_0)$

$$\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1),$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are lifts of γ_1 and γ_2 starting at $\tilde{x}$. Well, if F is the homotopy between γ_1 and γ_2 then F has a unique lift $\tilde{F}$ satisfying $\tilde{F}(0,0) = \tilde{x}$. In other words, $\tilde{F} : [0,1] \times [0,1] \longrightarrow \tilde{X}$ is the unique continuous map such that

$$p \circ \tilde{F} = F, \quad \tilde{F}(0,0) = \tilde{x}$$

In particular the image set $\{\tilde{F}(s,1)\}$ as s runs through $[0,1]$, must be a connected subset of $\tilde{X}$. But since F is a homotopy of loops based at x_0,

$$F(s,1) = p \circ \tilde{F}(s,1) = x_0, \quad \text{for all } s \in [0,1].$$

Hence $\{\tilde{F}(s,1)/s \in [0,1]\} \subset p^{-1}(x_0)$ which means $\{\tilde{F}(s,1)/s \in [0,1]\}$ is a singleton since $p^{-1}(x_0)$ is discrete. In particular,

$$\tilde{F}(0,1) = \tilde{F}(1,1), \quad \text{that is,} \quad \tilde{\gamma}_1(1) = \tilde{\gamma}_2(1).$$

Next, we show that (15.1) defines a right group action. First let us note that if $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ are three points in $p^{-1}(x_0)$ and $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ is a pair of paths joining $\tilde{x}_1$ to $\tilde{x}_2$ and $\tilde{x}_2$ to $\tilde{x}_3$ respectively then

$$p \circ (\tilde{\gamma}_1 * \tilde{\gamma}_2) = (p \circ \tilde{\gamma}_1) * (p \circ \tilde{\gamma}_2).$$

Now let γ_1 and γ_2 be two loops in X based at x_0. Assume that $\tilde{\gamma}_1$ is the unique lift of γ_1 starting at $\tilde{x}_1$ and $\tilde{\gamma}_2$ is the unique lift of γ_2 starting at the point $\tilde{x}_2 = \tilde{\gamma}_1(1)$ then the juxtaposition $\tilde{\gamma}_1 * \tilde{\gamma}_2$ is defined and is the unique lift of $\gamma_1 * \gamma_2$ starting at $\tilde{x}_1$. Thus,

$$\tilde{x}_1 \cdot ([\gamma_1][\gamma_2]) = \tilde{x}_1 \cdot [\gamma_1 * \gamma_2] = \tilde{\gamma}_1 * \tilde{\gamma}_2(1) = \tilde{\gamma}_2(1)$$

On the other hand,

$$\tilde{\gamma}_2(1) = \tilde{x}_2 \cdot [\gamma_2] = (\tilde{x}_1 \cdot [\gamma_1]) \cdot [\gamma_2].$$

Note that if we had tried to operate from the left we would instead get an anti-action. This is one of the instances where it is important to have the book-keeping done correctly from the very outset.

Finally the constant loop ε_{x_0} at x_0 lifts as the constant loop starting at $\tilde{x}_1 \in p^{-1}(x_0)$ and so (17.1) implies

$$\tilde{x}_1 \cdot [\varepsilon_{x_0}] = \tilde{x}_1, \quad \tilde{x}_1 \in p^{-1}(x_0).$$

We now examine the issues related to this group action namely, its transitivity and the stabilizer subgroups of various points of $p^{-1}(x_0)$.

Theorem 17.2: (i) The group action defined in theorem (17.1) is transitive.

(ii) For $x_0 \in X$ and each $\tilde{x} \in p^{-1}(x_0)$, the stabilizer of $\tilde{x}$ is the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}))$.

(iii) The family $\{p_*(\pi_1(\tilde{X}, \tilde{x}))/\tilde{x} \in p^{-1}(x_0)\}$ forms a complete conjugacy class of subgroups of $\pi_1(X, x_0)$.

(iv) $|p^{-1}(x_0)| = [\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$

Proofs: Statement (iv) follows from (ii). Assertion (iii) is a general fact about transitive group actions. To prove that the group action is transitive, let $\tilde{x}_1$ and $\tilde{x}_2$ be two points in the fiber $p^{-1}(x_0)$ and $\tilde{\gamma}$ be a path in $\tilde{X}$ joining $\tilde{x}_1$ and $\tilde{x}_2$. The image path $\gamma = p \circ \tilde{\gamma}$ is then a loop in X based at x_0 and so represents an element of $\pi_1(X, x_0)$. Also $\tilde{\gamma}$ being the lift of γ starting at $\tilde{x}_1$, we see that

$$\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}(1) = \tilde{x}_2.$$

Turning now to the proof of (ii), let $\tilde{x}_0 \in p^{-1}(x_0)$ and γ be an arbitrary loop in X based at x_0. Then $[\gamma]$ belongs to the stabilizer of $\tilde{x}_0$ if and only if its lift starting at $\tilde{x}_0$ terminates at the same point $\tilde{x}_0$. That is if and only if γ lifts as a loop based at $\tilde{x}_0$. But this is equivalent to saying $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ and $\gamma = p_*[\tilde{\gamma}]$. Conversely if $[\gamma] \in \pi_1(X, x_0)$ is the image under p_* of $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ then $\tilde{\gamma}$ is a loop homotopic to a lift of γ starting at $\tilde{x}_0$. But any two such lifts have the same terminal point which means

$$\tilde{x}_0 \cdot [\gamma] = \tilde{\gamma}(1) = \tilde{x}_0.$$

That is to say $[\gamma]$ belongs to the stabilizer of $\tilde{x}_0$ and that completes the proof.

Corollary 17.3: The fundamental group of $\mathbb{R}P^n$ $(n \geq 2)$ is the cyclic group of order two.

Proof: We know that the fundamental group of S^n is the trivial group and the standard quotient map $\eta : S^n \longrightarrow \mathbb{R}P^n$ is a covering projection. So from (iv) of the preceding theorem we get

$$|\eta^{-1}(x_0)| = 2 = [\pi_1(\mathbb{R}P^n, x_0) : \eta_*(\pi_1(S^n, \tilde{x}_0))]$$

From which follows that

$$|\pi_1(\mathbb{R}P^n, x_0)| = 2$$

and that completes the proof.

Regular coverings: These are coverings that exhibit symmetry. The precise meaning will be clarified in theorem (19.4) below and theorem (19.2) in lecture 19.

Theorem 17.4: For a covering projection $p : \tilde{X} \longrightarrow X$ with path connected X and $\tilde{X}$ the following are equivalent:

(i) The subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$, where $p(\tilde{x}_0) = x_0$.

(ii) For any loop in X based at x_0, either all its lifts are closed loops or none of the lifts is closed.

Proof: We begin with the observation that the condition spelled out in (i) is independent of the choice of x_0 and also independent of the choice of $\tilde{x}_0 \in p^{-1}(x_0)$. Well, changing the element $\tilde{x}_0$ in the fiber would give a conjugate subgroup but the normality hypothesis says that the conjugacy class of the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a singleton. Second, a group isomorphism must take a normal subgroup to a normal subgroup and so the condition (i) does not depend on the choice of the base point x_0.

Proof that (i) implies (ii). Suppose that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$ and γ is an arbitrary loop in X based at x_0. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two lifts of γ with initial points $\tilde{x}_1, \tilde{x}_2$ such that $\tilde{\gamma}_1$ a closed loop. Then

$$\tilde{x}_1 \cdot [\gamma] = \tilde{x}_1$$

which means $[\gamma]$ is in the stabilizer of $\tilde{x}_1$ and hence, by (iii) of theorem (17.2), $[\gamma]$ belongs to the stabilizer of $\tilde{x}_2$. Thus

$$\tilde{x}_2 \cdot [\gamma] = \tilde{\gamma}_2(1) = \tilde{x}_2$$

and we see that $\tilde{\gamma}_2$ is also closed.

Proof that (ii) implies (i). If $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is not a normal subgroup of $\pi_1(X, x_0)$ then it has at least two distinct conjugates which, by virtue of theorem (17.2), must be the stabilizers of say $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. Thus there exists $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = \text{Stab}\ \tilde{x}_1$ but $[\gamma] \notin p_*(\pi_1(\tilde{X}, \tilde{x}_2)) = \text{Stab}\ \tilde{x}_2$. In other words

$$\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}_1(1) = \tilde{x}_1, \quad \tilde{x}_2 \cdot [\gamma] = \tilde{\gamma}_2(1) \neq \tilde{x}_2,$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the lifts of γ starting at $\tilde{x}_1$ and $\tilde{x}_2$ respectively. Thus the lift $\tilde{\gamma}_1$ of γ is closed whereas the lift $\tilde{\gamma}_2$ is not closed.

Definition 17.2: A covering projection $p : \tilde{X} \longrightarrow X$ with path connected X and $\tilde{X}$ is said to be regular if it satisfies one of the equivalent conditions stated in theorem (15.4).

Corollary 17.5: If $\pi_1(X, x_0)$ is abelian then every covering of X is regular.

To construct an example of a non-regular covering we need a space with non-abelian fundamental group. We shall see an example in lecture 19 (exercise 3).

Exercises

1. Describe a path in S^n whose image under the standard map represents the generator of $\pi_1(\mathbb{R}P^n, x_0)$.

2. Let C_0 be the unit circle in the complex plane and $\omega_1, \omega_2, \ldots, \omega_n$ denote the $n-$th roots of unity and at each of these a circle C_j of small radius touches the unit circle externally. Construct a continuous map p from the union of these $n+1$ circles onto the figure eight loop such that p is a regular covering. Hint: Take one lobe of the figure eight to be the unit circle C_0 and define $p(z) = z^n$ for $z \in C_0$. Let L be the other lobe of figure eight touching the lobe C_0 at say the point 1. For each j let $p_j : C_j \longrightarrow L$ be any homeomorphism such that $p_j(\omega_j) = 1$. Use gluing lemma to glue these maps to obtain the desired covering.

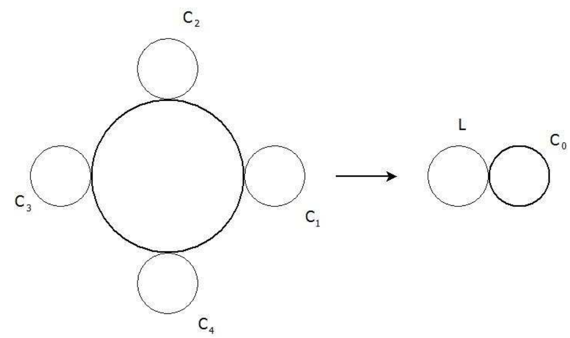


Figure 15: Covering of the figure eight loop

3. For the covering projection of the preceding exercise determine the action of the fundamental group of the base on a fiber assuming that the loops C_0 and L (based at 1) generate the fundamental group.

4. Consider the covering projection of exercise 6, lecture 15. Show by studying the lifts of various loops based at $(1, 1)$ that the covering is regular. We shall see another proof of regularity of this covering in lecture 19.

5. For the covering considered in the preceding exercise, determine the lifts of the loops

$$\gamma_1 : t \mapsto 1 - \exp(2\pi i t), \quad \gamma_2 : t \mapsto -1 + \exp(2\pi i t).$$

Find the lift of $\gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}$ and deduce that the fundamental group of the figure eight space is non-abelian.

6. Show that the figure eight loop $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ is not a retract of the torus $S^1 \times S^1$. Show that the figure eight loop is a deformation retract of the torus minus a point.

Lecture XVIII - The lifting criterion

We have already discussed the lifting problem and examined its significance in the light of complex analysis. We have seen in connection with the exponential map/squaring map that the existence of a lift of the inclusion map of a domain Ω into $\mathbb{C}-\{0\}$ is equivalent to the existence of a continuous branch of the logarithm/square-root function on Ω. Thus it is desirable to have a necessary and sufficient condition for the existence of lifts. We prove one such theorem in this lecture which provides an elegant necessary and sufficient condition.

Theorem 18.1: Let X and Y be connected locally path connected spaces, $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a covering projection and $f : (Y, y_0) \longrightarrow (X, x_0)$ is a continuous function. A lift $\tilde{f} : Y \longrightarrow \tilde{X}$ satisfying $\tilde{f}(y_0) = \tilde{x}_0$ exists if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)). \tag{18.1}$$

In particular, if Y is simply connected, that is if $\pi_1(Y, y_0)$ is trivial, then (18.1) holds and the lift $\tilde{f} : Y \longrightarrow \tilde{X}$ satisfying $\tilde{f}(y_0) = \tilde{x}_0$ exists.

Proof: To prove that the condition (18.1) is necessary, let us assume that a the lift exists. Then $p \circ \tilde{f} = f$ and $p_* \circ \tilde{f}_* = f_*$ whereby,

$$f_*(\pi_1(Y, y_0)) = p_*\Big(\tilde{f}_*(\pi_1(Y, y_0))\Big) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

We now turn to the proof of sufficiency of (18.1). To construct the lift $\tilde{f}$ let $y \in Y$ and γ be a path in Y joining y_0 and y. Take the lift of $f \circ \gamma : [0, 1] \longrightarrow X$ starting at $\tilde{x}_0$ and we declare

$$\tilde{f}(y) = \widetilde{f \circ \gamma}(1).$$

To show that the function $\tilde{f}$ is well-defined, take two paths γ_1 and γ_2 joining y_0 and y in Y and form the closed loop $\gamma_1 * \gamma_2^{-1}$ at y_0. Then $f \circ (\gamma_1 * \gamma_2^{-1})$ is a loop in X based at x_0 and so

$$[f \circ (\gamma_1 * \gamma_2^{-1})] \in f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Choose a loop σ in $\tilde{X}$ based at $\tilde{x}_0$ such that $p_*([\sigma]) = [f \circ (\gamma_1 * \gamma_2^{-1})]$. In other words, the loop $(f \circ \gamma_1) * (f \circ \gamma_2^{-1})$ is homotopic to $p \circ \sigma$. By the covering homotopy lemma, The lift of $(f \circ \gamma_1) * (f \circ \gamma_2^{-1})$ starting at $\tilde{x}_0$ which will be denoted by τ, is homotopic to σ. As a result, τ is also closed loop at $\tilde{x}_0$. Let $\widetilde{f \circ \gamma_1}$ be the lift of $f \circ \gamma_1$ starting at $\tilde{x}_0$ and $\widetilde{f \circ \gamma_2^{-1}}$ be the lift of $f \circ \gamma_2^{-1}$ starting at the terminal point $\widetilde{f \circ \gamma_1}(1)$. Observe that

$$\tau(t) = \begin{cases} \widetilde{f \circ \gamma_1}(2t) & 0 \leq t \leq 1/2 \\ \widetilde{f \circ \gamma_2^{-1}}(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

We now look at the projection of the two paths $\tau(s/2)$ and $\tau\left(\frac{2-s}{2}\right)$ $(0 \leq s \leq 1)$:

$$p \circ \tau(s/2) = f \circ \gamma_1(s), \quad 0 \leq s \leq 1$$

and

$$p \circ \tau\left(\frac{2-s}{2}\right) = f \circ \gamma_2(s), \quad 0 \leq s \leq 1.$$

The paths $\tau(s/2)$ and $\tau\left(\frac{2-s}{2}\right)$ $(0 \leq s \leq 1)$ are thus lifts of $f \circ \gamma_1$ and $f \circ \gamma_2$, both starting at $\tilde{x}_0$ since τ is a closed loop. Hence

$$\widetilde{f \circ \gamma_1}(1) = \tau(1/2) = \widetilde{f \circ \gamma_2}(1)$$

proving that $\tilde{f}(y)$ is well-defined.

Continuity of the lift $\tilde{f}$: Let $y \in Y$ be arbitrary, and let $f(y) = x$ and $\tilde{f}(y) = \tilde{x}$. Choose an evenly covered neighborhood U of x and $\tilde{U}$ be the sheet containing $\tilde{x}$ lying above U. By continuity of f we obtain a neighborhood V of y in Y such that $f(V) \subset U$ and hence $\tilde{f}(V) \subset p^{-1}(U)$ (since $p \circ \tilde{f} = f$).

Now if we assume that $\tilde{f}$ maps the neighborhood V into $\tilde{U}$, then the following would be valid:

$$\tilde{f} = \left(p\Big|_{\tilde{U}}\right)^{-1} \circ f, \tag{18.2}$$

which would prove the continuity of $\tilde{f}$. To prove that $\tilde{f}(V) \subset \tilde{U}$, we shall assume that the neighborhoods U, V and $\tilde{U}$ are path connected and invoke the construction of $\tilde{f}$. Choose a path γ in Y joining y_0 and y and for each $z \in V$ pick a path η joining y and z and then we get the path $\gamma * \eta$ joining y_0 and z. Lift $f \circ \gamma$ and $f \circ \eta$ to paths in $\tilde{X}$ starting at $\tilde{x}_0$ and $\widetilde{f \circ \gamma}(1)$ respectively. Since $f \circ \eta$ lies in U, its lift must lie entirely in $\tilde{U}$ and hence

$$\tilde{f}(z) = f \circ \widetilde{(\gamma * \eta)}(1) = \widetilde{f \circ \eta}(1) \in \tilde{U}.$$

Theorem 18.2 (Uniqueness of simply connected covers): Suppose that $p_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x_0)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x_0)$ are covering projections such that both $\tilde{X}_1$ and $\tilde{X}_2$ are simply connected and locally path connected. Then there is a homeomorphism $\psi : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that

$$p_2 \circ \psi = p_1. \quad INSERTDIAGRAM$$

Proof: Since $\tilde{X}_1$ is simply connected the map p_1 has a lift $\phi_1 : \tilde{X}_1 \longrightarrow \tilde{X}_2$ with respect to the covering projection $p_2 : \tilde{X}_2 \longrightarrow X$, such that $\phi_1(\tilde{x}_1) = \tilde{x}_2$. Likewise there exists a lift $\phi_2 : \tilde{X}_2 \longrightarrow \tilde{X}_1$ of the map p_2 with respect to the covering $p_1 : \tilde{X}_1 \longrightarrow X$, such that $\phi_2(\tilde{x}_2) = \tilde{x}_1$. From $p_1 \circ \phi_2 = p_2$ and $p_2 \circ \phi_1 = p_1$ follows $p_1 \circ (\phi_2 \circ \phi_1) = p_1$ and $(\phi_2 \circ \phi_1)(\tilde{x}_1) = \tilde{x}_1$. Thus, the identity map on $\tilde{X}_1$ and $\phi_2 \circ \phi_1 : \tilde{X}_1 \longrightarrow \tilde{X}_1$ are both lifts of $p_1 : \tilde{X}_1 \longrightarrow X$ with respect to itself. By uniqueness of lifts we conclude that $\phi_2 \circ \phi_1$ is the identity map on $\tilde{X}_1$. Likewise $\phi_1 \circ \phi_2$ is the identity map on $\tilde{X}_2$. $\square$

Example 18.1 (Some applications to complex analysis): (i) Let Ω be a *simply connected* open subset of $\mathbb{C} - \{0\}$ and $j : \Omega \longrightarrow \mathbb{C} - \{0\}$ be the inclusion and $\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$ be the exponential map. Then p is a covering projection with respect to which j has a lift $\tilde{j} : \Omega \longrightarrow \mathbb{C}$ which means

$$\exp(\tilde{j}(z)) = z, \quad z \in \Omega \tag{18.3}$$

Thus there is a continuous branch of the logarithm on any simply connected open subset of $\mathbb{C} - \{0\}$. In the exercises the student is asked to show that any continuous lift is holomorphic.

(ii) Consider the map $S : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ given by $S(z) = z^2$. Let $\Omega = \mathbb{C} - [0, 1/2]$ and $f : \Omega \longrightarrow \mathbb{C} - \{0\}$ be given by

$$f(z) = z(2z - 1). \tag{16.4}$$

Let us determine the induced map $f_* : \pi_1(\Omega, 1) \longrightarrow \pi_1(\mathbb{C} - \{0\}, 1)$. The group $\pi_1(\Omega, 1)$ is the infinite cyclic group generated by the homotopy class of the loop $\gamma(t) = \exp(2\pi i t)$. Since $\mathbb{C} - \{0\}$ is a topological group under multiplication of complex numbers, we may apply corollary (12.2) to get

$$[f \circ \gamma(t)] = [\gamma(t)] + [2\gamma(t) - 1]. \tag{18.5}$$

The additive notation is used for the infinite cyclic group. The last equation may be rewritten as

$$[f \circ \gamma(t)] = [\gamma(t)] + \Big[\gamma(t)\Big(2 - \frac{1}{\gamma(t)}\Big)\Big] = 2[\gamma(t)] + \Big[2 - \frac{1}{\gamma(t)}\Big] = 2, \tag{18.6}$$

since $|\gamma(t)| = 1$ and the loop $\Big(2 - \frac{1}{\gamma(t)}\Big)$ can be contracted to the constant loop in $\mathbb{C} - \{0\}$. Hence

$$f_*(\pi_1(\mathbb{C} - [0, 1/2], 1) = 2\mathbb{Z} = S_*(\mathbb{C} - \{0\}, 1). \tag{18.7}$$

The lifting criterion holds and f has a unique lift $\tilde{f}$ such that $\tilde{f}(1) = 1$. This lift is the continuous branch of $\sqrt{z(2z-1)}$ defined on Ω. In exercise 3, the student is asked to show that the lift $\tilde{f}$ is holomorphic. Note that the space Ω is not simply connected.

The next example is Picard's theorem which is a corollary of the following highly non-trivial result.

Theorem 16.3: The open unit disc is a covering space for the plane with two points removed.

Theorem 16.4 (The Little Picard Theorem): An entire function that misses two or more points is a constant.

Proof: Suppose an entire function f misses two points p and q. The map $f : \mathbb{C} \longrightarrow \mathbb{C} - \{p, q\}$ lifts to a map $\tilde{f} : \mathbb{C} \longrightarrow \{z \in \mathbb{C}/|z| < 1\}$. As before the lift is holomorphic and hence is an entire function taking its values in the unit disc. By Liouville's theorem, $\tilde{f}$ is constant and so must f.

Exercises:

1. For the map S in example (18.3) show that S_* is the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by $x \mapsto 2x$.
2. Suppose G is a path connected topological group with unit element e and $p : \tilde{G} \longrightarrow G$ is a covering map. For any choice of $\tilde{e} \in p^{-1}(e)$ show that there is a group operation on $\tilde{G}$ with unit element $\tilde{e}$ that makes $\tilde{G}$ into a topological group and p is a continuous group homomorphism.
3. Show that if Ω is an open subset of $\mathbb{C} - \{0\}$ on which a continuous branch of the logarithm exists then this branch is automatically holomorphic. Likewise show that the continuous branch of $\sqrt{z(2z-1)}$ on $\mathbb{C} - [0, 1/2]$ obtained in the lecture is holomorphic.
4. Use the fact that S^{n-1} is not a retract of S^n to prove that $\mathbb{R}P^{n-1}$ is not a retract of $\mathbb{R}P^n$.
5. Show that any continuous map $S^n \longrightarrow S^1$ is homotopic to the constant map if $n \geq 2$. What about maps from the projective spaces $\mathbb{R}P^n \longrightarrow S^1$ $(n \geq 2)$?

Lecture XIX - Deck Transformations

Given a covering projection $p : \tilde{X} \longrightarrow X$, the deck transformations are, roughly speaking, the symmetries of the covering space. Thus it should not come as a surprise that they play a crucial part in the theory of covering spaces. In this lecture all spaces are assumed to be connected and locally path connected.

Definition 19.1 (Deck transformations): Let $p : \widetilde{X} \longrightarrow X$ be a covering projection. A deck transformation is a homeomorphism $\phi : \widetilde{X} \longrightarrow \widetilde{X}$ such that $p \circ \phi = p$, that is to say ϕ is a lift of p.

Examples 19.1: (i) For the covering space $\text{ex} : \mathbb{R} \longrightarrow S^1$ given by $\text{ex}(t) = \exp(2\pi i t)$ the deck transformations are the maps

$$T_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad T_n(x) = x + n, \quad n \in \mathbb{Z}$$

(ii) For the two sheeted covering $p : S^n \longrightarrow \mathbb{R}P^n$ the deck transformations are the identity map and the antipodal map.

The following theorem summarizes the most basic properties of the group of deck transformations.

Theorem 19.1: Let $p : \widetilde{X} \longrightarrow X$ be a covering projection and ϕ be a deck transformation. Then

(i) ϕ is uniquely determined by its value at one point of $\widetilde{X}$

(ii) $\phi(\tilde{x}_0) \in p^{-1}(x_0)$ whenever $\tilde{x}_0 \in x_0$.

(iii) If $\phi(\tilde{x}_1) = \tilde{x}_2$, where $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ then

$$p_* \pi_1(\widetilde{X}, \tilde{x}_1) = p_* \pi_1(\widetilde{X}, \tilde{x}_2) \tag{19.1}$$

(iv) Conversely if (1) holds then there exists a unique deck transformation ϕ such that $\phi(\tilde{x}_1) = \tilde{x}_2$

Proof: Statement (i) follows from the uniqueness of lifts. Statement (ii) follows immediately from the definition. To prove (iii) apply the lifting criterion (necessity) to both ϕ and ϕ^{-1}. To prove (iv) apply lifting criterion (sufficiency) to get continuous functions $\phi : \widetilde{X} \longrightarrow \widetilde{X}$ and $\psi : \widetilde{X} \longrightarrow \widetilde{X}$ such that

$$p \circ \phi = p, \ \phi(\tilde{x}_1) = \tilde{x}_2; \quad p \circ \psi = p, \ \psi(\tilde{x}_2) = \tilde{x}_1.$$

Then $\phi \circ \psi$ and $\psi \circ \phi$ are both lifts of the map $p : \widetilde{X} \longrightarrow X$ such that

$$\phi \circ \psi(\tilde{x}_2) = \tilde{x}_2, \quad \psi \circ \phi(\tilde{x}_1) = \tilde{x}_1$$

The identity map on $\widetilde{X}$ is also a lift of p with these initial conditions. By uniqueness, we see that both $\phi \circ \psi$ and $\psi \circ \phi$ must be the identity map on $\widetilde{X}$ proving that ϕ and ψ are homeomorphisms. The uniqueness clause follows from the uniqueness of lifts. □

Remark: If $\phi : \widetilde{X} \longrightarrow \widetilde{X}$ is a *continuous* map such that $p \circ \phi = p$, then prove that ϕ is a homeomorphism in the following cases:

(i) $\pi_1(\widetilde{X})$ is a finite group (ii) $p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ has finite index in $\pi_1(X, x_0)$ (iii) $\widetilde{X}$ is a regular cover of X. Is this true in general? The point is that if H is a subgroup of G and $gHg^{-1} \subset H$ then it follows $gHg^{-1} = H$ in case H is finite or has finite index or is normal.

Definition 19.2: The set of deck transformations of a covering projection $p : \widetilde{X} \longrightarrow X$ forms a group under composition of maps denoted by $\text{Deck}(\widetilde{X}, X)$.

Action of Deck($\widetilde{X}, X$) on the fibers $p^{-1}(x_0)$: We fix a base point $x_0 \in X$. Since each deck transformation is a bijection, it is a permutation of the fiber $p^{-1}(x_0)$ and so acts on $p^{-1}(x_0)$ as a group of permutations:

$$(\phi, \tilde{x}_0) \mapsto \phi(\tilde{x}_0)$$

We study this action closely and relate it to the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$. We first look at the case of regular coverings

Theorem 19.2: The covering $p : \widetilde{X} \longrightarrow X$ is a regular covering if and only if the action of $\text{Deck}(\widetilde{X}, X)$ is transitive on $p^{-1}(x_0)$.

Proof: Let $\tilde{x}_1$ and $\tilde{x}_2$ be two arbitrary points of $p^{-1}(x_0)$. The action of $\text{Deck}(\widetilde{X}, X)$ is transitive on $p^{-1}(x_0)$ if and only if there is a $\phi \in \text{Deck}(\widetilde{X}, X)$ carrying $\tilde{x}_1$ to $\tilde{x}_2$, which is the case if and only if (19.1) holds. This in turn implies that the conjugacy class

$$\left\{ p_*\pi_1(\widetilde{X}, \tilde{x}_0) : \tilde{x}_0 \in p^{-1}(x_0) \right\}$$

reduces to a singleton and conversely, in other words, if and only if the covering is regular. □

We now relate the (perhaps intransitive) action of $\text{Deck}(\widetilde{X}, X)$ on $p^{-1}(x_0)$ with the transitive action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$. Pick $\phi \in \text{Deck}(\widetilde{X}, X)$ and $\phi(\tilde{x}_1) = \tilde{x}_2$. Then on the one hand (19.1) must hold while since $p*\pi_1(\widetilde{X}, \tilde{x}_1) = \text{stab } \tilde{x}_1$ (for the action of $\pi_1(X, x_0)$), we have on the other hand

$$\text{stab } \tilde{x}_1 = \text{stab } \tilde{x}_2 = g(\text{stab } \tilde{x}_1)g^{-1}, \tag{19.2}$$

for some $g \in \pi_1(X, x_0)$. In fact (19.2) states that g belongs to the normalizer

$$N(\text{stab } \tilde{x}_1) = N(p*(\pi_1(\widetilde{X}, \tilde{x}_1)) \subset \pi_1(X, x_0).$$

This suggests that we must relate ϕ to the element $g \in N(p*(\pi_1(\widetilde{X}, \tilde{x}_1))$. However since there may be several such elements g it is expedient to define the map in the opposite direction.

Let $g \in N(p*(\pi_1(\widetilde{X}, \tilde{x}_1)) \subset \pi_1(X, x_0)$ and $\tilde{x}_1 \cdot g = \tilde{x}_2$. Then (19.1) holds since g is in the normalizer of stab $\tilde{x}_1$. There is a unique $\phi_g \in \text{Deck}(\widetilde{X}, X)$ such that $\phi_g(\tilde{x}_1) = \tilde{x}_2 = \tilde{x}_1 \cdot g$. The map

$$\psi \; : \; N(p*(\pi_1(\widetilde{X}, \tilde{x}_1)) \longrightarrow \text{Deck}(\widetilde{X}, X), \quad g \mapsto \phi_g \tag{19.3}$$

is a homomorphism. To see that it is surjective, let $\phi \in \text{Deck}(\widetilde{X}, X)$. There is a $g \in \pi_1(X, x_0)$ such that

$$\tilde{x}_1 \cdot g = \phi(\tilde{x}_1)$$

then stab $\tilde{x}_1$ and stab $\phi(\tilde{x}_1)$ are conjugate by g but they are also equal by (iii) of Theorem (19.1), whereby we conclude g is in the normalizer $N(p*(\pi_1(\widetilde{X}, \tilde{x}_1))$ and $\phi = \phi_g$. To determine the kernel of ψ, observe that $\phi_g = \text{id}$ if and only if

$$\phi_g(\tilde{x}_1) = \tilde{x}_1 \cdot g$$

that is, if and only if $g \in$ stab $\tilde{x}_1$. But stab $\tilde{x}_1 = p*(\pi_1(\widetilde{X}, \tilde{x}_1))$. Summarizing these observations,

Theorem 19.3: We the group isomorphism

$$\text{Deck}(\widetilde{X}, X) \cong N(p*(\pi_1(\widetilde{X}, \tilde{x}_1))/p*(\pi_1(\widetilde{X}, \tilde{x}_1)). \tag{19.4}$$

Corollary 19.4: If $p : \widetilde{X} \longrightarrow X$ is a regular covering then

$$\text{Deck}(\widetilde{X}, X) \cong \pi_1(X, x_0)/p*(\pi_1(\widetilde{X}, \tilde{x}_1)). \tag{19.5}$$

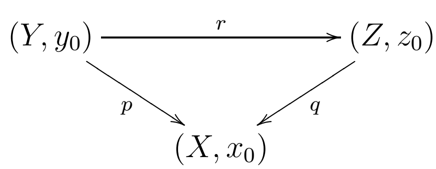
Corollary 19.5: If $\widetilde{X}$ is a simply connected covering of X then

$$\text{Deck}(\widetilde{X}, X) \cong \pi_1(X, x_0). \tag{19.6}$$

Corollary 19.6: $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$

Existence of a simply connected covering space: Despite being an important theme, we shall not discuss this in any detail in this elementary course but make a few remarks about it. Most of the spaces that we shall encounter are reasonably well-behaved and indeed many of them such $SO(n, \mathbb{R})$, S^3 and the projective spaces are smooth manifolds. Given the existence of a simply connected covering - called a universal covering[4], one can develop a Galois correspondence for covering spaces which asserts the existence of a unique (upto isomorphism) covering corresponding to each conjugacy class of subgroups of $\pi_1(X, x_0)$.

Definition 19.3: Let us consider a fixed connected topological space X with a specified base point $x_0 \in X$. A homomorphism between two coverings $p : (Y, y_0) \longrightarrow (X, x_0)$ and $q : (Z, z_0) \longrightarrow (X, x))$ is a surjective continuous map $r : (Y, y_0) \longrightarrow (Z, z_0)$ such that $q \circ r = p$ or diagrammatically,



The definition enables us to form a category of coverings of a given space X with a specified base point $x_0 \in X$. To obtain a satisfactory theory one must impose some additional assumption on X such as local connectedness. In other words r is a lift of p with respect to the covering map q. The universal covering is then defined in terms of a universal property.

[4]Actually the notion of a universal covering is more general than the notion of a simply connected coverings but the two notions coincide for all reasonable spaces and certainly for all spaces that we shall deal with.

Definition 19.4: The universal covering is a covering $e : (E, e_0) \longrightarrow (X, x_0)$ such that for every covering $p : (Y, y_0) \longrightarrow (X, x_0)$ there is a unique homomorphism $\psi : (E, e_0) \longrightarrow (Y, y_0)$, that is a continuous surjection ψ such that $p \circ \psi = e$.

The universal covering if it exists is unique and one can establish the existence of a universal covering for a reasonable nice class of topological spaces X.

Exercises

1. Suppose that G and $\tilde{G}$ are topological groups and $p : \tilde{G} \longrightarrow G$ is a covering projection that is also a group homomorphism then $\ker p = \text{Deck}(\tilde{G}, G)$.

2. Determine the deck transformations for the covering
$$\sin : \mathbb{C} - \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\} \longrightarrow \mathbb{C} - \{\pm 1\}$$

3. Determine the deck transformations for the covering
$$p : \mathbb{C} - \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} - \{\pm 2\}$$
given by $p(z) = z^3 - 3z$. Show that this covering is not regular. Hint: Use Riemann's removable singularities theorem to show that a deck transformation must be analytic on the whole plane.

4. If p is a prime, what can you say about the group of deck transformations of a p-sheeted covering space?

5. Show using the universal property that the universal covering, if it exists is unique upto isomorphism of covering projections.

Lecture XX - Orbit Spaces

Many interesting spaces in geometry arise as the space of orbits under the action of groups. We have seen examples of this already in lecture 4. An important special case is when the group action is discrete such as the case of the multiplicative group $\{\pm 1\}$ on the sphere S^n resulting in the real projective space $\mathbb{R}P^n$.

Properly discontinuous group actions: Recall that a group is said to act freely if there are no fixed points of the action. That is to say, if G acts on S such that if $g \cdot s = s$ for all $s \in S$ then $g = 1$. We now define a stronger notion when the group acts on a topological space.

Definition 20.1: Let Y be a topological space on which a group G acts. We say that the action is properly discontinuous if each point $y \in Y$ has a neighborhood U such that for any pair of *distinct* elements $g', g'' \in G$,

$$g'U \cap g''U = \emptyset.$$

Theorem 20.1: If a group G acts properly discontinuously on a topological space then the group action must then be free.

Proof: We shall show that if $g \cdot y = y$ for some $y \in Y$ and $g \in G$ then $g = 1$. If $g \neq 1$, choose a neighborhood U of y as in definition (20.1) which in particular implies $g \cdot U \cap U = \emptyset$. But $y \in g \cdot U \cap U$ and we get a contradiction.

The set of all orbits of the action with its quotient topology is denoted by Y/G and the following theorem expresses the covering properties of the quotient map

$$\eta : Y \longrightarrow Y/G.$$

Note that for each $g \in G$, the map $y \mapsto g \cdot y$ is a bijective map. If each of these maps is a homeomorphism of Y onto itself, we say that G acts as a group of homeomorphisms on Y.

Theorem 20.2: Let Y be a Hausdorff space and G be a group of acting properly discontinuously on Y as a group of homeomorphisms. Then,

(i) The orbit space Y/G is Hausdorff.

(ii) The quotient map $\eta : Y \longrightarrow Y/G$ is a covering projection.

(iii) G is the group of deck-transformations for the covering projection $\eta : Y \longrightarrow Y/G$.

(iv) In case Y is simply connected, $\pi_1(Y/G)$ is isomorphic to G.

Proof: Pick distinct points $\bar{y}, \bar{z} \in Y/G$ and let U and V be disjoint neighborhoods of y and z such that for every pair of distinct elements $g', g'' \in G$,

$$g'U \cap g''U = \emptyset, \quad g'V \cap g''V = \emptyset.$$

Then

$$\eta^{-1}(\eta(U)) = \bigcup_{g \in G} gU, \quad \eta^{-1}(\eta(V)) = \bigcup_{g \in G} gV.$$

Since G is a group of homeomorphisms, it follows from the definition of quotient topology that $\eta(U)$ and $\eta(V)$ are open sets containing $\bar{y}$ and $\bar{z}$. It is easy to see that $\eta(U)$ and $\eta(V)$ are disjoint and (i) follows and also that η is an open mapping. Now η restricted to each gU is a continuous, open bijection, that is a homeomorphism onto $\eta(U)$ and so (ii) follows. Conclusion (iv) follows from (iii). To prove (iii) first observe that the map

$$\phi_g : y \mapsto g \cdot y, \quad y \in Y$$

is a deck transformation for each $g \in G$. The map

$$\psi : g \mapsto \phi_g$$

is easily seen to be a group homomorphism. To see that it is surjective, let ϕ be a deck transformation and y_1 be a given point in Y and $\phi(y_1) = y_2$. Since y_1 and y_2 are in the same fiber, there is a unique element g of the group such that $g \cdot y_1 = y_2$. Then the deck transformations ϕ and ϕ_g agree at y_1 and so are identical which means $\psi(g) = \phi$ proving surjectivity. If $g \in \ker \psi$ then

$$\phi_g(y) = g \cdot y = y, \quad \forall y \in Y.$$

Since the action is properly discontinuous (and hence fixed point free) this forces $g = 1$.

Definition 20.2 (Lens spaces): Let $Y = S^3 = \{(z, w) \in \mathbb{C}^2 / |z|^2 + |w|^2 = 1\}$ and p be a prime, q be an integer relatively prime to p. The action of $\mathbb{Z}_p$ on S^3 given by

$$\exp(2\pi ik/p) \cdot (z_1, z_2) = (\exp(2\pi ik/p) z_1, \exp(2\pi ikq/p) z_2)$$

is fixed point free and hence properly discontinuous. The orbit space is called the lens space denoted by $L(p, q)$. Theorem (20.2) now implies

$$\pi_1(L(p, q)) = \mathbb{Z}_p.$$

Definition 20.3 (Generalized lens spaces): Let $q_1, q_2, \ldots, q_n$ be relatively prime to p. Define the action of the cyclic group $\mathbb{Z}_p$ on S^{2n+1} by

$$(z_0, z_1, \ldots, z_n) \mapsto \left(z_0 \exp\left(\frac{2\pi i}{p}\right), z_1 \exp\left(\frac{2\pi i q_1}{p}\right), \ldots, \exp\left(\frac{2\pi i q_n}{p}\right)\right),$$

where $S^{2n+1} = \{(z, w_1, w_2, \ldots, w_n) \in \mathbb{C}^{n+1} / |z|^2 + |w_1|^2 + \cdots + |w_n|^2 = 1\}$. The resulting orbit space is denoted by $L(p, q_1, q_2, \ldots, q_n)$ and its fundamental group is $\mathbb{Z}_p$ since the action is properly discontinuous.

The Mobius band: Consider the strip $Y = [0,1] \times \mathbb{R}$ and let $S : Y \longrightarrow Y$ be the homeomorphism

$$S(x,y) = (1-x, y+1)$$

of Y. Then S generates an infinite cyclic group of homeomorphisms of Y acting properly discontinuously on Y. The resulting orbit space is the Möbius band. It is an exercise to show that the cylinder is a double cover of the Möbius band.

Klein's bottle: Let $Y = \mathbb{R}^2$ and G be the group generated by the affine maps T and S given by

$$T(x,y) = (x+1, y), \quad S(x,y) = (1-x, y+1).$$

Note that T and S are isometies and the group generated by these acts properly discontinuously on $\mathbb{R}^2$. The orbit space is the Klein's bottle K. Thus $\pi_1(K) = G$. Now

$$TS(x,y) = (2-x, y+1), \quad ST(x,y) = (-x, y+1)$$

The fundamental group of the Klein's bottle is non-abelian. Note that $TST = S$. There are no other independent relations and the fundamental group of the Klein's bottle is the group on two generators T and S with one relation $TST = S$. Summarizing we have

Theorem 20.3: The fundamental group of the Klein's bottle is the non-abelian group with two generators S and T with the relation $TST = S$.

Exercises:

1. Suppose that G is a finite group acting freely on a Hausdorff space then the action is properly discontinuous and hence deduce that the group action in the example of the generalized Lens space is properly discontinuous.

2. Suppose that $p : \tilde{X} \longrightarrow X$ is a covering projection and $\tilde{X}$ is locally path connected and simply connected. Show that if U is an evenly covered open set in X and $\tilde{U}$ is a sheet lying above it then $\phi(\tilde{U}) \cap \tilde{U} = \emptyset$ for every $\phi \in \text{Deck}(\tilde{X}, X)$ and $\phi \neq \text{id}_{\tilde{X}}$. Deduce that the group of deck transformations acts properly discontinuously on $\tilde{X}$. How does this relate to theorem 17.2?

3. Does the fundamental group of Klein's bottle have elements of finite order? Identify this group with a familiar group that we have already encountered in lecture 8. What is its abelianization? Hint: First show that $T^n S^m = S^m T^{(-1)^m n}$. Now for an element α in the group, define height of α to be the $y-$coordinate of $\alpha(0,0)$. What is the height of a torsion element and what does this imply about the sum of the indices of powers of S occuring in it?

4. Show that the torus is obtained as the orbit space of a group of homeomorphisms acting properly discontinuously on $\mathbb{R}^2$. Write out these homeomorphisms explicitly.

5. Show that the torus is a double cover of the Klein's bottle. Hence the fundamental group of the Klein's bottle must contain a subgroup of index two. Determine this subgroup.

6. Show that the cylinder is a two-sheeted cover of the Möbius band.

7. Suppose that G is a topological group, H is a discrete subgroup of G. Show that there exists a neighborhood U of the identity such that $U = U^{-1}$, $U \cap H = \{1\}$ and that $\{hU/h \in H\}$ is a family of disjoint open sets. Deduce that the quotient map $\eta : G \longrightarrow G/H$ is a covering projection. Also show that G/h is Hausdorff.

Lecture XXI - Test - III

1. Show that a homeomorphism of E^2 onto itself must preserve the boundary. That is it must map a boundary point to a boundary point.

2. Show that for fixed $k \in \mathbb{N}$, the $k-$th roots of unity acts properly discontinuously on $\mathbb{C} - \{0\}$.

3. Let G be the infinite grid

$$G = \{(x, y) \in \mathbb{R}^2 \;/\; x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}.$$

   Consider the covering map from G onto the figure eight loop $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ given by

$$p(x, y) = (\exp(2\pi i x),\; \exp(2\pi i y)).$$

   Determine the deck transformations of this covering. Is this a regular covering?

4. Suppose $p : \tilde{X} \longrightarrow X$ is a covering projection and $\tilde{X}$ is path connected, show that $p^{-1}(y)$ and $p^{-1}(z)$ have the same cardinality for every pair $y, z \in X$.

5. Given a covering projection $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$, describe the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ and deduce that the cardinality of $p^{-1}(x_0)$ is the index of the subgroup $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

Solutions to Test - III

1. Suppose that $\phi : E^2 \longrightarrow E^2$ is a homeomorphism and maps a boundary point p into an interior point q. Then the restriction $\phi : E^2 - \{p\} \longrightarrow E^2 - \{q\}$ is a homeomorphism but $\pi_1(E^2 - \{q\})$ is isomorphic to $\mathbb{Z}$ whereas $E^2 - \{p\}$ is star-shaped and so has trivial fundamental group.

2. Check that the action is free and use the result of exercise 1 of lecture 20. Thus we have to show that if a finite group G acts freely on a Hausdorff space X then the action is properly discontinuous. Well, let $p \in X$ and since the action is free,
$$g' \cdot p \neq g'' \cdot p, \qquad \text{when } g' \neq g''.$$
Choose for each $g \in G$ an open neighborhood U_g of gp such that the finite family $\{U_g\}$ consists of pairwise disjoint sets. Let
$$V = \bigcap_{g \in G} g^{-1} U_g.$$
Then V is an open neighborhood of p and it is easy to see that $g'V \cap g''V = \emptyset$ whenever $g' \neq g''$.

3. It is elementary to verify that the the deck-transformations are translations namely for each pair $(n, m) \in \mathbb{Z}^2$,
$$(x, y) \mapsto (x + n, y + m), \quad (x, y) \in \mathbb{R}^2.$$
These deck transformations act transitively on each fiber and so the covering is regular.

4. See the proof of theorem 16.4.

5. See lecture 17.

Lecture XXII - Fundamental group of $SO(3,\mathbb{R})$ and $SO(4,\mathbb{R})$

For many applications, it is important to know the fundamental groups of the classical groups. We shall discuss in detail the orthogonal groups $SO(3,\mathbb{R})$ and $SO(4,\mathbb{R})$ since their underlying topological spaces are easily described. Indeed $SO(3,\mathbb{R})$ is the three dimensional real projective space and $SO(4,\mathbb{R})$, as a topological space, is the product of the three dimensional real projective space and the three dimensional sphere S^3. To unravel the structure of these spaces it is convenient to use quaternions. We shall assume some basic familiarity with quaternions (see [1]). We shall also use some basic facts from multi-variable calculus. The student who is unfamiliar with these parts of multi-variable calculus may omit these parts of the proof.

Theorem 22.1 The unit sphere S^3 is the double cover of the space $SO(3,\mathbb{R})$ and as a topological space is homeomorphic to $\mathbb{R}P^3$. In particular $\pi_1(SO(3,\mathbb{R}))$ is the cyclic group of order two.

The proof will be split into several lemmas. We begin by setting up a few notations which would remain in force throughout the lecture. We shall regard S^3 as the set of all unit quaternions forming a subgroup of the multiplicative group of non-zero quaternions.

Definition 22.1: A pure quaternion is one whose real part is zero. Thus a quaternion is pure q if and only if $\overline{q} = -q$, where the bar denotes the conjugate of q. We denote the set of all pure quaternions by Π. Thus Π is a three dimensional real vector space with the Euclidean norm inherited from $\mathbb{R}^4$.

We now list three lemmas whose proofs are left for the reader as easy exercises in linear algebra. It is useful to recall that a linear map of $\mathbb{R}^n$ to itself which preserves the Euclidean norm is an orthogonal transformation.

Lemma 22.2: If q is a pure quaternion then so is $x^{-1}qx$ for any non-zero quaternion x.

Thus, each non-zero quaternion x defines a non-singular linear map $T_x : \Pi \longrightarrow \Pi$ namely

$$T_x(q) = x^{-1}qx \tag{22.1}$$

Lemma 22.3: The linear map $T_x : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by equation (22.1) preserves the Euclidean norm and so defines an element of $O(4,\mathbb{R})$. Its restriction to Π still denoted by T_x maps Π onto itself and so may be identified as an element of $O(3,\mathbb{R})$. The map

$$\psi : x \mapsto T_x \tag{22.2}$$

is a group homomorphism from the multiplicative group of non-zero quaternions into $O(4,\mathbb{R})$.

Lemma 22.4: The kernel of ψ is the set of non-zero real numbers. In particular the kernel of the map

$$\psi : S^3 \longrightarrow O(3, \mathbb{R}) \tag{22.3}$$

obtained by restricting ψ to S^3 is the two element group ± 1.

Lemma 22.5: The image of $\psi : S^3 \longrightarrow O(3, \mathbb{R})$ is a compact connected subgroup of $SO(3, \mathbb{R})$.

Proof: Since S^3 is compact and connected, the image of the map $\psi : S^3 \longrightarrow O(3, \mathbb{R})$ is a compact and connected subgroup of $O(3, \mathbb{R})$. Now $O(3, \mathbb{R})$ is disconnected with two components and so the image must lie entirely in one of these components. Since $\psi(1)$ is the identity map this connected subgroup meets $SO(3, \mathbb{R})$ and so must be contained entirely in $SO(3, \mathbb{R})$.

Slightly more difficult is the proof that the image of S^3 under ψ is the whole of $SO(3, \mathbb{R})$. It is possible to give an argument which uses only linear algebra but we prefer to follow a slightly more sophisticated approach using the inverse function theorem. The student who is uncomfortable may merely skim through the argument and take the result on faith.

Lemma 22.6: The group homomorphism $\psi : S^3 \longrightarrow SO(3, \mathbb{R})$ is surjective and is a covering projection. As a topological space, $SO(3, \mathbb{R})$ is homeomorphic to $\mathbb{R}P^3$.

Proof: Once we show that $\psi : S^3 \longrightarrow SO(3, \mathbb{R})$ is surjective it follows from lemma (22.4) and the definition of real projective spaces that $SO(3, \mathbb{R})$ and $\mathbb{R}P^3$ are homeomorphic.

To prove the surjectivity of ψ, note that S^3 and $SO(3, \mathbb{R})$ are three dimensional manifolds and ψ is a smooth map. We show that the derivative $D\psi(1)$ is an invertible linear map and so by the inverse function theorem the image must contain a neighborhood of the identity. We merely have to recall from lecture 5 that if a subgroup H of a connected topological group G contains a neighborhood of the identity then $H = G$.

We now turn to the proof that $D\psi(1)$ is a surjective linear transformation. We shall regard ψ as a map from $\mathbb{R}^4$ to $SO(3, \mathbb{R}) \subset M(3, \mathbb{R})$ and compute its derivative at 1. For a quaternion h with sufficiently small norm,

$$\psi(1 + h)v - \psi(1)v = \|1 + h\|^{-2}(v + hv + v\overline{h}) - v + O(\|h\|^2) = -2h_0 v + hv + v\overline{h} + O(\|h\|^2),$$

where h_0 denotes the real part of h. We see that $D\psi(1)$ is the linear map $\mathbb{R}^4 \longrightarrow M(3, \mathbb{R})$ given by

$$h \mapsto -2h_0(\cdot) + h(\cdot) + (\cdot)\overline{h}. \tag{22.4}$$

The kernel of this linear map contains 1 and so is at-least one dimensional. It is exactly one dimensional since $D\psi(1)i$, $D\psi(1)j$ and $D\psi(1)k$ are linearly independent (skew-symmetric) matrices.

Remark: The curves σ_1, σ_2 and σ_3 given by

$$\sigma_1(t) = \cos t + i \sin t, \quad \sigma_2(t) = \cos t + j \sin t, \quad \sigma_3(t) = \cos t + k \sin t$$

lie on S^3 and pass through the point 1. Differentiating and setting $t = 0$ confirms that the vectors i, j, k span the tangent space to S^3 at 1. Thus $D\psi(1)i, D\psi(1)j$ and $D\psi(1)k$ span the image of $D\psi(1)$. We leave it to the reader to check, by calculating the derivatives of $\psi \circ \sigma_j$ $(j = 1, 2, 3)$ at $t = 0$, that $D\psi(1)i, D\psi(1)j$ and $D\psi(1)k$ are linearly independent.

Topological structure of $SO(4,\mathbb{R})$: Regard $L \in SO(4,\mathbb{R})$ as a linear transformation on the space $\mathbb{R}^4$ of all quaternions. In particular, $L(1)$ is a non-zero quaternion and we may define the linear map $L' : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ via the prescription

$$L'(x) = L(x)L(1)^{-1}, \quad x \in \mathbb{R}^4.$$

Lemma 22.7: The map L' preserves Euclidean distance and maps Π to itself.

Proof: The fact that it is distance preserving is clear so that it is an orthogonal transformation. Since L' also fixes the real axis by orthogonality it must map Π to itself.

Theorem 22.8: As a topological space, $SO(4,\mathbb{R})$ is homeomorphic to the product $S^3 \times SO(3,\mathbb{R})$.

Proof: We show that the map $\phi : SO(4,\mathbb{R}) \longrightarrow S^3 \times O(3,\mathbb{R})$ given by $\phi(L) = (L(1), L')$, where L' is defined as in the previous lemma, is a homeomorphism. The map L' is an element of $O(3,\mathbb{R})$ since it maps Π to itself and preserves Euclidean norm. Further, $L(1)$ is obviously a unit quaternion. The image of ϕ is a compact connected subspace of $S^3 \times O(3,\mathbb{R})$ and sends the identity element to the pair $(1, \mathrm{id})$ which means the image must be contained in $S^3 \times SO(3,\mathbb{R})$. It is an exercise that the map is bijective. Since the space $SO(4,\mathbb{R})$ is compact and $S^3 \times SO(3,\mathbb{R})$ is Hausdorff, it follows that ϕ is a homeomorphism.

Corollary 22.9: The fundamental group of $SO(4,\mathbb{R})$ is the cyclic group of order two. $\square$

Exercises

1. Show that the sphere S^3 is isomorphic (as a topological group) to $SU(2,\mathbb{C})$.
2. Show that the center of the group of non-zero quaternions is the set of non-zero real numbers. In the light of this explain why $\ker D\psi(1)$ in lemma (22.6) is non-trivial.
3. Explain why the map ϕ defined in theorem (22.8) is bijective.
4. Verify the properties of the map T_A in the proof of theorem (22.10). Also fill in the details concerning the properties of the map ϕ (except for the claims made concerning its derivative).
5. Use exercise 4 to find a generator of $\pi_1(SO(3,\mathbb{R}))$. Let $i : SO(2,\mathbb{R}) \longrightarrow SO(3,\mathbb{R})$ be given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \in SO(2,\mathbb{R}).$$

   Show that $i_* : \pi_1(SO(2,\mathbb{R})) \longrightarrow \pi_1(SO(3,\mathbb{R}))$ is surjective.

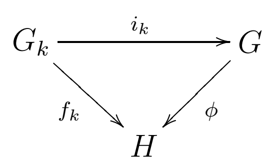
Lectures - XXIII and XXIV Coproducts and Pushouts

We now discuss further categorical constructions that are essential for the formulation of the Seifert Van Kampen theorem. We first discuss the notion of coproducts which is a prerequisite for a proof of the existence of push-outs. The coproduct is popularly known as the free product in the context of groups but we shall also use the term coproduct which seems more appropriate from a categorical point of view ([11], p. 71). The notion of coproducts has already been introduced in the exercises to lecture 7 for the categories **Top** and **AbGr** where it is popularly known as the disjoint union and the direct sum respectively. However the construction is more complicated in the category **Gr**. The coproduct is defined in terms of a universal property.

Definition 23.1: Given two groups G_1 and G_2, their coproduct is a group G together with a pair of group homomorphisms $i_1 : G_1 \longrightarrow G$ and $i_2 : G_2 \longrightarrow G$ such that given any group H and group homomorphisms $f_1 : G_1 \longrightarrow H$ and $f_2 : G_2 \longrightarrow H$ there exists a **unique** homomorphism $\phi : G \longrightarrow H$ such that

$$\phi \circ i_1 = f_1, \quad \phi \circ i_2 = f_2 \tag{23.1}$$

summarized in the following diagram ($k = 1, 2.$):



The definition immediately generalizes to any arbitrary (not necessarily finite) collection of groups. The uniqueness clause in the definition is important and the following theorem hinges upon it.

Theorem 23.1: If the coproduct (free product) exists then it is unique upto isomorphism. Denoting the coproduct by $G_1 * G_2$, the maps i_1 and i_2 are injective and so G_1 and G_2 may be regarded as subgroups of $G_1 * G_2$.

Proof: To establish uniqueness, suppose that G' is another candidate for the coproduct with the associated homomorphisms $j_1 : G_1 \longrightarrow G'$ and $j_2 : G_2 \longrightarrow G'$ satisfying the universal property. Taking $f_1 = j_1$ and $f_2 = j_2$ in the definition, there exists a homomorphism $\phi : G \longrightarrow G'$ such that

$$\phi \circ i_1 = j_1, \quad \phi \circ i_2 = j_2.$$

But since G' is also a coproduct we obtain reciprocally a group homomorphism $\psi : G' \longrightarrow G$ such that

$$\psi \circ j_1 = i_1, \quad \psi \circ j_2 = i_2.$$

Combining the two we get $(\psi \circ \phi) \circ i_1 = i_1$ and $(\psi \circ \phi) \circ i_2 = i_2$. We see that the identity map id_G as well as $\psi \circ \phi$ satisfy the universal property with $H = G$, $f_1 = i_1$ and $f_2 = i_2$. The uniqueness clause in the definition of the coproduct gives $\psi \circ \phi = \mathrm{id}_G$ Interchanging the roles of G and G' we get $\phi \circ \psi = \mathrm{id}_{G'}$. We leave it to the student to show that the maps i_1 and i_2 are injective.

Theorem 23.2: Coproducts exist in the category **Gr**.

Proof: We shall merely provide a sketch of the argument. Let G_1 and G_2 be two given groups. A word is by definition a finite sequence $(x_1, x_2, \dots x_n)$ such that each x_i $(i = 1, 2 \dots, n)$ belongs to one of the groups, no pair of adjacent terms of the sequence belong to the same group and none of the x_i is the identity element of either of the groups. We call the integer n the length of the word and also include the empty word of length zero. Denoting by W is the set of all words, the idea is to define a binary operation of *juxtaposition* of words. The empty word would serve as the identity and the inverse of a word $(x_1, x_2, \dots, x_n)$ would be the word $(x_n^{-1}, x_{n-1}^{-1}, \dots, x_1^{-1})$. One would hope that the operation of juxtaposition would make W a group. This however would not quite suffice. The juxtaposition of two words $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_m)$ may result in a sequence that does not qualify to be called a word for the simple reason that x_n and y_1 may belong to the same group. When this happens we may try to replace the juxtaposed string by the smaller string

$$(x_1, x_2, \dots, x_{n-1}, z, y_2, \dots, y_m)$$

where $z = x_n y_1$. If z is not the unit element we do get a legitimate word but if z is the unit element of one of the groups we must drop it altogether obtaining instead the still smaller string

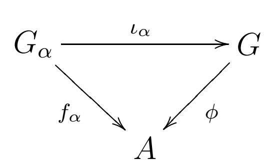
$$(x_1, x_2, \dots, x_{n-1}, y_2, \dots, y_m)$$

If x_{n-1} and y_2 belong to the same group the above process must continue and thus in finitely many steps we obtain a legitimate word that ought to be the product of the two given words. To check that we do get a group that qualifies as the coproduct of the given groups can be tedious. The reader may consult [11], pp 72-73.

We now introduce the notion of a direct sum of abelian groups which will play a crucial role in the second part of the course.

Definition 23.2 (Coproduct of abelian groups or the direct sum): Given a family of abelian groups $\{G_\alpha \ / \ \alpha \in \Lambda\}$, their coproduct or direct sum is an abelian group G together with a family of group homomorphisms $\{\iota_\alpha : G_\alpha \longrightarrow G \ / \ \alpha \in \Lambda\}$ such that the following universal property holds.

Given any abelian group A and a family of group homomorphisms $f_\alpha : G_\alpha \longrightarrow A$, there exists a *unique* group homomorphism $\phi : G \longrightarrow A$ such that each of the diagrams commutes:



Theorem 23.3: Coproducts exist in the category **AbGr** and it is unique.

Proof: We use the additive notation and shall use the same symbol 0 to denote the identity element of all the groups. The cartesian product $\prod G_\alpha$ is a group with respect to component-wise addition and we consider the subgroup $\bigoplus G_\alpha$ given by

$$\bigoplus_{\alpha\in\Lambda} G_\alpha = \Big\{ (x_\alpha)_\alpha \in \prod_{\alpha\in\Lambda} G_\alpha \ / \ x_\alpha = 0 \text{ for all but finitely many indices } \alpha \Big\}.$$

For each $\beta \in \Lambda$ we define the standard inclusion map

$$\iota_\beta : G_\beta \longrightarrow \bigoplus_{\alpha\in\Lambda} G_\alpha$$

such that $\iota_\beta(x)$ has entry x in position β and all other coordinates are zero. We leave it to the reader to check that the group $\bigoplus_\alpha G_\alpha$ together with the family $\{\iota_\alpha : G_\alpha \longrightarrow G \ / \ \alpha \in \Lambda\}$ satisfies all the requirements.

Definition 23.3 (free groups): The coproduct in the category **Gr** (known as the free product) of k copies of $\mathbb{Z}$ is called the free group on k generators.

We shall denote a free group on k generators by F_k or if there is a need to specify the generators $a_1, a_2, \ldots, a_k$ we shall use the notation $F[a_1, a_2, \ldots, a_k]$.

Theorem 23.4: Any group H having k generators is a homomorphic image of F_k.

Proof: Let H be generated by $x_1, x_2, \ldots, x_k$ and for each $j = 1, 2, \ldots, k$ let G_j be the infinite cyclic group with generator a_j, regarded as a subgroup of F_k. Applying the definition of the coproduct to the collection of group homomorphisms $f_j : G_j \longrightarrow H$ defined by

$$f_j(a_j) = x_j, \quad j = 1, 2 \ldots, k,$$

we get a group homomorphism $\phi : F_k \longrightarrow H$ such that $\phi(a_j) = f_j(a_j) = x_j$. It is clear that ϕ is surjective and the proof is complete.

Generators and relations: Denoting by B the set of generators $a_1, a_2, \ldots, a_k$ of F_k, any collection S of words

$$a_{i_1}^{n_1} a_{i_2}^{n_2} \ldots, a_{i_p}^{n_p}, \quad a_{i_j} \in B,\ n_j \in \mathbb{Z},\ 1 \leq j \leq p. \tag{23.2}$$

gives rise to a group $F_k/\langle S\rangle$ where $\langle S\rangle$ denotes the normal subgroup generated by S. Conversely, let H be a finitely generated group and ϕ be as in the theorem. We take a set R of words (23.2) generating the kernel of ϕ and write

$$H = F_k/\langle R\rangle. \tag{23.3}$$

The elements of R are called *relators* and the set of equations

$$a_{i_1}^{n_1} a_{i_2}^{n_2} \ldots, a_{i_p}^{n_p} = 1 \tag{21.4}$$

obtained by setting each relator to 1 are called the relations for the group with respect to ϕ. The list of generators $\{a_1, a_2, \ldots, a_k\}$ and relations among them uniquely specifies H through equation (23.3). If a relation in the list (23.4) is a consequence of others, for example if one of them is the product of two others, we may clearly drop it from the list thereby shortening the list. In practice one would try to keep the list of relations down to a minimum. Such a description of H is called a presentation of the group H through generators and relations. A group in general has many presentations and it is usually very difficult to decide whether or not two presentations represent the same group.

Example 23.1 (Presentation of some groups): We describe some of the commonly occurring groups in terms of generators and relations. Some of these would appear as fundamental groups of spaces that we have already encountered or would do so in the next few lectures.

1. If we take the free group on two generators a, b and take $H = \mathbb{Z} \times \mathbb{Z}$ then every commutator $a^m b^n a^{-m} b^{-n}$ is a relator and hence each of the equations $a^m b^n a^{-m} b^{-n} = 1$ is a relation. However, all of them may be derived from the single relation $aba^{-1}b^{-1} = 1$. For example, we derive the relation $a^2ba^{-2}b^{-1} = 1$ as follows
$$a^2ba^{-2}b^{-1} = a(aba^{-1}b^{-1})ba^{-1}b^{-1} = aba^{-1}b^{-1} = 1.$$
Thus $\mathbb{Z} \times \mathbb{Z}$ has presentation
$$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle \tag{23.5}$$

2. The cyclic group of order n has presentation
$$\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle \tag{23.6}$$

3. Recall from lecture 20 that the fundamental group of the Klein's bottle is given by the presentation
$$\mathbb{Z} \ltimes \mathbb{Z} = \langle a, b \mid aba = b \rangle \tag{23.7}$$

4. This example is from [15], p. [?]. Let us consider the group G given by the presentation
$$G = \langle a, b \mid a^2 = b^4 = 1,\ bab = a \rangle \tag{23.8}$$
To understand this group concretely, let us derive some consequences of the three displayed relations. Multiplying $bab = a$ on the left/right by a gives the relations $(ab)^2 = 1$ and $(ba)^2 = 1$. Further,
$$ab^3 = (ab)b^2 = b^3(bab)b^2 = b^3ab^2 = ba.$$
We conclude from this that G consists of the elements
$$\{1, a, b, b^2, b^3, ab, ba, ab^2\} \tag{23.9}$$
This however does not preclude further simplifications to a group of smaller order though it seems unlikely. The group has atleast three elements of order two and so if the elements listed in (21.9) are distinct then G must be the dihedral group D_4 of order eight if it is non-abelian or else must be an abelian group. In any case there must be atleast five elements of order two (why?). It is easy to see that ab^2 has order two. The map $f : a, b \longrightarrow D_4$ given by
$$f(a) = (13),\ f(b) = (1234)$$
respects the given relations since $(13)^2 = 1$, $(1234)^4 = 1$ and $(1234)(13)(1234) = (13)$. Hence f extends to a surjective group homomorphism $f : F_2 \longrightarrow D_4$. Since the kernel contains a^2, b^4 and bab we get a surjective group homomorphism $G \longrightarrow D_4$ and we conclude that G is indeed D_4.

Push-outs: The notion of push-outs is a convenient generalization of the coproduct and in the context of groups is also known as the free-product with amalgamation. In topology it is often referred to as the adjunction space though some authors in analogy with groups call it the amalgamated sum. We formulate this notion in general terms.

Definition 23.4: Suppose given a pair of morphisms $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ in a category $\mathcal{C}$, represented as a diagram:

$$\begin{array}{ccc} C & \xrightarrow{j_1} & A_1 \\ {\scriptstyle j_2}\downarrow & & \\ A_2 & & \end{array}$$

a push out is an object P in $\mathcal{C}$ together with a pair of morphisms $f_1 : A_1 \longrightarrow P$ and $f_2 : A_2 \longrightarrow P$ satisfying the following two conditions:

(i) $f_1 \circ j_1 = f_2 \circ j_2$

(ii) Universal property: Given any pair of morphisms $g_1 : A_1 \longrightarrow E$ and $g_2 : A_2 \longrightarrow E$ satisfying

$$g_1 \circ j_1 = g_2 \circ j_2$$

there exists a **unique** morphism $\phi : P \longrightarrow E$ such that

$$\phi \circ f_1 = g_1, \quad \phi \circ f_2 = g_2.$$

Remark: If P is a push-out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{j_1} & A_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & P \end{array}$$

is also known as a cocartesian square.

Theorem 23.5: If the push out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ exists in a given category, then it is unique.

Proof: If P' with morphisms $f_1' : A_1 \longrightarrow P'$ and $f_2' : A_2 \longrightarrow P'$ is another candidate we may apply the universal property to get a map $\phi : P \longrightarrow P'$ such that

$$\phi \circ f_1 = f_1', \quad \phi \circ f_2 = f_2'.$$

Reciprocally since P' is a push out, there is a map $\psi : P' \longrightarrow P$ such that

$$\psi \circ f_1' = f_1, \quad \psi \circ f_2' = f_2.$$

Combining we see that $(\psi \circ \phi) \circ f_1 = f_1$ and $(\psi \circ \phi) \circ f_2 = f_2$. We see that both $\psi \circ \phi$ and id_P satisfy the universal property with $E = P$, $g_1 = f_1$ and $g_2 = f_2$. The uniqueness clause in the definition gives $\psi \circ \phi = \mathrm{id}_P$. Likewise we get $\phi \circ \psi = \mathrm{id}_{P'}$ and the proof is complete.

Example: Let us now work in the category **Top** and recast the gluing lemma in terms of the push-out construction. Take a pair of open sets G_1, G_2 in a topological space X and the inclusions

$$j_1 : G_1 \cap G_2 \longrightarrow G_1, \quad j_2 : G_1 \cap G_2 \longrightarrow G_2.$$

The push out for this pair is the space $G_1 \cup G_2$ together with inclusion maps

$$i_1 : G_1 \longrightarrow G_1 \cup G_2, \quad i_2 : G_2 \longrightarrow G_1 \cup G_2$$

To see this suppose that Y is a topological space and $f_1 : G_1 \longrightarrow Y$ and $f_2 : G_2 \longrightarrow Y$ are a pair of continuous maps such that $f_1 \circ j_1 = f_2 \circ j_2$ then

$$f_1\Big|_{G_1 \cap G_2} = f_2\Big|_{G_1 \cap G_2}$$

The gluing lemma now says that there exists a unique map $\psi : G_1 \cup G_2 \longrightarrow Y$ such that

$$\psi\Big|_{G_1} = f_1, \quad \psi\Big|_{G_2} = f_2$$

which means $\psi \circ i_1 = f_1$ and $\psi \circ i_2 = f_2$ as desired. Instead of a pair of open subsets of a topological space one could choose a pair of closed sets.

Existence of push outs: We begin with the coproduct of A_1 and A_2 and perform some identifications. We examine the three categories **Gr**, **AbGr** and **Top** and show that the push-out exists in each of them. It may be noted that the popular term for the push out in the category of groups is *free product with amalgamation.*

Theorem 23.6: Push-outs exist in the categories **Gr**, **AbGr** and **Top**.

Proof: Let us begin with **Gr** and a given pair of morphisms $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$. Let G be the coproduct of the groups A_1 and A_2. We regard A_1 and A_2 as subgroups of G. Let N be the normal subgroup of G generated by

$$\{j_1(c) j_2(c)^{-1} / c \in C\}$$

and $\eta : G \longrightarrow G/N$ be the quotient map. We claim that G/N qualifies as the push-out with the associated homomorphisms

$$f_1 = \eta \circ i_1, \quad f_2 = \eta \circ i_2$$

where i_1 and i_2 are the inclusions of A_1, A_2 in G. Since $\eta(j_1(c)) = \eta(j_2(c))$, we see that $f_1 \circ j_1 = f_2 \circ j_2$. To check the universal property, let $g_1 : A_1 \longrightarrow H$ and $g_2 : A_2 \longrightarrow H$ be a pair of morphisms such that

$$g_1 \circ j_1 = g_2 \circ j_2 \tag{23.10}$$

Aside from (23.10), by definition of coproduct, there exists a unique homomorphism $\psi : G \longrightarrow H$ such that $\psi \circ i_1 = g_1$ and $\psi \circ i_2 = g_2$ from which follows easily that the kernel of ψ contains N. Let $\overline{\psi} : G/N \longrightarrow H$ be the unique map such that $\overline{\psi} \circ \eta = \psi$. Then

$$\overline{\psi} \circ \eta \circ i_1 = \psi \circ i_1 = g_1, \quad \overline{\psi} \circ \eta \circ i_2 = g_2.$$

which means $\overline{\psi} \circ f_1 = g_1$ and $\overline{\psi} \circ f_2 = g_2$. That completes the job of verifying that G/N is indeed the push-out. Note that we have only used the definition of coproducts and the most basic property of quotients. As a result the proof goes through verbatim for the other two situations as we shall see. Leaving aside the case of abelian groups we pass on to the category **Top**.

Well, changing notations to suit the need, let $h_1 : Z \longrightarrow X$ and $h_2 : Z \longrightarrow Y$ be a pair of continuous functions and $X \sqcup Y$ be the disjoint union of X and Y, and $i_1 : X \longrightarrow X \sqcup Y$, $i_2 : Y \longrightarrow X \sqcup Y$ be the canonical inclusions. For each $z \in Z$ we identify the points $(i_1 \circ h_1)(z)$ and $(i_2 \circ h_2)(z)$ in $X \sqcup Y$ and W be the quotient space with the projection map

$$\eta : X \sqcup Y \longrightarrow W = (X \sqcup Y)/\sim$$

We claim that W qualifies to be the push-out with associated morphisms $f_1 = \eta \circ i_1 : X \longrightarrow W$ and $f_2 = \eta \circ i_2 : Y \longrightarrow W$. To check the first condition observe that since $(i_1 \circ h_1)(z)$ and $(i_2 \circ h_2)(z)$ are identified, $\eta(i_1(h_1(z))) = \eta(i_2(h_2(z)))$ which means $f_1 \circ h_1 = f_2 \circ h_2$. Turning now to the universal property let $g_1 : X \longrightarrow T$ and $g_2 : Y \longrightarrow T$ be two continuous maps such that

$$g_1 \circ h_1 = g_2 \circ h_2. \tag{23.11}$$

Aside from (23.11), since $X \sqcup Y$ is the coproduct in **Top**, there is a unique continuous map $\psi : X \sqcup Y \longrightarrow T$ such that $\psi \circ i_1 = g_1$ and $\psi \circ i_2 = g_2$. Now (23.11) implies that ψ respects the identification and so there is a unique $\overline{\psi} : (X \sqcup Y)/\sim \longrightarrow T$ such that $\overline{\psi} \circ \eta = \psi$. By the universal property of the quotient, $\overline{\psi}$ is continuous and

$$\overline{\psi} \circ f_1 = \overline{\psi} \circ \eta \circ i_1 = \psi \circ i_1 = g_1,$$

and likewise $\overline{\psi} \circ f_2 = g_2$. That suffices for a proof.

Exercises

1. Show that the maps i_1 and i_2 in definition (23.1) are injective and that the images of i_1 and i_2 generate $G_1 * G_2$. Hint: Use the universal property with $H = G_1$, $f_1 = i_1$ and $i_2 = 1$.

2. Show that abelianizing a free group on k generators results in a group isomorphic to the direct sum of k copies of $\mathbb{Z}$. Use the fact that the coproduct in **AbGr** is the direct sum.

3. Is there a surjective group homomorphism from the direct sum $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$? Prove that if k and l are distinct positive integers, the free group on k generators is not isomorphic to the free group on l generators.

4. Show that $\langle a, c \mid a^2c^2 = 1\rangle$ is also a presentation of the fundamental group of the Klein's bottle.

5. Construct the push-out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ in the category **AbGr**?

6. Suppose that C is the trivial group in the definition of push-out in the category **Gr**, show that the resulting group is the coproduct of the two given groups. What happens in the category **AbGr**? Describe explicitly the construction of the group specifying the subgroup that is being factored out.

Lecture - XXV Adjunction Spaces

The notion of push-outs in the category **Top** leads to an important class of spaces known as adjunction spaces. We shall see that most of the important spaces encountered are adjunction spaces. This lecture may be regarded as one on important examples of topological spaces.

Definition 25.1: Given a topological space X, a closed subset A and a continuous map $A \longrightarrow B$ we define an equivalence relation on the disjoint sum (coproduct) $X \sqcup B$ as follows

$$b \sim x \quad \text{if and only if} \quad x \in A \text{ and } f(x) = b.$$

Thus a point $x \in A$ is identified with its image $f(x) \in B$. There are no other identifications besides this. The quotient space under this equivalence relation is called the adjunction space or the space obtained by attaching X to B via the map f. The space is denoted by $X \sqcup_f B$. Thus

$$X \sqcup_f B = (X \sqcup B)/\sim$$

The situation may be pictorially described as

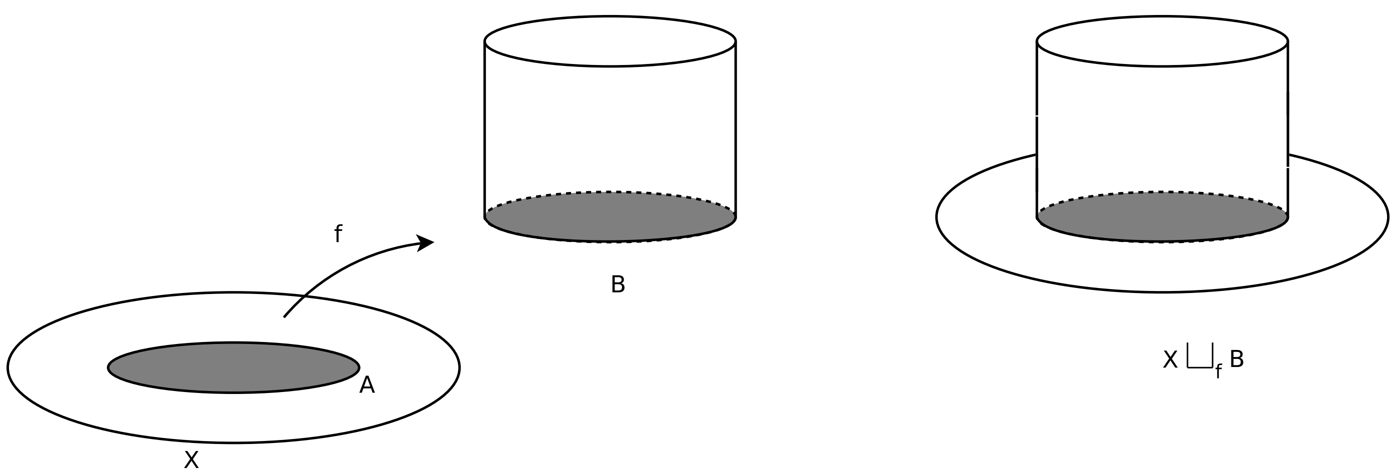


Figure 16: Adjunction Space

Example 25.1: Take $X = S^1$, $A = \{1\} \subset X$, $B = S^1$ and $f : A \longrightarrow B$ as $f(1) = 1$. The resulting space is the wedge of two circles $S^1 \vee S^1$.

Example 25.2 We now take $X = E^2$ the closed unit disc in the plane, $A = S^1$ the boundary of E^2, $B = \{1\}$ and f to be the constant map from S^1 to the singleton set B. The adjunction space is obtained by collapsing the boundary of E^2 to the single point B. The resulting space is S^2.

Before discussing further examples we relate this to the push-out construction.

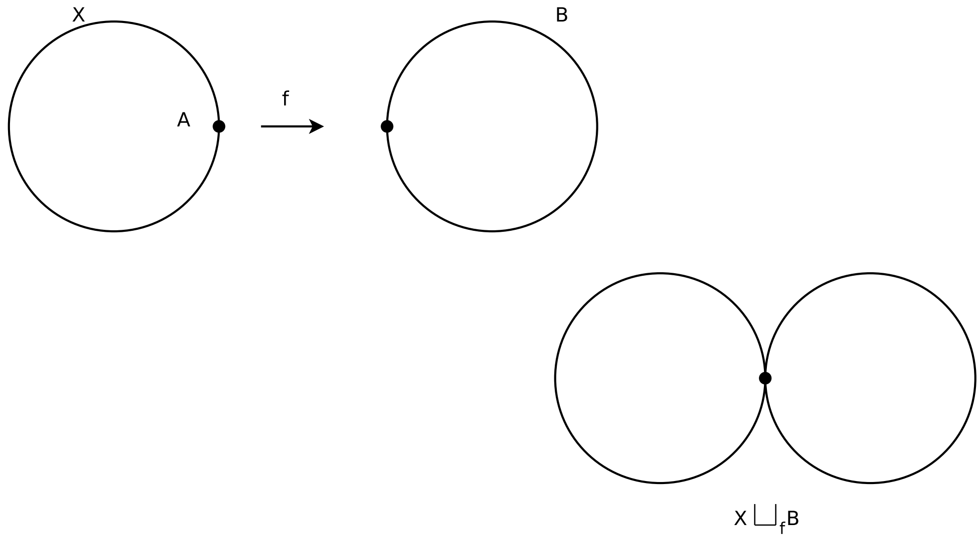


Figure 17: Wedge of two circles

Theorem 25.1: Let X and B be topological spaces, A be a closed subspace of X and $f : A \longrightarrow B$ be a continuous map. Then the space $X \sqcup_f B$ is the push-out for the following diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

where $i : A \longrightarrow X$ denotes the inclusion map.

Proof: We first define the associated maps $h_1 : X \longrightarrow X \sqcup_f B$ and $h_2 : X \longrightarrow X \sqcup_f B$. Let $\eta : X \sqcup B \longrightarrow X \sqcup_f B$ be the quotient map and $i_X : X \longrightarrow X \sqcup B$ and $i_B : B \longrightarrow X \sqcup B$ denote the inclusions. Then the associated maps h_1 and h_2 given by

$$h_1 = \eta \circ i_X, \quad h_2 = \eta \circ i_B. \tag{25.1}$$

For any $a \in A$ we have

$$h_1 \circ i(a) = \eta(i_X(a)) = \eta(a), \quad h_2 \circ f(a) = \eta(i_B(f(a))) = \eta(f(a))$$

Recalling the identifications we see that $h_1 \circ i = h_2 \circ f$. We now check the universal property. Suppose Z is a topological space and $g_1 : X \longrightarrow Z$, $g_2 : B \longrightarrow Z$ are continuous maps such that

$$g_1 \circ i = g_2 \circ f \tag{25.2}$$

Define the continuous map $\phi : X \sqcup B \longrightarrow Z$ as

$$\phi(x) = \begin{cases} g_1(x) & \text{if } x \in X \\ g_2(x) & \text{if } x \in B. \end{cases}$$

Condition (25.2) now shows that there is a unique map $\overline{\phi} : (X \sqcup_f B)/\sim \longrightarrow Z$ such that

$$\overline{\phi} \circ \eta = \phi. \tag{25.3}$$

The universal property of the quotient implies that $\overline{\phi}$ is continuous. Equations (25.1)-(25.3) immediately give

$$\overline{\phi} \circ h_1 = g_1, \quad \overline{\phi} \circ h_2 = g_2. \tag{25.4}$$

thereby completing the verification of the universal property.

Corollary 25.2: The square

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ f\downarrow & & \downarrow h_1 \\ B & \xrightarrow{h_2} & X \sqcup_f B \end{array}$$

is a push-out where h_1 and h_2 are defined as in (25.1).

Proof: This is just a summary of the details of the maps involved.

Definition 25.2: An $n-$cell is any space that is homeomorphic to the closed unit ball E^n in $\mathbb{R}^n$.
Thus the square I^2 is an example of a two cell and the hemisphere

$$\{(x_1, x_2, \ldots, x_n) \in S^{n-1}/x_n \geq 0\}$$

is an $n-1$ cell.

Example 25.3 (The torus and the Klein's bottle): We now show that the Klein's bottle and the torus are obtained by attaching a two cell to the figure eight space $S^1 \vee S^1$. In both cases we take $X = I^2$ to be the two cell, $A = \dot{I}^2$ the boundary of I^2 and $B = S^1 \vee S^1$ regarded as a subset of $S^1 \times S^1$ namely $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$. The distinguishing factor is that the attaching map $f : A \longrightarrow B$ is different in the two cases.

1. For the torus we define $f : A \longrightarrow B$ to be the continuous surjection

$$\begin{aligned} f(x,1) &= f(x,0) = (e^{2\pi ix}, 1), \quad x \in [0,1] \\ f(1,y) &= f(0,y) = (1, e^{2\pi iy}), \quad y \in [0,1] \end{aligned}$$

It is geometrically clear that $X \sqcup_f B$ is a torus but we demonstrate this formally owing to the importance of the type of argument involved. Let $p : I^2 \longrightarrow S^1 \times S^1$ be the quotient map, $i_X : X \longrightarrow X \sqcup B$ the inclusion map and $\eta : X \sqcup B \longrightarrow X \sqcup_f B$ the quotient map. The map

$$\phi : S^1 \times S^1 \longrightarrow X \sqcup_f B$$

given by $\phi(\exp(2\pi ix), \exp(2\pi iy)) = (\eta \circ i_X)(x,y)$ is well-defined, bijective and its continuity follows from the fact that $\phi \circ p = i_X \circ \eta$ and $i_X \circ \eta$ is continuous. Finally the compactness of $S^1 \times S^1$ and the fact that $X \sqcup_f B$ is Hausdorff shows that ϕ is a homeomorphism.

2. The argument for the Klein's bottle proceeds along similar lines and we merely indicate the attaching map $f : I^2 \longrightarrow S^1 \vee S^1$ namely,

$$\begin{aligned} f(x,1) &= f(x,0) = (e^{2\pi i x}, 1), \quad x \in [0,1] \\ f(1,y) &= f(0,1-y) = (1, e^{2\pi i y}), \quad y \in [0,1]. \end{aligned}$$

3. It is sometimes convenient to take the closed unit disc E^2 as the two cell. But the attaching map $f : S^1 \longrightarrow S^1 \vee S^1$ would be slightly more complicated to write down. For the Klein's bottle the attaching map is given by

$$f(z) = \begin{cases} (z^4, 1) & 0 \leq \arg z \leq \pi/2 \\ (1, z^4) & \pi/2 \leq \arg z \leq \pi \\ (-\overline{z}^4, 1) & \pi \leq \arg z \leq 3\pi/2 \\ (1, -\overline{z}^4) & 3\pi/2 \leq \arg z \leq 2\pi \end{cases} \tag{25.5}$$

For the torus the attaching map is obtained from (25.5) by suppressing the negative signs in the last two expressions. The student is invited to work out a similar construction for the double torus as well.

Example 25.4 (The projective plane): This is obtained by attaching a two cell to the circle. For the two cell we take the closed unit disc E^2 in the complex plane and its boundary as A. The attaching map is given by $f(z) = z^2$. We leave it to the reader to prove that the resulting adjunction space is indeed $\mathbb{R}P^2$.

Example 25.5 (Real projective spaces): We take the space X to be the closed unit disc E^n in $\mathbb{R}^n$ and A as its boundary. The space B is the lower dimensional projective space $\mathbb{R}P^{n-1}$. The attaching map is the quotient map $p : S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$. We leave the proof of the following result to the reader.

Theorem 25.3: The space $E^n \sqcup_p \mathbb{R}P^{n-1}$ is homeomorphic to the real projective space $\mathbb{R}P^n$. Thus $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell.

Definition 25.3 (The cone over a space): Let X be a topological space. The cone $C(X)$ over X is the quotient space

$$C(X) = (X \times [0,1])/(X \times \{0\})$$

We have an obvious inclusion map $i : X \longrightarrow C(X)$ given by $i(x) = [x, 1]$ where the square bracket

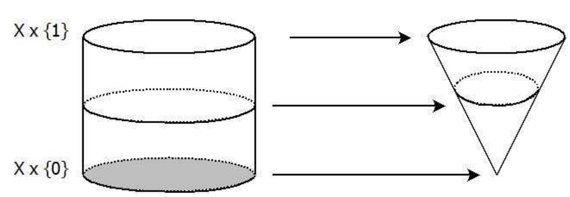


Figure 18: Cone over a space

refers to the image of $(x, 1) \in X \times [0,1]$ in the quotient $C(X)$.

Theorem 25.5: A continuous map $f : X \longrightarrow Y$ is homotopic to a constant map if and only if f extends continuously to a map $F : C(X) \longrightarrow Y$ namely $F \circ i = f$.

Proof: The proof writes itself out. Suppose that $G : X \times [0,1] \longrightarrow Y$ is a homotopy between f and the constant map taking the value y_0 say,

$$G(x,1) = f(x), \quad G(x,0) = f(y_0), \qquad \text{for all } \; x \in X.$$

The second equation in (25.12) says that G respects the identification made on $X \times [0,1]$ to yield $(X \times [0,1])/(X \times \{0\})$ whereby we conclude the existence of a map $F : C(X) \longrightarrow Y$ satisfying $F \circ \eta = G$. This map F is continuous by the universal property and the first equation in (25.12) gives $F[x,1] = G(x,1) = f(x)$. The proof of necessity is complete.

Conversely suppose given a continuous map $f : X \longrightarrow Y$ such that there is a $G : C(X) \longrightarrow Y$ with $F \circ i = G$. Denoting by η the quotient map $X \times [0,1] \longrightarrow C(X)$, the map $G \circ \eta$ provides a homotopy between f the constant map. $\square$

Exercises

1. We have obtained S^2 by attaching E^2 to a singleton with the attaching map as the constant map on the boundary of E^2. Discuss how would you obtain S^n analogously as an adjunction space.

2. Show that if X and B are connected/path-connected then $X \sqcup_f B$ is connected/path-connected.

3. Describe the push out resulting from the diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{i_1} & E^n \\ {\scriptstyle i_2}\big\downarrow & & \\ E^n & & \end{array}$$

4. Show that $S^m \times S^n$ results from attaching an $n+m$ cell to $S^n \vee S^m$. Hint: Let I denote $[0,1]$ and define a map $f : \partial(I^n \times I^m) \longrightarrow S^n \vee S^m$ as follows

$$f(z) = \begin{cases} (\eta_1(x), y_0) & \text{if } \; x \in \partial I^n \\ (x_0, \eta_2(y)) & \text{if } \; y \in \partial I^m \end{cases}$$

and $\eta_1 : I^n \longrightarrow S^n$ and $\eta_2 : I^m \longrightarrow S^m$ are the quotient maps of exercise 1.

5. Prove theorem (25.3).

6. Fill in the details in examples (25.4) and (25.5).

Lecture - XXVI Seifert Van Kampen theorem and knots

This is one of the most famous theorem concerning the fundamental group which serves as a tool for computations and applications to combinatorial group theory. If U and V are path connected open subsets of a topological space such that $U \cap V$ is path connected, the theorem provides information on the geometry of $U \cup V$ in terms of the geometry of U, V and $U \cap V$. In precise terms it states that the π_1 functor maps the push-out diagram of pointed topological spaces with $x_0 \in U \cap V$,

$$\begin{array}{ccc} (U \cap V, x_0) & \xrightarrow{i_1} & (U, x_0) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ (V, x_0) & \xrightarrow{j_2} & (U \cup V, x_0) \end{array}$$

to the push-out diagram of groups:

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i_{1*}} & \pi_1(U, x_0) \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle j_{1*}} \\ \pi_1(V, x_0) & \xrightarrow{j_{2*}} & \pi_1(U \cup V, x_0) \end{array}$$

thereby giving a precise description of the group $\pi_1(U \cup V, x_0)$ in terms of the groups $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$. Thus $\pi_1(U \cup V, x_0)$ is the free product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ amalgamated along $\pi_1(U \cap V, x_0)$. The theorem enables us to calculate quickly the fundamental groups of several important spaces.

Theorem 26.1 (Seifert and Van Kampen - version I): Let U, V be open path connected subsets of a topological space such that $U \cap V$ is path connected. Let $x_0 \in U \cap V$ and $i_1 : U \cap V \longrightarrow U$, $i_2 : U \cap V \longrightarrow V$ denote the inclusion maps. Then $\pi_1(U \cup V, x_0)$ is the free product (coproduct) of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ amalgamated along $\pi_1(U \cap V, x_0)$ with respect to the maps i_{1*} and i_{2*}. That is to say if N is the normal subgroup

$$N = \langle i_{1*}[\gamma](i_{2*}[\gamma])^{-1} : [\gamma] \in \pi_1(U \cap V, x_0) \rangle \tag{26.1}$$

then the fundamental group of $U \cup V$ is given by

$$\pi_1(U \cup V, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0)/N. \tag{26.2}$$

Considering $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ as subgroups of $\pi_1(U) * \pi_1(V)$, their images in the quotient group generate $\pi_1(U \cup V, x_0)$.

The result may be elegantly stated using a push-out diagram namely,

Theorem 26.2 (Seifert and Van Kampen - version II): Let U, V be open path connected subsets of a topological space such that $U \cap V$ is path connected. Let $x_0 \in U \cap V$ and $i_1 : U \cap V \longrightarrow U$, $i_2 : U \cap V \longrightarrow V$ denote the inclusion maps. Then the push-out data

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i_{1*}} & \pi_1(U, x_0) \\ {\scriptstyle i_{2*}}\big\downarrow & & \\ \pi_1(V, x_0) & & \end{array}$$

may be completed to yield the push-out square

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i_{1*}} & \pi_1(U, x_0) \\ {\scriptstyle i_{2*}}\big\downarrow & & \big\downarrow{\scriptstyle j_{1*}} \\ \pi_1(V, x_0) & \xrightarrow{j_{2*}} & \pi_1(U \cup V, x_0) \end{array}$$

where the maps $j_1 : U \longrightarrow U \cup V$ and $j_2 : V \longrightarrow U \cup V$ are inclusions.

The proof is neatly presented on pages 110-113 of the book by J. Vick and need not be repeated here. Instead we move on to its applications to the computation of the fundamental groups of certain spaces.

Corollary 26.3: Suppose that U, V are open path-connected, simply connected subsets of a topological space such that $U \cap V$ is path connected then $U \cup V$ is simply connected.

Fundamental groups of spheres: An important example of this is the case $U = S^n - \{\mathbf{e}_n\}$ and $V = \{\mathbf{e}_n\}$. When $n \geq 2$, the spaces U and V are homeomorphic to $\mathbb{R}^n$ via the stereo-graphic projection and since $U \cap V$ is path connected we conclude that $U \cup V = S^n$ is simply connected.

Corollary 26.4: Suppose that U, V are open path-connected subsets of a topological space such that $U \cap V$ is simply connected then

$$\pi_1(U \cap V, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0).$$

Wedge of two circles: Let us consider the space $S^1 \vee S^1$ given by the union of two circles of radius one in the plane touching each other externally at the origin. We take U and V to be the open sets obtained by deleting one of the points of each lobe (not the common point!). Then the circle is a deformation retract of both U and V and $U \cap V$ deformation retracts to the origin. Thus

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}. \tag{26.3}$$

The last clause in theorem (26.1) also provides the generators of the fundamental group. Assuming the circles to centered at ± 1, the generators are given by the homotopy classes of the loops

$$\pm 1 + \exp(2\pi i t) \tag{26.4}$$

The generalization to a wedge of n circles is left as an exercise.

Corollary 26.5 Suppose that U, V are open path-connected subsets of a topological space such that $U \cap V$ and U are simply connected then with a base point $x_0 \in U \cap V$,

$$\pi_1(U \cup V, x_0) = \pi_1(V, x_0).$$

We turn to an important example to illustrate the use of this corollary. Regard $\mathbb{R}^3$ as a subset of S^3 via the stereo-graphic projection and K be a compact subset of $\mathbb{R}^3$ such that the complement $\mathbb{R}^3 - K$ is connected. We then have the following result.

Theorem 26.6: $\pi_1(S^3 - K) = \pi_1(\mathbb{R}^3 - K)$.

Proof: Let $p \in S^3$ denote the north-pole using which we project $S^3 - \{p\}$ stereo-graphically onto $\mathbb{R}^3$. Since K is compact there is a neighborhood U of p in S^3 homeomorphic to a ball which does not intersect K. Taking $V = S^3 - (K \cup \{p\}) = \mathbb{R}^3 - K$, we see that $U \cup V = S^3 - K$ and $U \cap V$ deformation retracts to S^2. The result now follows from the previous theorem.

Corollary 26.7: Suppose that U, V are open path-connected subsets of a topological space such that U is simply connected and $i : U \cap V \longrightarrow V$ is the inclusion map then, taking a base point $x_0 \in U \cap V$,

$$\pi_1(U \cup V, x_0) = \pi_1(V, x_0)/\langle \text{Im } i_* \rangle,$$

where $\langle \text{Im } i_* \rangle$ denotes the normal subgroup generated by the image of i_*.

Proof: The subgroup N in (26.1) reduces to $\langle \text{Im } i_* \rangle$.

The projective plane: We work this example out in meticulous detail. Such details will be progressively cut down and left for the students to fill in as we go along. The projective plane $\mathbb{R}P^2$ is obtained by attaching a two cell E^2 to S^1 using the map given in complex form as $f(z) = z^2$. Let p denote the center of E^2 and $\eta : E^2 \longrightarrow \mathbb{R}P^2$ be the quotient map. Taking U to be the interior of E^2 and $V = \mathbb{R}P^2 - \{p\}$ we apply corollary (26.7). For computing the image of i_* we take a generator for the infinite cyclic group $\pi_1(U \cap V, y_0)$ with base point $y_0 = 1/2$. The generator is the equivalence class of the loop

$$\gamma(t) = \frac{1}{2}\exp(2\pi i t), \quad 0 \leq t \leq 1. \tag{26.5}$$

We also need a base point x_0 sitting on the loop Γ given by

$$\Gamma(t) = \eta(\exp(i\pi t)), \quad 0 \leq t \leq 1, \tag{26.6}$$

which generates $\pi_1(\mathbb{R}P^2 - \{p\}, x_0)$. Taking a path β joining y_0 and x_0 we get a generator for the infinite cyclic group $\pi_1(\mathbb{R}P^2 - \{p\}, y_0)$ namely, the class of the loop $\beta * \Gamma * \beta^{-1}$. Having set the stage we are ready to compute $i_*[\gamma]$ namely, the homotopy class of the loop γ in $\mathbb{R}P^2 - \{p\}$. This loop γ based at y_0 is homotopic to the loop

$$\beta * \Gamma * \Gamma * \beta^{-1}. \tag{26.7}$$

The required homotopy is $\eta \circ F$ where F is a map of a rectangle onto a suitable annulus (see exercise (1)). Introducing a $\beta^{-1} * \beta$ we get

$$i_*[\gamma] = [\beta * \Gamma * \beta^{-1}][\beta * \Gamma * \beta^{-1}] \tag{26.8}$$

or in additive notation it is the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by $n \mapsto 2n$. We conclude from corollary (26.7) that $\pi_1(\mathbb{R}P^2)$ is the cyclic group of order two.

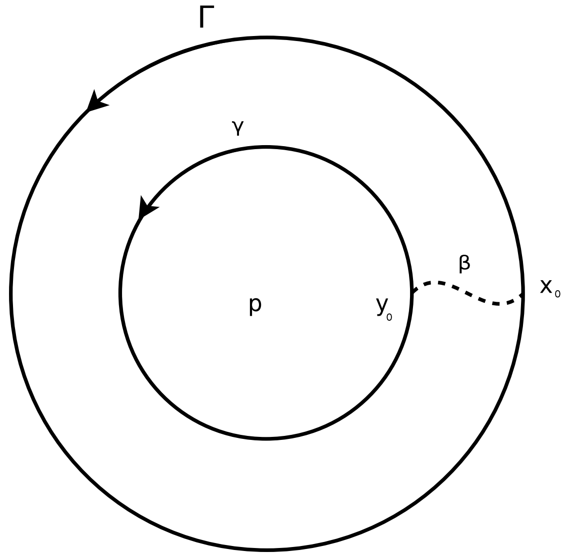


Figure 19: Computing $\pi_1(\mathbb{R}P^2)$

The torus and the Klein's bottle: We proceed along the same lines using the convenient form (25.5). Denoting by X either the torus or the Klein's bottle and p to be the origin, we see that $X-\{p\}$ deformation retracts to the figure eight loop and, in analogy with (26.6), the generators for the free group $\pi_1(X-\{p\})$ are given by

$$\Gamma_1(t) = \eta(\exp(i\pi t/2)), \quad \Gamma_2(t) = \eta(\exp(i\pi(t+1)/2)), \quad 0 \leq t \leq 1$$

We take U to be the open unit disc, V to be $X - \{p\}$ and the class of (26.5) as the generator for $\pi_1(U \cap V, y_0)$ where the base point y_0 is 1/2. Taking an auxiliary path β joining y_0 and the point $x_0 = 1$ common to both $\Gamma_1(t)$ and $\Gamma_2(t)$, we get the generators

$$[\beta * \Gamma_1 * \beta^{-1}] \quad \text{and} \quad [\beta * \Gamma_2 * \beta^{-1}] \tag{26.9}$$

for $\pi_1(X - \{p\}, y_0)$. The deformation of the previous example (exercise (1)) can be employed here again and this time we get

$$i_*[\gamma] = [\beta * \Gamma_1 * \beta^{-1}][\beta * \Gamma_2 * \beta^{-1}][\beta * \Gamma_1^{-1} * \beta^{-1}][\beta * \Gamma_2^{-1} * \beta^{-1}] \tag{26.10}$$

for the torus whereas for the Klein's bottle we get instead

$$i_*[\gamma] = [\beta * \Gamma_1 * \beta^{-1}][\beta * \Gamma_2 * \beta^{-1}][\beta * \Gamma_1 * \beta^{-1}][\beta * \Gamma_2^{-1} * \beta^{-1}] \tag{26.11}$$

One could also work with the other models described in example (25.3) where the spaces are obtained by identifying the opposite edges of a square. The homotopy $\eta \circ F$ of the last example would have to be modified to $\eta \circ G \circ F$ where G is a certain homeomorphism from the unit disc onto the square $[0,1] \times [0,1]$.

Denoting the generators (26.9) of $\pi_1(V, y_0)$ by S and T we are ready to apply corollary (26.7) since (26.10) gives us the image of the map i_*. The fundamental group of the torus is then

$$\langle S, T : ST = TS \rangle \cong \mathbb{Z} \times \mathbb{Z} \tag{26.12}$$

and the fundamental group of the Klein's bottle is

$$\langle S, T : TST = S \rangle \cong \mathbb{Z} \ltimes \mathbb{Z}. \tag{26.13}$$

The double torus: By writing out the attaching map $S^1 \longrightarrow S^1 \vee S^1 \vee S^1 \vee S^1$ akin to (25.5) or else using the identification of the sides of a regular octagon as described in lecture 4, the reader is invited to prove that the fundamental group of the double torus is

$$\langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} = 1 \rangle \tag{26.14}$$

Fundamental groups of some adjunction spaces: The method used in the last few examples may be adapted to prove a general theorem about the fundamental group of the adjunction space $X \sqcup_f E^k$ obtained by attaching E^k to a given space X via a map $f : S^{k-1} \longrightarrow X$. As in the case of the projective plane, Klein's bottle and torus the crucial point is to obtain some specific information about the induced map f_*. We shall merely state the result and suppress the proof.

Theorem 26.8: Let $X \sqcup_f E^k$ be the space obtained by attaching a k cell to a path connected space X via a map $f : S^{k-1} \longrightarrow X$. Then for any choice of base point in $f(S^{k-1})$,

(i) $\pi_1(X \sqcup_f E^k, x_0) = \pi_1(X, x_0)$ if $k \geq 3$.

(i) $\pi_1(X \sqcup_f E^2, x_0) = \pi_1(X, x_0)/\langle \text{im } f_* \rangle$.

Exercises

1. Fill in the details in the computation of the fundamental group of the projective plane, Klein's bottle and the torus done in the lecture by providing a careful proof of equations (26.8), (26.10) and (26.11). Hint: Use polar coordinates. Continuously shrink the path β to the point x_0.

2. Show that the fundamental group of the wedge of n copies of S^1 is the free group on n generators. Calculate the fundamental group of the truncated grid
$$\{(x, y) \in \mathbb{R}^2 / x \in \mathbb{Z} \text{ or } y \in \mathbb{Z},\ 0 \leq x \leq n,\ 0 \leq y \leq n\}.$$

3. Determine the generators of double torus by expressing it as a union of open sets each of which is a torus from which a tiny closed disc has been removed.

4. Let C be the union of the two *unlinked* circles
$$\begin{aligned} (x-2)^2 + y^2 &= 1,\ z = 0, \\ (x+2)^2 + y^2 &= 1,\ z = 0. \end{aligned}$$
in $\mathbb{R}^3$. Show that $\pi_1(\mathbb{R}^3 - C)$ is the free group on two generators.

5. Calculate the fundamental groups of the following spaces
   - (i) $\mathbb{R}^4$ minus a line.
   - (ii) $\mathbb{R}^4$ minus a two dimensional linear subspace.
   - (iii) $\mathbb{R}^4$ minus two parallel lines.
   - (iv) $\mathbb{R}^4$ minus two intersecting lines.

(v) $\mathbb{R}^3$ minus the coordinate axes

(vi) $\mathbb{C}^2 - \{(z_1, z_2)/z_1 z_2 = 0\}$

(vii) $\mathbb{R}^3$ minus finitely many points.

Lecture - XXVII (Test IV)

1. Use the Seifert Van Kampen theorem to compute the fundamental group of the double torus.
2. Let K be a compact subset of $\mathbb{R}^3$ and regard S^3 as the one point compactification of $\mathbb{R}^3$. Show that $\pi_1(\mathbb{R}^3 - K) = \pi_1(S^3 - K)$.
3. If C is the circle in $\mathbb{R}^3$ given by the pair of equations

$$x^2 + z^2 = 1, \quad z = 0,$$

   show that $\pi_1(\mathbb{R}^3 - C) = \mathbb{Z}$. Let C' be the circle given by

$$(y-1)^2 + z^2 + 1, \quad x = 0.$$

   Show that $\pi_1(\mathbb{R}^3 - C \cup C') = \mathbb{Z} \oplus \mathbb{Z}$. Hint: Use stereographic projection.
4. Show that the complement of a line in $\mathbb{R}^4$ is simply connected.
5. Calculate the fundamental group of $\mathbb{C}^2 - \{(z_1, z_2)/z_1 z_2 = 0\}$.

Solutions to Test IV

1. The procedure parallels the computation of the fundamental groups of the projective plane, Klein's bottle and the torus worked out in lecture 26.

2. See theorem (26.6)

3. First recall the details of stereographic projection from lecture 2. Observe that S^3 minus a is homeomorphic to $\mathbb{R}^3$ and by symmetry any point may be chosen as the *north pole* to carry out the stereographic projection. It is easily verified that under the stereographic projection a circle (intersection of $S^3 \subset \mathbb{R}^4$ with a two dimensional linear subspace of $\mathbb{R}^4$) through the north pole is mapped onto a straight line in $\mathbb{R}^3 \times \{-1\}$. Regard S^3 as the one point compactification of $\mathbb{R}^3$ with q as the point at infinity. By theorem (26.6),

$$\pi_1(\mathbb{R}^3 - C) = \pi_1(S^3 - C).$$

To calclulate $\pi_1(S^3 - C)$, we take a point $p \in C$ as the point at infinity and apply theorem (26.6). Check that the conditions of theorem (26.6) are satisfied. Then $S^3 - C$ is homeomorphic to $\mathbb{R}^3$ minus a straight line and this deformation retracts to S^1. Hence

$$\pi_1(S^3 - C) = \mathbb{Z}.$$

For the second part the same argument shows that

$$\pi_1(\mathbb{R}^3 - C \cup C') = \pi_1(\mathbb{R}^3 - (C' \cup Z))$$

where $Z = \{(x, y, 0)\ ;\ x, y \in \mathbb{R}\}$.

4. Take the line to be the $\{(x, 0, 0, 0)\ :\ x \in \mathbb{R}\}$ and denote by X the complement of the line in $\mathbb{R}^4$. The map $F : X \times [0, 1] \longrightarrow X$ given by

$$F(\mathbf{x}, t) = \frac{t\mathbf{x}}{\|\mathbf{x}\|} + (1 - t)\mathbf{x}$$

is a deformation retraction from X onto S^3 minus a circle and so $\pi_1(X) = \mathbb{Z}$.

5. The argument used in the previous example shows that the space in question deformation retracts to S^3 minus two simply linked circles and so the fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$.

Lecture - XXVIII Introductory remarks on homology theory

In the first part of the course we focused on the fundamental group and its basic properties. We discussed an elegant solution of the lifting problem for covering projections in terms of the fundamental group. While the theory of fundamental groups and covering spaces is fairly adequate for many applications in low dimensional geometry and other parts of mathematics such as the theory of function of one complex variable, it is quite ineffective when higher dimensional objects are involved. For instance the ball B^n and the sphere S^{n-1} both have trivial fundamental group ($n \geq 3$) which renders it useless for proving the higher dimensional analogues of the Brouwer's fixed point theorem.

Homology theory provides a functor that is quite convenient for understanding the geometry of "higher dimensional objects" which has the added advantage of being easily computable (at least for a large class of interesting spaces). While the fundamental group functor respect products, the homology groups of $X \times Y$ are not so easily described in terms of the homology groups of X and Y. A covering projection is a very special case of a fiber bundle with discrete fibers. We have seen that in the case of a covering projection $p : \tilde{X} \longrightarrow X$ we have a relationship between $\pi_1(X)$ and $\pi_1(\tilde{X})$. The story is decidedly more complicated with homology groups. For instance some work is required to compute the homology groups of the real projective spaces $\mathbb{R}P^n$. Homotopy theory is better suited for studying fibrations where the use of homology would entail the formidable machinery of "spectral sequences". However, on the computational side there is a very useful substitute for the Seifert Van Kampen theorem in homology known as the Mayer Vietoris sequence. We shall use it to calculate efficiently the homology groups of a large number of spaces.

There are several approaches to the homology theory, the oldest being the simplicial theory. Homology theory evolved over several decades through the early part of the twentieth century becoming progressively abstract.

The theory we discuss in this course is known as the singular homology theory and would appear somewhat non-intuitive in the beginning but we hope that the examples and applications presented would enable the students to digest the material. Singular homology theory appeared rather late in the development of algebraic topology and is a culmination of efforts spanning a few decades by several eminent topologists. In the intervening years several seemingly different homology theories developed the oldest and most intuitive being simplicial homology theory that applies to the restricted class of simplicial complexes. However the topological invariance is highly non-trivial and beset with technical complications.

Some motivation for singular homology: Let us recall some of the notions in the theory of contour integrals in elementary complex analysis. Given a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$ one defines a line integral

$$\int_\gamma f(z)dz \tag{28.1}$$

along a path[5] $\gamma : [a, b] \longrightarrow \Omega$ lying in the domain Ω. If the path γ is the juxtaposition of several paths $\gamma_1, \gamma_2, \dots, \gamma_k$ then one knows that

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_k} f(z)dz \tag{28.2}$$

[5]The path would have to satisfy some regularity condition such as being piecewise continuously differentiable. However since this is merely supposed to be a motivation we shall brush aside these technicalities.

Thus one can *break* the path γ into several pieces, compute the integral over the individual pieces and add the results. One can also reparametrize the pieces and regard all the pieces γ_j as being maps from $[0,1]$. In view of all these, it seems meaningful to write

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k \tag{28.3}$$

in place of

$$\gamma_1 * \gamma_2 * \cdots * \gamma_k.$$

We see that the rigidity present in the theory of the fundamental group where one deals with homotopy classes of loops all of which are based at a given point, is now significantly relaxed.

Also, one checks that integration along the inverse path reverses the sign:

$$\int_{\gamma^{-1}} f(z)dz = -\int_\gamma f(z)dz \tag{28.4}$$

Taking a specific example with $f(z) = 1/z$ and integrating along two concentric circles γ_1, γ_2 traced counter clockwise, we see that

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz. \tag{28.5}$$

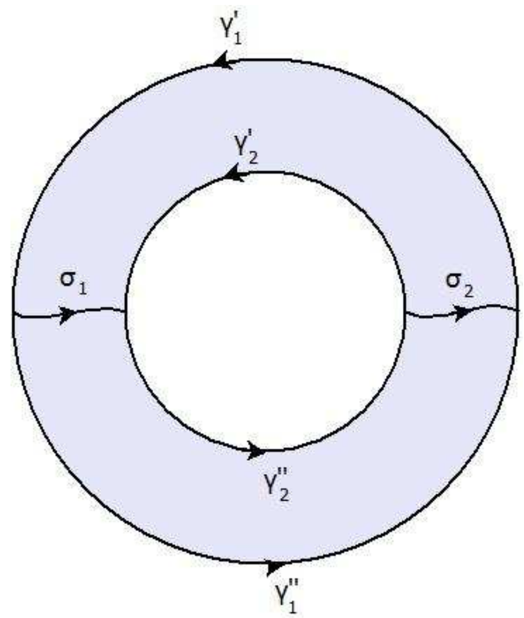


Figure 20:

Using (28.1) and (28.4) this may be rewritten as

$$\int_{\gamma_1 - \gamma_2} f(z)dz = 0, \tag{28.6}$$

where, in keeping with the additive notation (28.3) we have written $-\gamma_2$ in place of γ_2^{-1}. Equation (28.6) is interesting since γ_1 and γ_2 are the two pieces of the boundary of the annular region A bounded by them, where the function f is holomorphic. Equation (28.6) suggests that the two paths γ_1 and

γ_2 ought to be regarded as being *equivalent* with regard to f or more precisely with regard to A since nothing changes if f is replaced by any other function holomorphic in an neighborhood of A. However (28.6) fails for $f(z) = (z-p)^{-1}$, where p is any point in the interior of A. This is a reflection of the fact that the paths γ_1 and γ_2 do not constitute the full boundary of the punctured annulus $A - \{p\}$ which is where $(z-p)^{-1}$ is holomorphic.

In doing contour integrals one occasionally introduces auxiliary paths such as σ_j $(j = 1, 2)$ indicated in the figure below and writes the integral (28.6) over $\gamma_1 - \gamma_2$ as the sum

$$(\gamma_1' + \sigma_1 - \gamma_2' + \sigma_2) + (\gamma_1'' - \sigma_1 - \gamma_2'' - \sigma_2) \tag{28.7}$$

Each of the two parenthesis indicates a boundary of one of the halves of the annulus and so each ought to *equivalent* to a null path or in other words, the equivalence of γ_1 and γ_2 translates to $\gamma_1 - \gamma_2$ being equivalent to a *null* path. We write $\gamma_1 \sim \gamma_2$ to indicate the equivalence of $\gamma_1 - \gamma_2$ to a null-path.

These considerations suggest an underlying *calculus of paths* bounding regions in the plane. Indeed homology theory does develop such a calculus of paths as well as its higher dimensional analogues. Perhaps the student has encountered these higher dimensional analogues in connection with the Gauss' divergence theorem in vector calculus[6].

Note that the sum indicated in (28.3) is a formal sum we are lead to the free abelian group generated by the set of all piecewise smooth functions from $[0, 1]$ to Ω called the group of one chains. Thus $\gamma_1 - \gamma_2$ in (28.6) and $\gamma_1 + \gamma_2 + \cdots + \gamma_k$ displayed in (28.3) are examples of one chains. Note that the one chain appearing in (28.2) is *different* from γ though in the final stage of construction they would be identified. The Cauchy theory suggests that the chains whose pieces are all closed curves would play a distinguished role and these are examples of one cycles - a certain subgroup of the group of chains called the group of one cycles Z_1. If a chain such as $\gamma_1 - \gamma_2$ appearing in equation (28.6) is the oriented boundary of a sub-domain we would regard it as being equivalent to zero and we would call such chains as boundaries. These form a subgroup of Z_1 known as the group of boundaries B. The equivalence relation is thus $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 - \gamma_2 \in B$. Passing to the quotient of Z via this equivalence relation or in algebraic terms, passing to the quotient group Z/B would give us the first homology group of the space Ω. All these heuristics are rigorously defined in the next couple of lectures. We shall of course have to dispense with the notion of piecewise smoothness and talk of continuous paths $\gamma : [0, 1] \longrightarrow X$ called *singular one simplexes* and and their formal linear combinations with integer coefficients called *singular one chains* . To develop a calculus of higher dimensional chains, one has the option of introducing *singular cubes* namely continuous maps $[0, 1]^n \longrightarrow X$, which is the approach taken by W. Massey. This however necessitates certain preliminary reductions but has some distinct advantages later particularly in applications of homology theory to the study of homotopy groups. We shall follow the traditional approach, as in J. Vick's book and use singular simplices instead.

[6]For a discussion along the lines of vector calculus see [11]

Lectures - XXIX/XXX The singular chain complex and homology groups

The program of developing a calculus of chains is now formalized in this lecture. We introduce a new algebraic category of chain complexes and maps between them and prove the fundamental theorem about these algebraic gadgets. In particular, to each chain complex is associated a sequence of groups called the homology groups. Given a topological space X we associate a chain complex to it and obtain the homology functors from the category **Top** to the category **AbGr**. Thus we lay in this lecture the foundations for a systematic calculus of chains and cycles putting the heuristic ideas of the last lecture on a rigorous footing.

Definition 29.1 (The standard simplex): The standard $n-$simplex denoted by Δ_n is the convex hull of the $n+1$ the standard unit vectors in $\mathbb{R}^{n+1}$. Denoting the standard unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}$, their convex hull is the set

$$\Delta_n = \{(t_1, t_2, \ldots, t_{n+1}),\ \ t_1 \geq 0,\ t_2 \geq 0, \ldots, t_{n+1} \geq 0,\ t_1 + t_2 + \cdots + t_{n+1} = 1\}.$$

We take the standard zero simplex Δ_0 to be the point $\mathbf{e}_1$.

Thus Δ_2 is the equilateral triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ and the one simplex Δ_1 is the line segment in $\mathbb{R}^2$ joining the points $(1,0)$ and $(0,1)$.

Note that Δ_2 contains three copies of Δ_1 namely the sides of the equilateral triangle. Likewise Δ_3 contains four copies of Δ_2, the four faces of the regular tetrahedron. To formalize this idea, we introduce $(n+1)$ affine maps $\Delta_{n-1} \longrightarrow \Delta_n$ called the face maps. For $i = 1, 2, 3$, the $i-$th *face* of Δ_2 is the face opposite to vertex $\mathbf{e}_i$ and consists of all points (t_1, t_2, t_3) with non-negative entries and $t_1 + t_2 + t_3 = 1$ such that the $i-$th coordinate t_i vanishes.

Now suppose that $(t_1, t_2, \ldots, t_{n+1})$ denotes denotes a typical point on the last face of Δ_n. Then since t_{n+1} vanishes, we see that $(t_1, t_2, \ldots, t_n)$ is a typical point on Δ_{n-1}. Turning the argument around we define the map

$$\begin{aligned} \Delta_{n-1} &\longrightarrow \Delta_n \\ (t_1, t_2, \ldots, t_n) &\mapsto (t_1, t_2, \ldots, t_n, 0), \end{aligned}$$

where the t_i are all non-negtive and $\sum t_i = 1$, and call it the standard n-th face map. The $i-$th face map $(0 \leq i \leq n)$ would be

$$\begin{aligned} \Phi_i^n : \Delta_{n-1} &\longrightarrow \Delta_n \\ (t_1, t_2, \ldots, t_n) &\mapsto (t_1, t_2, \ldots, t_{i-1}, 0, t_i, \ldots, t_n), \end{aligned} \tag{29.1}$$

We leave it to the reader to write down explicitly the maps $\Phi_j^n \circ \Phi_i^{n-1} : \Delta_{n-2} \longrightarrow \Delta_n$ and prove the following result:

Lemma 29.1: Suppose that $0 \leq j < i \leq n$ then

$$\Phi_j^n \circ \Phi_{i-1}^{n-1} = \Phi_i^n \circ \Phi_j^{n-1} \tag{29.2}$$

Definition 29.2 (Singular chains): A singular $n-$simplex in a topological space X is a continuous map $\sigma : \Delta_n \longrightarrow X$. The free abelian group generated by the set of all singular $n-$simplices in X is called the group of singular $n-$chains in X. This group is denoted by $S_n(X)$ and a typical element of $S_n(X)$ is thus a formal sum

$$n_1\sigma_1 + n_2\sigma_2 + \cdots + n_k\sigma_k, \tag{29.3}$$

where the coefficients $n_1, n_2, \ldots, n_k$ are integers. For convenience we define $S_{-1}(X)$ to be the zero group.

The most important notion in homology theory is the algebraization of the notion of a boundary which applies to arbitrary singular simplices in an arbitrary topological space and not merely polyhedra in Euclidean spaces obtained by gluing together affine simplices. It is precisely this algebraization which provides considerable flexibility towards applications of homology theory.

Definition 29.3 (Boundary of a singular simplex): Given a singular $n-$simplex $\sigma : \Delta_n \longrightarrow X$, its $j-$th singular boundary is the singular $(n-1)$ simplex $\sigma \circ \Phi_j^n$ and the boundary $\partial_n\sigma$ of σ is then the $(n-1)$ chain given by the algebraic sum of its singular faces:

$$\partial_n\sigma = \sum_{j=0}^{n}(-1)^j(\sigma \circ \Phi_j^n). \tag{29.4}$$

The map ∂_n then extends as a group homomorphism $\sigma_n : S_n(X) \longrightarrow S_{n-1}(X)$. When $n = 0$ we define the boundary map ∂_0 to be the zero map.

The most important property of the maps ∂_n is the vanishing of $\partial_{n-1} \circ \partial_n$ which we now prove.

Theorem 29.2: For each $n \geq 1$, we have

$$\partial_{n-1} \circ \partial_n = 0. \tag{29.5}$$

Proof: It clearly suffices to check the result on the generators of $S_n(X)$. So let σ be an arbitrary singular n simplex. Using equation (29.4),

$$(\partial_{n-1} \circ \partial_n)\sigma = \partial_{n-1}\Big(\sum_{i=0}^{n}(-1)^i\sigma \circ \Phi_i^n\Big) = \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}).$$

To use lemma (29.1) we break the double sum in two pieces and write

$$(\partial_{n-1} \circ \partial_n)\sigma = \sum_{i\leq j}(-1)^{i+j}\sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}) + \sum_{j<i}(-1)^{i+j}\sigma \circ (\Phi_i^n \circ \Phi_j^{n-1})$$

Using (29.2) in the second piece we get

$$(\partial_{n-1} \circ \partial_n)\sigma = \sum_{i\leq j}(-1)^{i+j}\sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}) + \sum_{j<i}(-1)^{i+j}\sigma \circ (\Phi_j^n \circ \Phi_{i-1}^{n-1})$$

It may be noted that each of the two pieces is a sum of $n(n+1)/2$ terms (why?). Renaming $i-1$ as k in the second sum gives

$$(\partial_{n-1}\circ\partial_n)\sigma = \sum_{i\le j\le n-1}(-1)^{i+j}\sigma\circ(\Phi_i^n\circ\Phi_j^{n-1}) + \sum_{j\le k\le n-1}(-1)^{k+j-1}\sigma\circ(\Phi_j^n\circ\Phi_k^{n-1}) = 0$$

as desired.

Now suppose that X and Y are two topological spaces and $f: X \longrightarrow Y$ is a continuous map then $f\circ\sigma$ is a singular $n-$simplex in Y whenever σ is a singular $n-$simplex in X.

Definition 29.4: Given a continuous map $f: X \longrightarrow Y$, the map $f_\sharp: S_n(X) \longrightarrow S_n(Y)$ is the group homomorphism which is defined on singular n simplices σ via the prescription

$$f_\sharp(\sigma) = f\circ\sigma, \tag{29.6}$$

and extended as a group homomorphism from $S_n(X)$ to $S_n(Y)$. We ought to denote this map by $f_\sharp^n$ but we shall suppress the superscript to enhance readability.

Theorem 29.3: (i) For a continuous map $f: X \longrightarrow Y$, the maps $f_\sharp: S_n(X) \longrightarrow S_n(Y)$ satisfy

$$\partial_n\circ f_\sharp = f_\sharp\circ\partial_n \tag{29.7}$$

The $f_\sharp$ on the right hand side obviously refers to the map $S_{n-1}(X) \longrightarrow S_{n-1}(Y)$ and ∂_n refers to the boundary operator on $S_n(Y)$ on the left hand side whereas it refers to the boundary operator on $S_n(X)$ on the right hand side.

(ii) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are two continuous maps then the maps $f_\sharp: S_n(X) \longrightarrow S_n(Y)$ and $g_\sharp: S_n(Y) \longrightarrow S_n(Z)$ satisfy

$$(g\circ f)_\sharp = g_\sharp\circ f_\sharp \tag{29.8}$$

Proof: We shall only prove (29.7). It suffices to check these on singular simplices. So let $\sigma: \Delta_n \longrightarrow X$ be a singular n simplex in X. Using (29.4) we get

$$(f_\sharp\circ\partial_n)\sigma = f_\sharp\Big(\sum_{i=0}^{n}(-1)^i\sigma\circ\Phi_i^n\Big) = \sum_{i=0}^{n}(-1)^i((f\circ\sigma)\circ\Phi_i^n) = \partial_n(f\circ\sigma) = \partial_n(f_\sharp(\sigma)) = (\partial_n\circ f_\sharp)\sigma.$$

The category of chain complexes: We have associated to each topological space X a sequence $\{S_n(X)\}$ of *free* abelian groups and group homomorphisms $\partial_n: S_n(X) \longrightarrow S_{n-1}(X)$ satisfying (29.7). It is useful to describe these in an abstract setting where the groups in question need not be free abelian and prove some general results about them. This paragraph serves as an algebraic prerequisite for the study homology theory.

Definition 29.5: (i) A differential chain complex is a sequence $\{G_n/n = 0,1,2\dots\}$ of abelian groups together with a sequence of group homomorphism $\partial_n : G_n \longrightarrow G_{n-1}$ called the *boundary operator* satisfying the condition

$$\partial_n\circ\partial_{n+1} = 0, \quad n = 0,1,2,\dots \tag{29.9}$$

with the convention $G_{-1} = \{0\}$ and $\partial_0 = 0$. We shall use the letter G to denote this chain complex.

(ii) For a chain complex G, we define the subgroup $Z_n(G)$ of n-cycles to be the kernel of ∂_n namely,

$$Z_n(G) = \{z \in G_n / \partial_n(z) = 0\} \tag{29.10}$$

and the subgroup of n-boundaries as the image of ∂_{n+1} namely

$$B_n(G) = \{\partial_{n+1}(x) / x \in G_{n+1}\}. \tag{29.11}$$

From (29.9) it is clear that $B_n(G) \subset Z_n(G)$ and also $Z_0(G) = G_0$.

(iii) The quotient group

$$H_n(G) = Z_n(G)/B_n(G) \tag{29.12}$$

is called the n-th homology of the chain complex G. If $z_n \in Z_n(G)$ is a cycle the symbol $\overline{z_n}$ refers to the coset of z_n in the quotient group $H_n(G)$, called the *homology class* of z_n. We shall simplify notations whenever feasible and write Z_n in place of $Z_n(G)$, B_n instead of $B_n(G)$ and sometimes ∂z in place of the cumbersome $\partial_n(z)$.

Given two chain complexes G and K one would like to study maps between them. These are the chain maps which we now define.

Definition 29.6: Given two chain complexes G and K with boundary maps $\partial' : G_n \longrightarrow G_{n-1}$ and $\partial'' : K_n \longrightarrow K_{n-1}$, a chain map $\phi : G \longrightarrow K$ is a sequence of group homomorphisms $\phi_n : G_n \longrightarrow K_n$ $(n = 0, 1, 2, \dots)$ such that

$$\partial''_n \circ \phi_n = \phi_{n-1} \circ \partial'_n \tag{29.13}$$

Equation (29.13) may be summarized by declaring that the following diagram commutes:

$$\begin{array}{ccc} G_n & \xrightarrow{\partial'_n} & G_{n-1} \\ {\scriptstyle \phi_n}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{n-1}} \\ K_n & \xrightarrow{\partial''_n} & K_{n-1} \end{array} \tag{29.14}$$

Theorem 29.4: A chain map $\phi : G \longrightarrow K$ induces for each $n = 0, 1, 2, \dots$, a group homomorphism $H_n(\phi) : H_n(G) \longrightarrow H_n(K)$ given by

$$\overline{x} \mapsto \overline{\phi_n(x)}.$$

Proof: Thanks to (29.7), ϕ_n maps $Z_n(G)$ into $Z_n(K)$ and $B_n(G)$ into $B_n(K)$. Thus the map induced on the quotient groups is a well defined group homomorphism.

Theorem 29.5: Suppose given a pair of chain maps $\phi : L \longrightarrow G$ and $G \longrightarrow K$, then the composite $\psi \circ \phi : L \longrightarrow K$ is a chain map and

$$H_n(\psi \circ \phi) = H_n(\psi) \circ H_n(\phi), \quad n = 0, 1, 2, \dots \tag{29.15}$$

In other words for each n we get a covariant functor H_n from the category of chain complexes to the category **AbGr**.

Proof: For each $x \in Z_n(L)$,

$$H_n(\psi \circ \phi)(\overline{x}) = \overline{\psi_n \circ \phi_n(x)} = H_n(\psi)(\overline{\phi_n(x)}) = H_n(\psi) \circ H_n(\phi)(\overline{x}).$$

We shall at some point as we go along, drop the primes and denote both sets of boundary maps by ∂_n or even ∂. Observe that if $z \in \ker \phi_n$ then

$$\phi_{n-1}(\partial_n z) = \partial_{n-1}\phi_n(z) = 0,$$

whereby we conclude that ∂_n maps $\ker \phi_n$ into $\ker \phi_{n-1}$ and we get a chain complex

$$\longrightarrow \ker \phi_{n+1} \xrightarrow{\partial_{n+1}} \ker \phi_n \xrightarrow{\partial_n} \ker \phi_{n-1} \longrightarrow$$

which we denote by $\ker \phi$. Likewise we get the chain complex

$$\longrightarrow \text{Im } \phi_{n+1} \xrightarrow{\partial_{n+1}} \text{Im } \phi_n \xrightarrow{\partial_n} \text{Im } \phi_{n-1} \longrightarrow$$

which we denote by Im ϕ. It is clear from (29.13) that ∂_n maps Im ϕ_n into Im ϕ_{n-1}.

The long exact homology sequence: We are now ready to prove the most basic result on chain complexes and their homologies. The symbol 0 in any diagram involving chain complexes refers to the zero chain complex in which all groups are zero and the boundary maps are all zero.

Definition 29.7: A short exact sequence of chain complexes consists of three chain complexes of abelian groups L, G and K and chain maps $f : L \longrightarrow G$ and $g : G \longrightarrow K$ such that

(i) For each n, the map f_n is injective.

(ii) For each n, the map g_n is surjective.

(iii) For each n, $\ker g_n =$Im f_n.

Thus for each n we have the diagram

$$\{0\} \longrightarrow L_n \xrightarrow{f_n} G_n \xrightarrow{g_n} K_n \longrightarrow \{0\} \tag{29.16}$$

We now write out two more parallel rows with n replaced by $n-1$ and $n+1$ and the boundary maps going across the rows:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{n+1} & \xrightarrow{f_{n+1}} & G_{n+1} & \xrightarrow{g_{n+1}} & K_{n+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{n+1} & & \downarrow \partial_{n+1} & & \downarrow \partial_{n+1} & & \\
0 & \longrightarrow & L_n & \xrightarrow{f_n} & G_n & \xrightarrow{g_n} & K_n & \longrightarrow & 0 \\
 & & \downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n & & \\
0 & \longrightarrow & L_{n-1} & \xrightarrow{f_{n-1}} & G_{n-1} & \xrightarrow{g_{n-1}} & K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

We now state and prove the fundamental result.

Theorem 29.6: A short exact sequence of complexes (29.16) induces a long exact sequence in homology

$$\longrightarrow H_n(L) \xrightarrow{H_n(f)} H_n(G) \xrightarrow{H_n(g)} H_n(K) \xrightarrow{\delta_n} H_{n-1}(L) \longrightarrow \tag{29.17}$$

where the map $\delta_n : H_n(K) \longrightarrow H_{n-1}(L)$ known as the *connecting homomorphism* is given by the formula

$$\delta_n \overline{k_n} = \overline{f_{n-1}^{-1} \partial_n g_n^{-1}(k_n)}, \quad k_n \in Z_n(K) \tag{29.18}$$

Here $\overline{k_n}$ refers to the homology class of $k_n \in Z_n(K)$ and $g^{-1}(k_n)$ refers to any pre-image of k_n.

Proof: We must first show that displayed formula (29.18) gives a well-defined map since several choices are being made. First, for $k_n \in Z_n(K)$ surjectivity of g_n shows that there exists $x_n \in G_n$ such that $g_n(x_n) = k_n$. Applying the boundary map ∂_n we see that $g_{n-1}(\partial x_n) = \partial k_n = 0$ which, by virtue of exactness of (29.16) and injectivity of f_{n-1}, shows there is a unique $y_{n-1} \in L_{n-1}$ such that

$$f_{n-1}(y_{n-1}) = \partial x_n. \tag{29.19}$$

We have to now show that y_{n-1} is a cycle in L. This is clear if $n = 1$ and so we assume $n \geq 2$. Applying the boundary map to (29.19) gives $f_{n-2}(\partial y_{n-1}) = 0$ from which we conclude, since f_{n-2} is injective, that $y_{n-1} \in Z_{n-1}(L)$. Hence the assignment

$$k_n \mapsto \overline{y_{n-1}}, \quad k_n \in Z_n(K) \tag{29.20}$$

is well defined once we show that it is independent of the choice of $x_n \in g_n^{-1}(k_n)$.

Second, we suppose that for a given $k_n \in Z_n(K)$, x'_n and x''_n are two members of $g_n^{-1}(k_n)$ then

$$x'_n - x''_n \in \ker g_n = \operatorname{im} f_n.$$

So there is a $u_n \in L_n$ such that $x'_n - x''_n = f_n(u_n)$. On the other hand, for these two choices there exist y'_{n-1} and y''_{n-1} in L_{n-1} such that (29.19) holds and so

$$f_{n-1}(y'_{n-1}) - f_{n-1}(y''_{n-1}) = \partial(x'_n - x''_n) = \partial f_n(u_n) = f_{n-1}(\partial u_n).$$

Injectivity of f_{n-1} implies y'_{n-1} and y''_{n-1} differ by a boundary and so define the same homology class.

Third, we must show that the same homology class results if we begin with two homologous cycles k'_n and k''_n. In this there exists $v_{n+1} \in K_{n+1}$ and $x_{n+1} \in G_{n+1}$ such that

$$k'_n - k''_n = \partial v_{n+1} = \partial g_{n+1}(x_{n+1}) = g_n(\partial x_{n+1}).$$

Let x'_n and x''_n be chosen from $g_n^{-1}(k'_n)$ and $g_n^{-1}(k''_n)$ respectively so that $g_n(x'_n - x''_n - \partial x_{n+1}) = 0$. By exactness of (29.16) there is a $w_n \in L_n$ such that $x'_n - x''_n - \partial x_{n+1} = f_n(w_n)$. Applying ∂ to this and and recalling (29.19) we see that the corresponding cycles y'_{n-1} and y''_{n-1} satisfy

$$f_{n-1}(y'_{n-1} - y''_{n-1}) = \partial f_n(w_n) = f_{n-1}(\partial w_n).$$

Since f_{n-1} is injective we see that the cycles y'_{n-1} and y''_{n-1} are homologous.

Exactness of (29.17): We first check the exactness at the junction $H_n(G)$. Since (29.15) implies $H_n(g) \circ H_n(f) = 0$, it suffices to prove ker $H_n(g) \subset$ im $H_n(f)$. So let $g_n(x_n) = \partial_{n+1} k_{n+1}$ for some $x_n \in Z_n(G)$ and $k_{n+1} \in K_{n+1}$. Since g_{n+1} is surjective we can find $x_{n+1} \in G_{n+1}$ such that $k_{n+1} = g_{n+1}(x_{n+1})$ and

$$g_n(x_n) = \partial_{n+1} g_{n+1}(x_{n+1}) = g_n(\partial_{n+1} x_{n+1}),$$

from which we conclude there exists $y_n \in L_n$ such that $x_n - \partial_{n+1} x_{n+1} = f_n(y_n)$. Applying the operator ∂_n and using injectivity of f_{n-1} we see that $y_n \in Z_n(L)$ and the result is established.

We now turn to the exactness at the junction $H_n(K)$. It is clear from (29.18) that $\delta_n(\overline{g_n x_n}) = 0$ for any $x_n \in Z_n(G)$ so that $\delta_n \circ H_n(g) = 0$. To prove ker $\delta_n(g) \subset$ im $H_n(g)$ let $k_n \in Z_n(K)$ such that $\delta_n(\overline{k_n}) = 0$. Equation (29.18) then implies, for any $x_n \in g_n^{-1}(k_n)$ there is $l_n \in L_n$ such that

$$f_{n-1}^{-1} \partial_n x_n = \partial_n l_n.$$

From this we get $x_n - f_n(l_n) = x'_n \in Z_n(G)$. Applying g_n we see that $k_n = g_n(x_n) = g_n(x'_n)$ whereby we conclude $\overline{k_n} \in$ im $H_n(g)$.

Finally we come to the exactness at the junction $H_{n-1}(L)$. From (29.18) follows $H_{n-1}(f) \circ \delta_n = 0$. To show ker $H_{n-1}(f) \subset$ im δ_n, pick a cycle l_{n-1} such that $f_{n-1}(l_{n-1})$ is a boundary say $\partial_n x_n$ for some $x_n \in G_n$. Applying g_{n-1} to the equation

$$f_{n-1} l_{n-1} = \partial_n x_n$$

gives a cycle $k_n = g_n(x_n) \in Z_n(K)$. From (29.18) we infer $\delta_n(\overline{k_n}) = \overline{l_{n-1}}$ and this suffices for a proof.

Exercises

1. Sketch Δ_n for $n = 1, 2, 3$. Show that Δ_n is a compact and connected subspace of $\mathbb{R}^{n+1}$.

2. Discuss the continuity of the maps (29.1). Prove lemma (29.1). what about the cases $i \leq j$?

3. Verify equation (29.8).

4. Determine the values of n ($n = 1, 2, \dots$) for which a constant function $\Delta_n \longrightarrow X$ an $n-$cycle.

5. Show that the family of all chain complexes forms a category in which the set of morphisms Mor(G, K) between any two chain complexes G and K is the set of all chain maps from G to K.

6. Naturality of (29.17)-(29.18). Assume given a commutative diagram of chain complexes with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & G & \xrightarrow{g} & K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & L' & \xrightarrow{f'} & G' & \xrightarrow{g'} & K' & \longrightarrow & 0
\end{array}$$

   Denoting by δ_n and δ'_n the connecting homomorphisms, sketch relevant diagrams and prove that

$$\delta'_n \circ H_n(\eta) = H_n(\psi) \circ \delta_n$$

Lecture - XXXI The homology groups and their functoriality

Having laid the algebraic foundations in the previous lecture we shall formally define the homology functors H_n, $n = 0, 1, 2, \dots$ from the category **Top** to the category **AbGr**. We shall discuss $H_0(X)$ completely and show that $H_0(X)$ is free abelian of rank equal to the number of path components of X. The groups $H_n(X)$ $(n \geq 1)$ vanish when X is a convex subset of $\mathbb{R}^n$. We shall prove this result using a technique that would be be considerably generalized in lecture 33. However the special case proved here for convex subsets would be needed in lecture 33. In the next lecture we shall see examples of topological spaces X for which $H_1(X)$ is non-trivial. However the reader would have to wait till lecture 34 to see more interesting examples.

The homology groups $H_n(X)$: Definitions (29.3)-(29.6) and theorems (29.2)-(29.5) from the previous lecture show that given a topological space X, the sequence of groups $S_n(X)$ and group homomorphisms $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ provide an example of a chain complex called *the singular chain complex.* If $f : X \longrightarrow Y$ is a continuous function, the sequence $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ $(n = 0, 1, 2, \dots)$ defines a chain map from the chain complex $S(X)$ to $S(Y)$. The general results on chain complexes when applied to this special case gives us the homology functors from **Top** to **AbGr**.

Definition 31.1: (i) The homology groups $H_n(X)$ of the space X are by definition the homology groups of the chain complex $S(X)$ namely

$$H_n(X) = Z_n(X)/B_n(X),$$

where $Z_n(X)$ is the kernel of the homomorphism $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ and $B_n(X)$ is the image of the homomorphism $\partial_{n+1} : S_{n+1}(X) \longrightarrow S_n(X)$.

(ii) Given a continuous map $f : X \longrightarrow Y$, the induced maps $H_n(f) : H_n(X) \longrightarrow H_n(Y)$ in homology are the homomorphisms

$$H_n(f) : \overline{\sigma} \mapsto \overline{f_\sharp(\sigma)}, \quad \sigma \in Z_n(X).$$

Theorem (29.5) in this context is reproduced below:

Theorem 31.1: (i) Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow W$ are continuous functions,

$$H_n(g \circ f) = H_n(g) \circ H_n(f), \quad n = 0, 1, 2, \dots.$$

The identity map on X induces the identity map on $H_n(X)$:

$$H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$$

In other words the $\{H_n/n = 0, 1, 2, \dots\}$ is a sequence of covariant functors from **Top** to **AbGr**. An immediate consequence is the following result.

Corollary 31.2: Suppose X and Y are homeomorphic, the groups $H_n(X)$ and $H_n(Y)$ are isomorphic for every $n = 0, 1, 2, \ldots$.

Another important consequence is the following result that parallels lemma (9.3).

Theorem 31.3: Suppose $r : X \longrightarrow A$ is a retraction, then for every $n = 0, 1, 2, \ldots$

$$H_n(r) : H_n(X) \longrightarrow H_n(A)$$

is surjective and

$$H_n(j) : H_n(A) \longrightarrow H_n(X)$$

is injective, where $j : A \longrightarrow X$ is the inclusion map.

The augmentation map $\epsilon : S_0(X) \longrightarrow \mathbb{Z}$: Since the standard Euclidean simplex Δ_0 is a singleton, each singular zero simplex $\Delta_0 \longrightarrow X$ can be identified with a point of X namely the image of the singular zero simplex. Thus we may think of a singular zero chain as an element of the free abelian group generated by the points of X, that is a formal expression

$$c_1 p_1 + c_2 p_2 + \cdots + c_k p_k, \tag{31.1}$$

where $p_1, p_2, \ldots, p_k$ are points of X and the coefficients $c_1, c_2, \ldots, c_k$ are integers.

Definition 31.2: The augmentation map $\epsilon : S_0(X) \longrightarrow \mathbb{Z}$ is the group homomorphism given by

$$c_1 p_1 + c_2 p_2 + \cdots + c_k p_k \mapsto c_1 + c_2 + \cdots + c_k.$$

If X is non-empty, the augmentation map is surjective. Since by definition, ∂_0 is the zero map and $Z_0(X) = S_0(X)$, we have to determine $B_0(X)$. The following theorem provides the answer.

Theorem 31.4: Suppose X is a path connected space then $B_0(X) = \ker \epsilon$. That is to say a singular zero chain (31.1) is a boundary if and only if the sum of its coefficients is zero. Thus, for a path connected space X,

$$H_0(X) \cong \mathbb{Z}.$$

Proof: We shall denote the ends of Δ_1 by a and b. If $\sigma : \Delta_1 \longrightarrow X$ is a singular one simplex then $\partial_1 \sigma = \sigma(b) - \sigma(a)$ which is obviously in $\ker \epsilon$ and we conclude that $B_0(X) \subset \ker \epsilon$. To prove the reverse inclusion, let σ be an arbitrary element of $\ker \epsilon$ given by (31.1). That is, the coefficients satisfy $c_1 + c_2 + \cdots + c_k = 0$. Pick any point $p \in X$ and for each j let $\sigma_j : \Delta_1 \longrightarrow X$ be a path in X joining p and p_j. We claim that σ is the boundary of the one chain $\tau = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_k\sigma_k$.

$$\begin{aligned}
\partial_1 \tau &= c_1(\sigma_1(b) - \sigma_1(a)) + c_2(\sigma_2(b) - \sigma_2(a)) + \cdots + c_k(\sigma_k(b) - \sigma_k(a)) \\
&= (c_1 p_1 + c_2 p_2 + \cdots + c_k p_k) - (c_1 + c_2 + \cdots + c_k)p = \sigma.
\end{aligned}$$

The last part follows from the fundamental theorem on group homomorphisms.

Theorem 31.5: If $\{X_\alpha / \alpha \in \Lambda\}$ is the family of path components of a topological space, then for each $k = 0, 1, 2, \ldots$

$$H_k(X) = \bigoplus_{\alpha \in \Lambda} H_k(X_\alpha)$$

Proof: We shall only sketch the proof leaving the details as an exercise. Note that if σ is a singular $k-$simplex, the image of σ must be contained in one of the components X_α of X and so may be regarded as a singular $k-$simplex in X_α. This gives a natural decomposition of $S_k(X)$ as a direct sum of the family $S_k(X_\alpha)$. To see that the boundary map ∂_k respects the decomposition note that the boundary of a singular simplex σ is a sum of finitely many $k-1$ singular simplexes each of which must map into the same component as σ. It is easy to deduce from this the decompositions $Z_k(X) = \bigoplus Z_k(X_\alpha)$ and $B_k(X) = \bigoplus B_k(X_\alpha)$.

Convex sets and barycentric coordinates: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ be given points. The convex hull of these points is the set consisting of all convex combinations $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k$, that is the coefficients $t_1, t_2, \ldots, t_k$ are non-negative and $t_1 + t_2 + \cdots + t_k = 1$. The convex hull of these points is clearly a convex set. The points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are said to be *affinely independent* if the $k-1$ vectors $\mathbf{v}_1 - \mathbf{v}_k, \mathbf{v}_2 - \mathbf{v}_k, \ldots, \mathbf{v}_{k-1} - \mathbf{v}_k$ are linearly independent (see exercise 4). The convex hull of a set of k affinely independent points is called the affine $k-$ simplex spanned by these points. The proof of the following result is left as an exercise.

Theorem 31.6: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are affinely independent then every point x in the convex hull of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ can be uniquely expressed as

$$x = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k, \tag{31.2}$$

where the coefficients t_j $(1 \leq j \leq k)$ are non-negative and $t_1 + t_2 + \cdots + t_k = 1$. These coefficients are called the *barycentric coordinates* of x.

We consider the standard n simplex Δ_n in $\mathbb{R}^{n+1}$ with summit $S = \mathbf{e}_{n+1}$. The figure below depicts a general point Q on the face Δ_{n-1} opposite to S and P an arbitrary point on the line segment joining Q and S. The reader may check that if $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are the barycentric coordinates of P then the coordinates of Q are given by the $n-$tuple

$$U(\lambda_1, \ldots, \lambda_{n+1}) = \Big(\frac{\lambda_1}{1-\lambda_{n+1}}, \frac{\lambda_2}{1-\lambda_{n+1}}, \ldots, \frac{\lambda_n}{1-\lambda_{n+1}}\Big) \tag{31.3}$$

Note that U is bounded but not continuous when $\lambda_{n+1} \longrightarrow 1$. As P approaches S the pyramid with

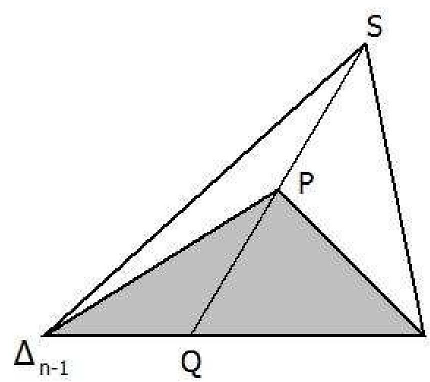


Figure 21:

base Δ_{n-1} and summit P fills up Δ_n. We are now in a position to prove the following theorem.

Theorem 31.7: Suppose X is a convex subset of a Euclidean space, $H_k(X) = 0$ for $k \geq 1$.

Proof: Choose a point $x_0 \in X$ and $F : X \times [0,1] \longrightarrow X$ be the homotopy $F(x,t) = (1-t)x + tx_0$. We shall define a group homomorphism $T : S_{n-1}(X) \longrightarrow S_n(X)$ satisfying a certain property (31.6) below. This is a special case of a *chain homotopy* that we shall encounter later in a more general context. Since $S_{n-1}(X)$ is a free abelian group generated by singular $(n-1)$ simplicies, it suffices to define T these. For a singular $(n-1)$ simplex $\sigma : \Delta_{n-1} \longrightarrow X$, define the continuous map $T\sigma : \Delta_n \longrightarrow X$ in terms of the barycentric coordinates using the expression (31.3) namely

$$(T\sigma)(\lambda_1, \dots, \lambda_{n+1}) = F((\sigma \circ U)(\lambda_1, \dots, \lambda_{n+1}), \lambda_{n+1}). \tag{31.4}$$

The continuity of $T\sigma$ is left as an exercise. Let us calculate the boundary of $T\sigma$ using equations (29.1) and (29.4). Recalling the notations used in lecture 29, one checks that $(T\sigma) \circ \Phi^n_n = \sigma$.

For $0 \leq j \leq n-1$, the $j-$th singular face is given by

$$(T\sigma) \circ \Phi^n_j(t_1, \dots, t_n) = F((\sigma \circ U)(t_1, \dots, t_{j-1}, 0, t_j, \dots, t_n), t_n). \tag{31.5}$$

On the other hand when $0 \leq j \leq n-1$,

$$T(\sigma \circ \Phi^{n-1}_j)(t_1, \dots, t_n) = T\Big(\sigma\Big(\frac{t_1}{1-t_n}, \dots, \frac{t_{j-1}}{1-t_n}, 0, \frac{t_j}{1-t_n}, \dots, \frac{t_{n-1}}{1-t_n}\Big),\ t_n\Big),$$

which may be rewritten as $T((\sigma \circ U)(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}), t_n)$, in agreement with the right hand side of (31.5). From equation (29.4) it follows that for $n \geq 1$,

$$\partial_n(T\sigma) - T(\partial_n\sigma) = \sigma, \quad \sigma \in S_n(X), \tag{31.6}$$

whereby we conclude that if $\sigma \in Z_n(X)$ then $\sigma = \partial_n(T\sigma) \in B_n(X)$. That is $Z_n(X) = B_n(X)$.

Exercises

1. Prove theorem (31.3).

2. Show that for a path connected space X, every singleton $\{p\}$ with $p \in X$ is a basis for $H_0(X)$.

3. Complete the proof of theorem (31.5).

4. Show that the set $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ is affinely independent if the vectors

$$\mathbf{v}_1 - \mathbf{v}_j, \dots, \mathbf{v}_{j-1} - \mathbf{v}_j, \mathbf{v}_{j+1} - \mathbf{v}_j, \dots, \mathbf{v}_{k-1} - \mathbf{v}_k$$

   are linearly independent for any j $(1 \leq j \leq k)$.

5. Prove theorem (31.6). Show that the barycentric coordinates are continuous functions of $\mathbf{x}$. All but the j-th barycentric coordinates of $\mathbf{v}_j$ vanish. The set of points in (31.2) obtained by setting $t_j = 0$ and varying the other coefficients is called the $j-$th face of the simplex spanned by the given points.

6. Check the continuity of the map $T\sigma$ in theorem (31.7).

Lecture - XXXII The abelianization of the fundamental group

In this lecture we shall establish a basic result relating the fundamental group $\pi_1(X, x_0)$ and the first homology group $H_1(X)$. The result is elegant and states that $H_1(X)$ is the abelianization of $\pi_1(X, x_0)$ when X is a path connected space. Further, the abelianization map is natural in the following sense. Suppose that $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ \Pi_X \downarrow & & \downarrow \Pi_Y \\ H_1(X) & \xrightarrow{H_1(f)} & H_1(Y) \end{array} \tag{32.1}$$

where $\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X)$ and $\Pi_Y : \pi_1(Y, y_0) \longrightarrow H_1(Y)$ are the quotient maps onto the respective abelianizations. We shall prove the main theorem (32.1) through a series of lemmas.

Theorem 32.1: Let X be a path connected topological space. There is a surjective group homomorphism

$$\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X) \tag{32.2}$$

whose kernel is the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$. Thus

$$H_1(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \tag{32.3}$$

Before taking up the proof which will be completed in several steps, we set up the map Π_X. Note that if γ is a loop in X based at x_0 then γ is a one cycle, that is to say $\gamma \in Z_1(X)$ and we denote its homology class in the quotient $H_1(X)$ by $\overline{\gamma}$. This suggests that we define $\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X)$ as

$$\Pi_X : [\gamma] \mapsto \overline{\gamma} \tag{32.4}$$

We shall show that the map is a well-defined surjective group homomorphism and determine its kernel. We do each of these as a separate lemma. Since homotopy of loops is a map from the square $[0, 1] \times [0, 1]$ whereas a singular two simplex is a map from Δ_2 to X we must first set up some standard maps from Δ_2 to the square with specific properties. The usual proofs seem slightly tricky and we shall try an approach that would be useful in the next lecture.

Divide the square $[0, 1] \times [0, 1]$ into two triangles by drawing a diagonal from $(0, 0)$ to $(1, 1)$. Let T_i $(i = 1, 2)$ be two affine homeomorphisms mapping Δ_2 onto the these two triangles given by

$$\begin{aligned} T_1(\widehat{\mathbf{e}}_1) &= (0, 0), \quad T_2(\widehat{\mathbf{e}}_1) = (0, 0), \\ T_1(\widehat{\mathbf{e}}_2) &= (1, 0), \quad T_2(\widehat{\mathbf{e}}_2) = (0, 1), \\ T_1(\widehat{\mathbf{e}}_3) &= (1, 1), \quad T_2(\widehat{\mathbf{e}}_3) = (1, 1). \end{aligned}$$

We shall regard the maps T_i $(i = 1, 2)$ as maps from Δ_2 into I^2 and use them to prove the following:

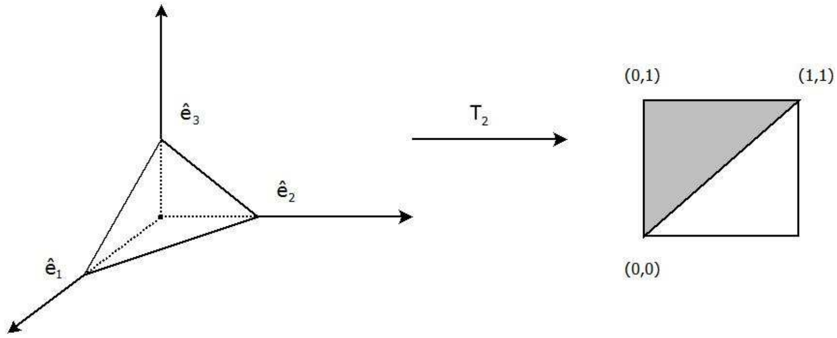


Figure 22:

Lemma 32.2: The map Π_X given by (32.4) is well defined.

Proof: Let γ_1 and γ_2 be two homotopic loops based at x_0 and let $F : I \times I \longrightarrow X$ be the homotopy fixing the base point x_0. Then $\sigma_i = F \circ T_i$ $(i = 1, 2)$ are two singular two simplicies. It is an exercise to compute the boundary of these two singular simplicies and we find easily

$$\partial(\sigma_1) = \partial(F \circ T_1) = \varepsilon_{x_0} + \gamma_1 - F(t,t)$$
$$\partial(\sigma_2) = \partial(F \circ T_2) = \varepsilon_{x_0} + \gamma_2 - F(t,t).$$

The one chain $\gamma_1 - \gamma_2$ is the boundary of the two chain $\sigma_1 - \sigma_2$ whence $\overline{\gamma}_1 = \overline{\gamma}_2$.

Lemma 32.3: The map Π_X given by (32.4) is a group homomorphism.

Proof: Let γ_1 and γ_2 be two loops in X based at x_0. We have to show that the one chain

$$\gamma_1 + \gamma_2 - \gamma_1 * \gamma_2$$

is a boundary of some singular two chain σ. The idea behind the construction is simple. We first define a map $\tilde{F} : I \times I \longrightarrow X$ whose restrictions to the four sides of the square are γ_1, γ_2, ε_{x_0} and $\gamma_1 * \gamma_2$. As in the previous lemma we shall employ the maps T_1, T_2 to construct our two chain σ.

We proceed as in lecture 7 by defining $\tilde{F}$ from the boundary of $I \times I$ to $[0,1]$, using Tietze's theorem to extend it to the whole of $I \times I$ and then composing with $\gamma_1 * \gamma_2$. So we define

$$\begin{aligned}
\tilde{F}(0,s) &= \frac{s}{2} \\
\tilde{F}(t,1) &= \frac{t+1}{2} \\
\tilde{F}(t,0) &= t \\
\tilde{F}(1,s) &= 1
\end{aligned}$$

and extend it continuously to $I \times I$. Let $F : I \times I \longrightarrow X$ be given by $F = (\gamma_1 * \gamma_2) \circ \tilde{F}$. The figure below depicts F along the boundary of I^2:

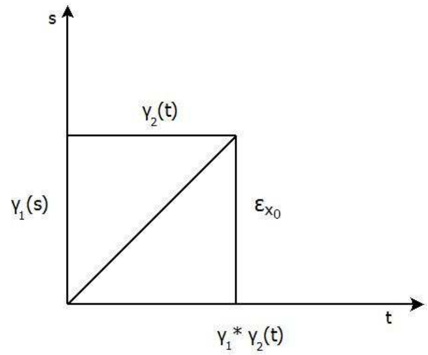


Figure 23:

It is now an easy matter to verify that the boundary of the two chain σ given by

$$\sigma = F \circ T_1 - F \circ T_2$$

is the one chain

$$\gamma_1 + \gamma_2 - \varepsilon_{x_0} - \gamma_1 * \gamma_2 \tag{32.5}$$

Now, if $\sigma' : \Delta_2 \longrightarrow X$ be the constant map taking value x_0 then $\partial\sigma' = \varepsilon_{x_0}$ whereby we conclude that

$$\gamma_1 + \gamma_2 - \gamma_1 * \gamma_2 = \partial\sigma + \partial\sigma', \tag{32.6}$$

which implies $\Pi_X([\gamma_1][\gamma_2]) = \Pi_X([\gamma_1 * \gamma_2]) = \Pi_X([\gamma_1]) + \Pi_X([\gamma_2])$. □

Lemma 32.4: The map Π_X given by (32.4) is surjective.

Proof: Let λ be a singular one cycle say $\lambda = \sum n_j\gamma_j$, where $n_j \in \mathbb{Z}$ and $\gamma_j : [0,1] \longrightarrow X$. Since $\partial\lambda = 0$,

$$\sum_{j=1}^{k} n_j(\gamma_j(1) - \gamma_j(0)) = 0. \tag{32.7}$$

The idea is to complete each of the paths γ_j into a loop at x_0 by means of paths joining x_0 to the ends $\gamma_j(0)$ and $\gamma_j(1)$. The only non-trivial part is the book-keeping which has to be done carefully. Let S denote the set of endpoints

$$S = \{\gamma_j(1),\ \gamma_j(0)/\ j = 1, 2 \ldots, k\}.$$

For each $p \in S$, if m_p denotes the sum of the coefficients of p in (32.7) then m_p must be zero. Taking a path β_p in X joining x_0 and $p \in S$ we construct for each j a loop η_j in X based at x_0 namely,

$$\eta_j = \beta_{\gamma_j(0)} * \gamma_j * \beta^{-1}_{\gamma_j(1)}.$$

Finally

$$\Pi_X(\eta_1^{n_1} * \eta_2^{n_2} * \cdots * \eta_k^{n_k}) = \sum_{j=1}^{k} n_j\gamma_j - \sum_{j=1}^{k} n_j(\beta_{\gamma_j(1)} - \beta_{\gamma_j(0)}) = \lambda$$

since

$$\sum_{j=1}^{k} n_j(\beta_{\gamma_j(1)} - \beta_{\gamma_j(0)}) = \sum_{p\in S} m_p\beta_p = 0.$$

Lemma 32.5: Suppose G is a group and $x_1, x_2, \ldots, x_k$ are *distinct* elements of G such that $x_i \neq x_j^{-1}$ if $i \neq j$. Let w be a word involving integer powers of $x_1, x_2, \ldots, x_k$ such that the sum of the exponents of each x_i is zero. Then w lies in the commutator subgroup of G.

Proof: We leave the easy proof for the student to work out.

Lemma 32.6: The kernel of the map Π_X is the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Proof: Since the Π_X is a map into an abelian group, its kernel contains the commutator subgroup. To prove the converse suppose that γ is a loop based at x_0 such that $[\gamma] \in \text{Ker}\, \Pi_X$. When considered as a singular one cycle it is a boundary of a singular two chain $\sum n_j \sigma_j$ where $\sigma_j : \Delta_2 \longrightarrow X$. Writing the boundary $\partial \sigma_j$ as a sum of its faces

$$\partial \sigma_j = \lambda_j + \mu_j + \nu_j$$

we see that

$$\sum_{j=1}^{k} n_j \partial \sigma_j = \sum_{j=1}^{k} n_j (\lambda_j + \mu_j + \nu_j) = \gamma. \tag{32.8}$$

We proceed as in lemma (32.4). Let S be the set distinct singular one simplicies in the list

$$\lambda_j, \mu_j, \nu_j \quad j = 1, 2, \ldots, k. \tag{32.9}$$

and choose auxiliary paths β_p joining x_0 and the endpoints p of each of the one simplicies in S. The loop γ also appears in the list (32.9) but since its ends are both x_0 there is no need to take the auxiliary paths β in this case. As in lemma (32.4), for each θ in the list (32.9), we denote by m_θ the sum of the

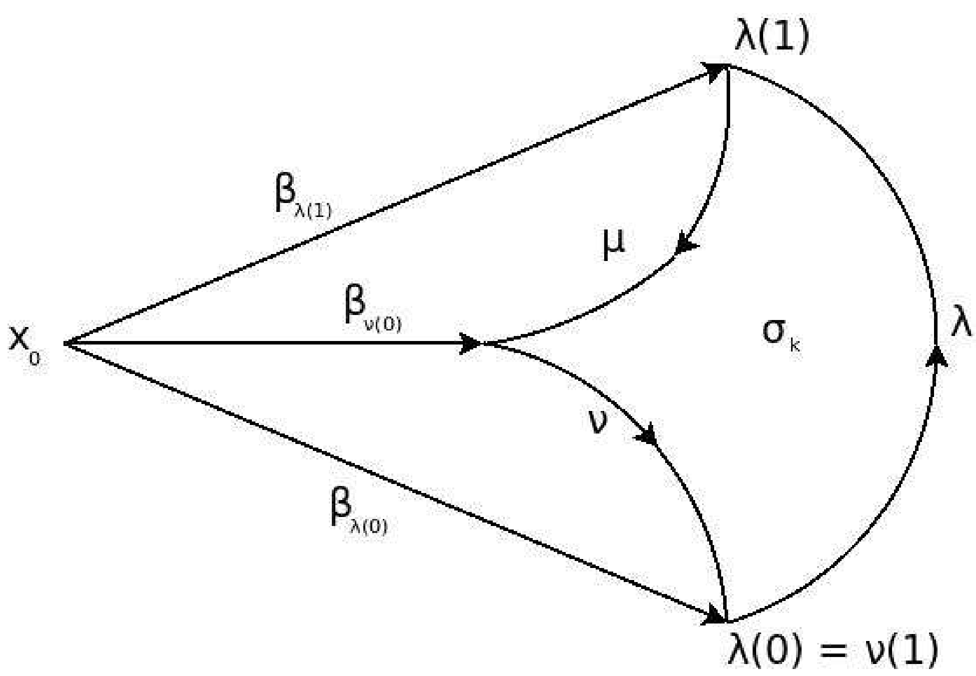


Figure 24:

coefficients of θ in (32.8) so that,

$$m_\theta = \begin{cases} 0 & \text{if } \theta \neq \gamma \\ 1 & \text{if } \theta = \gamma \end{cases} \tag{32.10}$$

For each two simplex σ_j we have the three loops (suppressing the subscript j)

$$\beta_{\lambda(0)} * \lambda * \beta^{-1}_{\lambda(1)}, \ \beta_{\mu(0)} * \mu * \beta^{-1}_{\mu(1)}, \ \beta_{\nu(0)} * \nu * \beta^{-1}_{\nu(1)},$$

whose juxtaposition η_j is easily seen to be homotopic to the trivial loop. For proving this one uses the equations $\lambda(1) = \mu(0)$, $\mu(1) = \nu(0)$ and $\nu(1) = \lambda(0)$. Corresponding to (32.8) we form the loop

$$\eta_1^{n_1} * \eta_2^{n_2} * \cdots * \eta_k^{n_k} * \gamma^{-1} \tag{32.11}$$

which is homotopic to γ^{-1} since the piece $\eta_1^{n_1} * \eta_2^{n_2} * \cdots * \eta_k^{n_k}$ is a juxtaposition of loops homotopic to the constant loop. On the other hand if we write out the expression (32.11) completely, we see that for each θ in the list (32.9), the factor $\beta * \theta * \beta^{-1}$ appears, probably in several positions, but the sum of its exponents is m_θ. In view of (32.10) and lemma (30.5) we see that the element of $\pi_1(X, x_0)$ represented by (32.11) lies in the commutator subgroup, that is to say, $[\gamma]^{-1}$ lies in the commutator subgroup of $\pi_1(X, x_0)$. The proof is complete.

Definition 32.1 (Natural transformation): Given a pair of functors $\pi : \mathcal{T} \longrightarrow \mathcal{G}$ and $H : \mathcal{T} \longrightarrow \mathcal{G}$, a natural transformation T between π and H is a function which assigns to each object X of $\mathcal{T}$ a morphism $\eta_X : \pi(X) \longrightarrow H(X)$ such that for each morphism $f : X \longrightarrow Y$ in $\mathcal{T}$, the following diagram commutes

$$\begin{array}{ccc} \pi(X) & \xrightarrow{\pi(f)} & \pi(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ H(X) & \xrightarrow{H(f)} & H(Y) \end{array} \tag{32.12}$$

The notation used in this definition is quite suggestive. The Poincaré-Hurewicz map provides a natural transformation between the functors π_1 and H_1.

Exercises

1. Verify the displayed results for $\partial\sigma_1$ and $\partial\sigma_2$ in lemma (32.2).
2. By writing out the boundary formula in detail verify equations (32.5) and (32.6).
3. Prove lemma (32.5).
4. Verify the naturality of Π_X by proving that the diagram (32.1) commutes.
5. Determine the first homology group of the Klein's bottle.
6. Determine the first homology groups of all the spaces described in the exercises to lecture 26.
7. In lecture 26 the fundamental group of the double torus was determined (see equation 26.14). Write the element $abcda^{-1}b^{-1}c^{-1}d^{-1}$ as a product of commutators and deduce that its abelianization is the free abelian group of rank four.

Lecture - XXXIII Homotopy invariance of homology

Homotopy of maps is one of the most important notions in topology and it is of interest to know what is its effect on the induced maps in homology. The result is simple and direct namely, if $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are a pair of homotopic maps then they induce the *same* maps in homology in every dimension. The further advantage here is that no base points are involved unlike the situation encountered in lecture 11 with the fundamental group. However the proof is not direct as one must algebraize the notion of homotopy in the context of chain maps. This leads to the notion of *chain homotopy* that we first define. We establish the purely algebraic result that a pair of chain homotopic maps induce equal maps in homology. We then proceed to relate the topological notion of homotopy of a pair of continuous maps f, g as above with the chain homotopy between the induced chain maps $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ and $g_\sharp : S_n(X) \longrightarrow S_n(Y)$. Some of these ideas have been implicitly used in the last lecture in the construction of the singular two chain σ in lemma (32.2). We shall follow the treatment in the book by T. Dieck[7] defining first the notion of the cross product which seems more transparent. The student who is familiar with differential forms may notice some similarities with wedge products and the exterior derivative. As in the theory of differential forms where the construction of the exterior derivative d is forced upon us through some of its properties, the cross product is determined by its properties described in theorem (33.1), as soon as one chooses for each pair (p, q) a *model chain* namely, the $p + q$ chain z in (33.4).

The cross product: This construction lies at the heart of the proof of Kunneth formula which relates the homology groups of $X \times Y$ in terms of the homologies of X and Y. The first step would be to relate the singular chain complex of $X \times Y$ with those of X and Y. This construction will be carried out naturally. Given a zero simplex $x \in X$ and a q simplex $\sigma : \Delta_q \longrightarrow Y$ in Y, $x \times \sigma$ denotes the singular q simplex in $X \times Y$ given by

$$\begin{aligned} x \times \sigma : \Delta_q &\longrightarrow X \times Y \\ t &\mapsto (x, \sigma(t)). \end{aligned}$$

Likewise given a q simplex τ in X and a zero simplex y in Y, one defines a q simplex $\tau \times y$ in $X \times Y$. For a pair of singular simplices $\sigma \in \Delta_p(X)$ and $\tau \in \Delta_q(Y)$ we call $p + q$ the total degree of the pair (σ, τ).

Theorem 33.1: There exists a bilinear map

$$\begin{aligned} S_p(X) \times S_q(Y) &\longrightarrow S_{p+q}(X \times Y) \\ (\sigma, \tau) &\mapsto \sigma \times \tau, \end{aligned}$$

[7]See also R. Stöcher and H. Zeischang, Algebraische Topologie, B. G. Teubner, Stuttgart (1988) 306-325.

with the following properties

(i) For zero simplices $x \in X$, $y \in Y$ and singular simplices $\sigma : \Delta_p \longrightarrow X$ and $\tau : \Delta_q \longrightarrow Y$ the products $x \times \tau$, $\sigma \times y$ are already defined above.

(ii) Naturality: Suppose that $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are two continuous maps and $f \times g : X \times Y \longrightarrow X' \times Y'$ denotes the product map $(f \times g)(x, y) = (f(x), g(y))$, then

$$(f \times g)_\sharp(\sigma \times \tau) = f_\sharp(\sigma) \times g_\sharp(\tau) \tag{33.1}$$

(iii) Generalized Leibnitz' rule: If $\sigma \in S_p(X)$ and $\tau \in S_q(Y)$ then

$$\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^p(\sigma \times \partial\tau) \tag{33.2}$$

Proof: The construction proceeds by induction on the total degree $p + q$ on pairs (σ, τ). It has already been carried out for the case when one of σ or τ is a zero simplex and in particular when the total degree $p + q$ is zero. Further, and for this case, conditions (ii) and (iii) hold trivially. Assume that the cross product

$$S_p(X) \times S_q(Y) \longrightarrow S_{p+q}(X \times Y) \tag{33.3}$$

has been defined for all pairs (p, q) such that $p + q < k$ satisfying (ii) and (iii). Now if $\sigma \in S_p(X)$ and $\tau \in S_q(Y)$ are such that $p + q = k$ then the right hand side of the formula in (iii) already makes sense and in particular this is so with the pair ι_p and ι_q. Thus we need a singular $p + q$ chain z such that

$$\partial z = \partial\iota_p \times \iota_q + (-1)^p(\iota_p \times \partial\iota_q). \tag{33.4}$$

Applying Leibnitz rule again to the right hand side one checks that it is a cycle. Since $\Delta_p \times \Delta_q$ is convex this cycle is also a boundary and so (33.4) has a (non-unique) solution $z \in S_{p+q}(\Delta_p \times \Delta_q)$. Once this choice is made the construction proceeds further as follows. Each $\sigma \in S_p(X)$ can be realized as $\sigma_\sharp(\iota_p)$ where $\sigma : \Delta_p \longrightarrow X$ and likewise for a singular q simplex τ in Y. But now equation (33.1) forces upon us the definition

$$\sigma \times \tau = \sigma_\sharp(\iota_p) \times \tau_\sharp(\iota_q) = (\sigma \times \tau)_\sharp(\iota_p \times \iota_q) = (\sigma \times \tau) \circ z, \tag{33.5}$$

where the $\sigma \times \tau$ appearing on the extreme left of (33.5) is the object we are defining whereas the σ and τ appearing in the middle and on the extreme right of (33.5) denote the functions $\sigma : \Delta_p \longrightarrow X$ and $\tau : \Delta_q \longrightarrow Y$. The easy verification of (33.1) is left for the reader. Proof of (33.2) runs as follows:

$$\begin{aligned}
\partial(\sigma \times \tau) &= \partial\big((\sigma \times \tau)_\sharp(\iota_p \times \iota_q)\big) \\
&= (\sigma \times \tau)_\sharp\partial(\iota_p \times \iota_q) \\
&= (\sigma \times \tau)_\sharp\partial z \\
&= (\sigma \times \tau)_\sharp\Big(\partial\iota_p \times \iota_q + (-1)^p(\iota_p \times \partial\iota_q)\Big)
\end{aligned}$$

Applying (33.1), which holds by induction hypothesis, and using the pair of equations $\sigma_\sharp\partial = \partial\sigma_\sharp$, $\tau_\sharp\partial = \partial\tau_\sharp$ we continue with our calculation:

$$\begin{aligned}
\partial(\sigma \times \tau) &= \sigma_\sharp(\partial\iota_p) \times \tau_\sharp(\iota_q) + (-1)^p(\sigma_\sharp(\iota_p) \times \tau_\sharp(\partial\iota_q)) \\
&= \partial\sigma \times \tau + (-1)^p(\sigma \times \partial\tau).
\end{aligned}$$

Having defined $\sigma \times \tau$ for singular simplices σ and τ, we can extend it as a bilinear map $S_p(X) \times S_q(Y) \longrightarrow S_{p+q}(X \times Y)$ since $S_p(X)$ and $S_q(Y)$ are free abelian groups.

Homotopy and chain homotopy: Chain homotopy is the algebraization of the topological notion of homotopic maps. Let $F : I \times X \longrightarrow Y$ be a homotopy between two continuous functions $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$. We use this map to define a sequence of maps

$$L_n : S_n(X) \longrightarrow S_{n+1}(Y) \tag{33.6}$$

satisfying the condition

$$\partial \circ L_n + L_{n-1} \circ \partial = f_\sharp - g_\sharp. \tag{33.7}$$

Let $u : \Delta_1 \longrightarrow I$ be the unique one simplex. For a singular n simplex σ in X, define

$$L_n(\sigma) = F_\sharp(u \times \sigma).$$

Then we compute using (33.1)-(33.2),

$$\begin{aligned} \partial(L_n(\sigma)) &= F_\sharp(\partial u \times \sigma) - F_\sharp(u \times \partial\sigma) \\ &= F_\sharp(\partial u \times \sigma) - L_{n-1}(\partial\sigma) \\ \therefore\ \partial(L_n(\sigma)) + L_{n-1}(\partial\sigma) &= F_\sharp(\{1\} \times \sigma) - F_\sharp(\{0\} \times \sigma) \\ \therefore\ \partial(L_n(\sigma)) + L_{n-1}(\partial\sigma) &= F(1, \sigma(\cdot)) - F(0, \sigma(\cdot)). \end{aligned}$$

So we have the important equation

$$\partial(L_n(\sigma)) + L_{n-1}(\partial\sigma) = g_\sharp(\sigma) - f_\sharp(\sigma), \quad \sigma \in S_n(X). \tag{33.8}$$

completing the proof of (33.7). The reader must go back to lemma (32.2) to observe some analogies. After these preparations we are ready to prove the following important result. Unlike theorems (11.2) - (11.5) we do not have to worry here about base points which makes life a lot easier.

Theorem 33.2: Homotopic maps $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ induce equal maps in homology. That is to say for each n we have

$$H_n(f) = H_n(g). \tag{33.9}$$

Proof: Taking $\sigma \in Z_n(X)$ in (33.8), the term $L_{n-1}(\partial\sigma)$ drops out and we immediately see that the cycles $f_\sharp(\sigma)$ and $g_\sharp(\sigma)$ differ by a boundary. The proof is complete.

We see that equation (33.7) is the algebraic analogue of homotopy of continuous maps. As this phenomenon would recur often, we give a formal definition and a name for it.

Definition 33.1: Given chain maps $\phi_n : C_n \longrightarrow D_n$ and $\psi_n : C_n \longrightarrow D_n$ $(n = 1, 2, \dots)$ between chain complexes C and D, a chain homotopy between ϕ and ψ is a sequence $L_n : C_n \longrightarrow D_{n+1}$ of group homomorphisms such that

$$\partial \circ L_n + L_{n-1} \circ \partial = \phi_n - \psi_n \tag{33.10}$$

It is easy to see that that *chain homotopy* is an equivalence relation on the family of chain maps. Recalling now the definition of homotopy equivalence (see lecture 11, definition 11.2) we state the very useful result which follows immediately from theorem (33.2).

Corollary 33.3: If X and Y have the same homotopy type, then $H_n(X) = H_n(Y)$ for $n = 0, 1, 2, \ldots$.

Exercises

1. Show that the $p + q - 1$ chain on the right hand side of (33.4) is a cycle.
2. Check that $\sigma \times \tau$ as defined by equation (33.5) satisfies (33.1).
3. Show that the product in theorem (33.1) defines a bilinear map $H_p(X) \times H_q(Y) \longrightarrow H_{p+q}(X \times Y)$.
4. Determine explicitly the two/three chain z satisfying (33.4) when

   (i) $p = 1$ and $q = 1$.

   (ii) $p = 1$ and $q = 2$.

   Hint: In the proof of lemma (32.2), we chopped the square into two triangles. When $q = 2$ we need to chop a prism into three pieces and map Δ_3 affinely onto each of them.
5. Use the map Π_X of the previous lecture to calculate the generators of $H_1(S^1 \times S^1)$.
6. Use equation (33.1) to determine the image of the pair of generating one cycles of the previous exercise under the map $H_1(S^1) \times H_1(S^1) \longrightarrow H_2(S^1 \times S^1)$.

Lecture - XXXIV Small simplicies

Recall that the Goursat lemma in complex analysis is proved by subdividing a triangle into four smaller triangles determined by the midpoints of the sides of the given triangle. The integral over the given triangle is then the sum of the integrals over the four little pieces. Likewise, in the proof of the classical Green's theorem (of which Cauchy's theorem is really a special case) one employs a subdivision into tiny squares. The contributions to the integral from an edge common to a pair of abutting triangles/squares cancel out.

A similar idea underlies the method of small simplicies where we perform a systematic subdivision operation known as *barycentric subdivision.* The barycentric subdivision enables us to replace a singular chain by a homotopic one in which the constituent singular simplicies are *small.* A small simplex is one whose image lies in an open set belonging to a prescribed open cover of the space. One achieves this through iterated barycentric subdivisions a process reminiscent of one used in the proof of the Goursat lemma. The fundamental theorem on small simplicies quickly leads us to the two fundamental results on algebraic topology - the excision theorem discussed in lecture 39 and the Mayer Vietoris sequence that we shall derive here and use in the next lecture.

Affine simplicies and barycentric subdivision: Given points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}$ in the standard $n-$simplex Δ_n, the continuous map $\sigma : \Delta_p \longrightarrow \Delta_n$ given in terms of the barycentric coordinates

$$\sum_{i=1}^{p+1} \lambda_i \mathbf{e}_i \mapsto \sum_{i=1}^{p+1} \lambda_i \mathbf{v}_i \tag{34.1}$$

is called an affine $p-$simplex and is denoted by $[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}]$. Note that the given need not be affinely independent. Each such σ is an element of $S_p(\Delta_n)$ and the subgroup generated by them is called the group of affine $p-$simplicies in Δ_n denoted by $A_p(\Delta_n)$. Thus $A_p(\Delta_n)$ is the set of all formal linear combinations with integer coefficients of affine simplicies. Since the face maps (29.1) are affine maps we conclude from exercise 2 that the boundary homomorphism $\partial_p : S_p(\Delta_n) \longrightarrow S_{p-1}(\Delta_n)$ maps $A_p(\Delta_n)$ into $A_{p-1}(\Delta_n)$ and so we get a subcomplex $\{A_p(\Delta_n)/p = 0, 1, 2, \dots\}$ with boundary maps as the restrictions of ∂_p to $A_p(\Delta_n)$.

If $\mathbf{b} \in \Delta_n$ is a given point the *cone* over the affine simplex $\sigma = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}]$ with vertex apex $\mathbf{b}$ is denoted by $K_{\mathbf{b}}\sigma$ and is defined as

$$K_{\mathbf{b}}\sigma = [\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}] \tag{34.2}$$

The cone $K_{\mathbf{b}}\sigma$ is thus an affine $p + 1$ simplex. If we start with a zero simplex namely, a point $\mathbf{v} \in S_n(\Delta_n)$, the cone over it is the line segment $[\mathbf{b}, \mathbf{v}]$. Since $A_p(\Delta_n)$ is a free abelian group generated by the affine p simplicies, we obtain by extension a group homomorphism $K_{\mathbf{b}} : A_p(\Delta_n) \longrightarrow A_{p+1}(\Delta_n)$. As in the proof of theorem (29.7) it is easy to compute the boundary of the cone $K_{\mathbf{b}}\sigma$ for any affine p simplex.

For a zero simplex $\sigma = [\mathbf{v}]$ we evidently have $\partial_1 K_{\mathbf{b}}(\sigma) = \sigma - [\mathbf{b}]$. We now calculate the faces of the affine $p+1$ simplex $K_{\mathbf{b}}(\sigma)$. If $j \geq 1$,

$$\begin{aligned}(K_{\mathbf{b}}\sigma \circ \Phi_j^p)(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) &= K_{\mathbf{b}}\sigma(\lambda_1, \ldots, \lambda_j, 0, \lambda_{j+1}, \ldots, \lambda_{p+1}) \\ &= [\mathbf{b}, \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{p+1}].\end{aligned}$$

This is the cone over the $j-$th face of $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$. Turning to the case $j = 0$,

$$(K_{\mathbf{b}}\sigma \circ \Phi_0^p)(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) = K_{\mathbf{b}}\sigma(0, \lambda_1, \ldots, \lambda_{p+1}) = [\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}].$$

Using equation (29.4) we immediately get the following result.

Theorem 34.1: The boundary of the cone $K_{\mathbf{b}}\sigma = [\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ is given by

$$\partial K_{\mathbf{b}}\sigma = \sigma - K_{\mathbf{b}}\partial\sigma \tag{34.3}$$

Hitherto the choice of the apex $\mathbf{b}$ of the cone was arbitrary but now we shall specialize it to be the barycenter of $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ that we now define.

Definition 34.1 (Barycenter of an affine simplex): (i) The barycenter of a zero simplex, that is a point, is the zero simplex itself.

(ii) The barycenter of an affine p-simplex $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ is the point

$$\frac{1}{p+1}\Big(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{p+1}\Big) \tag{34.4}$$

The barycenter of a one simplex is its midpoint and the barycenter of a two simplex is the centroid of the triangle determined by the vertices. Roughly speaking, the barycentric subdivision of a one simplex is obtained by subdividing the segment at its midpoint, or equivalently constructing the cone of each of the two endpoints with apex as the barycenter. To subdivide a two simplex, we first subdivide each of its three sides resulting in six one simplicies and taking the cone of each of the six pieces with apex as the barycenter of the two simplex. Figure below depicts these subdivisions. More precisely

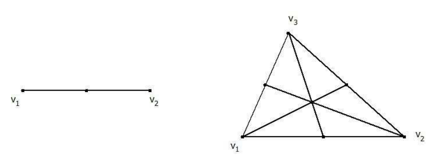


Figure 25:

the result of subdividing an affine $p-$simplex is a $p-$chain. The rough description above suggests an inductive definition.

Definition 34.2: The subdivision map $B : A_p(\Delta_n) \longrightarrow A_p(\Delta_n)$ is defined inductively as follows:

(i) For a zero simplex σ we define $B\sigma = \sigma$.

(ii) For $p \geq 1$, we assume that B is defined on $A_k(\Delta_n)$ for each $k \leq p-1$. For a $p-$simplex σ define

$$B\sigma = K_{\mathbf{b}}(B(\partial\sigma)), \tag{34.5}$$

the cone over the chain $B(\partial\sigma)$ with apex $\mathbf{b}$ as the barycenter of σ.

Theorem 34.2: The map $B : A_p(\Delta_n) \longrightarrow A_p(\Delta_n)$ is a chain map which is chain homotopic to the identity map.

Proof: If $p = 0$ and σ is a zero chain then $B\partial\sigma = 0$ whereas $\partial B\sigma = \partial\sigma = 0$. To handle the case $p > 0$ we assume inductively that for any $k-$chain σ with $k \leq p - 1$, the equation $\partial B\sigma = B\partial\sigma$ holds. To prove it for p chains, let σ be an arbitrary affine $p-$simplex. Equations (34.5) and (34.3) combine to give

$$\partial B\sigma = \partial K_{\mathbf{b}}(B\partial\sigma) = B\partial\sigma - K_{\mathbf{b}}(\partial B\partial\sigma) = B\partial\sigma - K_{\mathbf{b}}(B\partial\partial\sigma) = B\partial\sigma.$$

Note that induction hypothesis justifies $\partial B\partial\sigma = B\partial\partial\sigma$. We have now shown that for every p chain σ,

$$B\partial\sigma = \partial B\sigma. \tag{34.6}$$

We now construct a chain homotopy $J : A_p(\Delta_n) \longrightarrow A_{p+1}(\Delta_n)$ between B and the identity map. Equation (34.3) suggests a formula of the type

$$J\sigma = K_{\mathbf{b}} f(\sigma),$$

where $\mathbf{b}$ is the barycenter of σ and $f : A_p(\Delta_n) \longrightarrow A_p(\Delta_n)$ is to be determined. The condition that J is a chain homotopy between B and the identity now forces

$$f(\sigma) - K_{\mathbf{b}}\partial f(\sigma) = B\sigma - \sigma - J(\partial\sigma). \tag{34.7}$$

Clearly $f(\sigma) = 0$ for a zero simplex σ. If we assume inductively that $J : A_k(\Delta_n) \longrightarrow A_{k+1}(\Delta_n)$ has already been defined for $k \leq p-1$, the right hand side of (34.7) is then a known function. Let us refer to the term $K_{\mathbf{b}}\partial f(\sigma)$ in equation (34.7) as *junk*. Exercise 3 invites the reader to check that retaining the junk term is unnecessary. We set it equal to zero and define formally for a $p-$simplex σ,

$$J\sigma = \begin{cases} 0 & \text{if } p = 0, \\ K_{\mathbf{b}}(B\sigma - \sigma - J(\partial\sigma)) & \text{if } p \geq 1. \end{cases}$$

Let us now verify that this formula does the job. The case $p = 0$ is trivial and let us assume

$$\partial J\sigma + J(\partial\sigma) = B\sigma - \sigma$$

for any k chain such that $k \leq p - 1$. Using the formula of J we see that

$$\partial J\sigma = B\sigma - \sigma - J(\partial\sigma) - K_{\mathbf{b}}(\partial B\sigma - \partial\sigma - \partial J(\partial\sigma)). \tag{34.8}$$

By induction hypothesis $\partial J(\partial\sigma) = -J(\partial\partial\sigma) + B(\partial\sigma) - \partial\sigma$. Inserting this in (34.8) we get the desired result

$$\partial J\sigma = B\sigma - \sigma - J(\partial\sigma). \tag{34.9}$$

We shall now transfer the barycentric subdivision operator and the chain homotopy J to a chain map $\mathcal{B} : S_p(X) \longrightarrow S_p(X)$ and a chain homotopy $\mathcal{J} : S_p(X) \longrightarrow S_{p+1}(X)$. This will be unique subject to naturality.

Theorem 34.3: For each topological space X, there exists a unique chain map $\mathcal{B}_X : S_p(X) \longrightarrow S_p(X)$ and a chain homotopy $\mathcal{J}_X : S_p(X) \longrightarrow S_{p+1}(X)$ between $\mathcal{B}$ and the identity map, which satisfies the following two conditions.

(i) For a continuous map $f : X \longrightarrow Y$ between topological spaces X and Y,

$$\mathcal{B} \circ f_\sharp = f_\sharp \circ \mathcal{B}, \quad \mathcal{J} \circ f_\sharp = f_\sharp \circ \mathcal{J}.$$

(ii) $\mathcal{B}$ and $\mathcal{J}$ when restricted to the affine simplicies reduce to B and J respectively.

Proof: Let ι_p be the element of $A_p(\Delta_p)$ given by the identity map from Δ_p to itself. Since an arbitrary $\sigma \in S_p(X)$ can be written as $\sigma = \sigma_\sharp \iota_p$, condition (i) forces

$$\mathcal{B}\sigma = (\mathcal{B} \circ \sigma_\sharp)\iota_p = \sigma_\sharp(\mathcal{B}\iota_p) = \sigma_\sharp B \iota_p \tag{34.10}$$

since $\mathcal{B}\iota_p = B\iota_p$ by (ii). Thus the conditions (i) and (ii) determine $\mathcal{B}$ uniquely on the generators of the free abelian group $S_p(X)$. The same argument applies to $\mathcal{J}$.

We use (34.10) and (34.5) to show that B and $\mathcal{B}$ agree on any affine simplex $\sigma \in A_p(\Delta_n)$. Denoting by $\mathbf{g}$ the barycenter of ι_p and by $\mathbf{b}$ the barycenter of σ,

$$\mathcal{B}\sigma = \sigma_\sharp B\iota_p = \sigma_\sharp(K_{\mathbf{g}}(\partial_p \iota_p)).$$

Using exercise 2, this may be rewritten as

$$\mathcal{B}\sigma = K_{\sigma(\mathbf{g})}(\sigma_\sharp(\partial_p\iota_p)) = K_{\mathbf{b}}(\partial_p \sigma_\sharp \iota_p) = K_{\mathbf{b}}(\partial_p\sigma) = B\sigma.$$

The verification for $\mathcal{J}$ is similar. We now run through the proof that $\mathcal{B}$ is a chain map, which is now automatic. For an arbitrary $\sigma \in S_p(X)$, $\partial\mathcal{B}\sigma = \partial(\sigma_\sharp B\iota_p) = \sigma_\sharp(\partial B\iota_p)$. Since $\partial\iota_p$ is an affine chain and B is a chain map on the subcomplex of affine chains we get $\partial B\iota_p = B\partial\iota_p$. Applying $\sigma_\sharp$ to this gives $\partial\mathcal{B}\sigma = \sigma_\sharp(B\partial\iota_p)$. Working from the other end using the fact that $\sigma_\sharp$ is a chain map and $\mathcal{B}$ satisfies (i), we get

$$\mathcal{B}\partial\sigma = \mathcal{B}\partial(\sigma_\sharp\iota_p) = \mathcal{B}(\sigma_\sharp\partial\iota_p) = \sigma_\sharp\mathcal{B}(\partial\iota_p) = \sigma_\sharp B(\partial\iota_p).$$

Finally we show that $\mathcal{J}$ is a chain homotopy between $\mathcal{B}$ and the identity operator. For $\sigma \in S_p(X)$,

$$\mathcal{J}\partial\sigma = \mathcal{J}\partial\sigma_\sharp\iota_p = \mathcal{J}\sigma_\sharp\partial\iota_p = \sigma_\sharp\mathcal{J}\partial\iota_p = \sigma_\sharp J\partial\iota_p$$
$$\partial\mathcal{J}\sigma = \partial\mathcal{J}\sigma_\sharp\iota_p = \partial\sigma_\sharp\mathcal{J}\iota_p = \sigma_\sharp\partial\mathcal{J}\iota_p = \sigma_\sharp\partial J\iota_p.$$

We have used (i) and (ii) and the fact that $\sigma_\sharp$ is a chain map. Subtracting and using (34.9) we get the desired result. □

Theorem 34.4: (i) The diameter of an affine $p-$simplex $\sigma = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ is the length of its longest side namely,

$$\max_{i \neq j} \|\mathbf{v}_i - \mathbf{v}_j\|.$$

(ii) The diameter of any constituent simplex in the chain $B\sigma$ is $\left(\dfrac{p}{p+1}\right)\text{diam}\,(\sigma)$.

Proof: We leave (i) as an exercise for the reader. To prove (ii) we use induction on p setting aside the cases $p = 1, 2$ for the reader to investigate. Denoting by $\mathbf{b}$ the barycenter of σ, the reader may check that $\|\mathbf{b}-\mathbf{x}\| \leq p(p+1)^{-1}(\text{diam } \sigma)$ for any point $\mathbf{x}$ of σ. Let τ be one of the simplicies appearing in the chain $B\sigma$. Then the diameter of τ equals $\|w - z\|$ where w and z are two vertices of τ. If one of these is $\mathbf{b}$ then the result follows from the assertion in the previous sentence. If neither w nor z is $\mathbf{b}$ then they are both vertices of a face τ' of τ lying on a face σ' of σ. But τ' is then a constituent $(p-1)$ simplex of $B(\sigma')$ and by induction hypothesis, the result follows (how?). □

Definition 34.3: Given an open covering $\mathcal{U}$ of X, $S_n^{\mathcal{U}}(X)$ denotes the subgroup of $S_n(X)$ generated by all the singular simplicies $\sigma : \Delta_n \longrightarrow X$ such that $\sigma(\Delta_n) \subset U_\sigma$ for some open set U_σ in the covering $\mathcal{U}$. That is to say, $S_n^{\mathcal{U}}(X)$ is the free abelian group generated by *small simplicies,* namely those with images contained in one of the open sets in the given covering. It is clear that that the boundary homomorphism ∂_n maps $S_n^{\mathcal{U}}(X)$ into $S_{n-1}^{\mathcal{U}}(X)$ and the resulting subcomplex is denoted by $S^{\mathcal{U}}(X)$. The homology groups of the complex $S^{\mathcal{U}}(X)$ will be denoted by $H_n^{\mathcal{U}}(X)$.

Lemma 34.5: (i) Given an open cover $\mathcal{U}$ of X and a singular simplex $\sigma \in S_p(X)$, there exists a $k \in \mathbb{N}$ such that $\mathcal{B}^k\sigma \in S_p^{\mathcal{U}}(X)$. In other words each of the simplicies occurring in $\mathcal{B}^k\sigma$ has its image in one of the open sets of the cover $\mathcal{U}$.

(ii) If σ is a singular p simplex whose image lies in an open set $U \in \mathcal{U}$ then $\mathcal{J}\sigma \in S_{p+1}^{\mathcal{U}}(X)$ where $\mathcal{J}$ is the chain homotopy constructed in theorem (34.3).

Proof: (i) Choose a Lebesgue number for the open cover $\{\sigma^{-1}(U) \,/U \in \mathcal{U}\}$. According to theorem (34.3), the images of the simplicies occurring in the chain $\mathcal{B}^k\sigma$ are the same as the images under σ of the affine simplicies occurring in $B^k\iota_p$, where ι_p is the identity map of Δ_p. However, theorem (34.4) states that the simplicies occurring in $B^k\iota_p$ have diameters less than $(p(p+1)^{-1})^k$. Thus, if we choose k sufficiently large the image of each of the simplicies in $B^k\sigma$ would lie in one of the open sets of $\mathcal{U}$.

To prove (ii) we use the naturality of $\mathcal{J}$ and proceed as in the proof of theorem (34.4). Let $\sigma : \Delta_p \longrightarrow X$ have its image in $U \in \mathcal{U}$. Then $\mathcal{J}\sigma = \sigma_\sharp(J\iota_p)$. But we see immediately from the definition of J in theorem (34.3) that $J\iota_p$ is a $\mathbb{Z}-$linear combination:

$$J\iota_p = \sum c_k\lambda_k$$

where each λ_k is a (degenerate) affine (p+1) simplex contained in Δ_p and hence $\sigma_\sharp(\lambda_k)$ is a singular $(p+1)$ simplex with image contained in U. □

Theorem 34.6: The inclusion maps $S_n^{\mathcal{U}}(X) \longrightarrow S_n(X)$ $(n = 0, 1, 2, \ldots)$ define a chain map of complexes. Further, these inclusion maps induce isomorphisms in homology:

$$H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X), \quad n = 0, 1, 2, \ldots$$

Proof: The first assertion follows from the comments preceding lemma (34.5). To show that the inclusion maps induce an injective map on homologies, let $\sigma \in S_p^{\mathcal{U}}(X)$ be a singular chain such that $\sigma = \partial\eta$ for some $\eta \in S_{p+1}(X)$. Choose $k \in \mathbb{N}$ such that $\mathcal{B}^k\eta \in S_{p+1}^{\mathcal{U}}(X)$. We have to show that $\mathcal{B}^k\eta$ is a boundary in $S^{\mathcal{U}}$. By exercise 5, $\mathcal{B}^k$ is chain homotopic to the identity via a homotopy T_k say. Applying ∂ to

$$\mathcal{B}^k\eta - \eta = T_k\partial\eta + \partial T_k\eta,$$

we see that $\partial(\mathcal{B}^k\eta) - \sigma = \partial T_k\sigma$. By (ii) of lemma (34.5), $\partial T_k\sigma \in S_p^{\mathcal{U}}(X)$ which means σ is a boundary in $S_p^{\mathcal{U}}(X)$. To prove surjectivity, let σ be a cycle in $S(X)$ and $k \in \mathbb{N}$ be such that $\mathcal{B}^k\sigma \in S^{\mathcal{U}}(X)$. From $\mathcal{B}^k\sigma - \sigma = \partial T_k\sigma$ we conclude that σ is homologous to the cycle $\mathcal{B}^k\sigma$ in $S^{\mathcal{U}}(X)$. □

Theorem 34.7 (Mayer Vietoris sequence): (i) Let $\{U, V\}$ be an open covering of X,

$$\kappa' : H_k(U \cap V) \longrightarrow H_k(U), \quad \kappa'' : H_k(U \cap V) \longrightarrow H_k(V)$$

be the maps induced by inclusions. Further, let $q_n : H_n(U) \oplus H_n(V) \longrightarrow H_n(U \cup V)$ be the map:

$$(a, b) \mapsto j_{1*}a + j_{2*}b,$$

where j_{1*} and j_{2*} are induced by the respective inclusions $j_1 : U \longrightarrow U \cup V$ and $j_2 : V \longrightarrow U \cup V$. Then, the following long exact sequence known as the *Mayer Vietoris sequence* holds:

$$\longrightarrow H_n(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_n(U) \oplus H_n(V) \xrightarrow{q_n} H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow$$

(ii) A cycle $\zeta \in Z_n(U \cup V)$ may be represented (modulo boundaries) as $\zeta = \zeta_1 + \zeta_2$ for some $\zeta_1 \in S_n(U)$ and $\zeta_2 \in S_n(V)$ and the connecting homomorphism δ_n is given by

$$\delta_n \; : \; \zeta \mapsto \partial\zeta_1 = -\partial\zeta_2.$$

Proof: In the diagrams below, the Left hand square depicts a push-out square of inclusions which goes over to a push-out square of complexes on the right:

$$\begin{array}{ccc} U \cap V & \xrightarrow{i_1} & U \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ V & \xrightarrow{j_2} & U \cup V \end{array} \qquad\qquad \begin{array}{ccc} S(U \cap V) & \xrightarrow{i_1} & S(U) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ S(V) & \xrightarrow{j_2} & S^{\mathcal{U}}(U \cup V) \end{array}$$

The reader may check that the latter may be recast as a short exact sequence of chain complexes namely

$$0 \longrightarrow S(U \cap V) \xrightarrow{(i_1, -i_2)} S(U) \oplus S(V) \xrightarrow{j_1 + j_2} S^{\mathcal{U}}(U \cap V) \longrightarrow 0. \qquad (34.11)$$

The corresponding long exact sequence in homology gives

$$\longrightarrow H_n(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_n(U) \oplus H_n(V) \xrightarrow{Q_n} H_n^{\mathcal{U}}(U \cup V) \xrightarrow{D_n} H_{n-1}(U \cap V) \longrightarrow$$

The definition of κ', κ'' and exercise 6 enables us to replace Q_n and D_n by the composites

$$q_n \; : \; H_n(U) \oplus H_n(V) \xrightarrow{Q_n} H_n^{\mathcal{U}}(U \cup V) \xrightarrow{\lambda} H_n(U \cup V)$$

$$\delta_n \; : \; H_n(U \cap V) \xrightarrow{\lambda^{-1}} H_n^{\mathcal{U}}(U \cup V) \xrightarrow{D_n} H_n(U \cap V) \qquad (34.12)$$

where λ is the isomorphism given by theorem (34.6). The final result is the Mayer Vietoris sequence stated in the theorem. The second part is clear from (29.18). □

Theorem 34.8 (Naturality of the Mayer Vietoris sequence): Given a continuous map of pairs $f : (U, V) \longrightarrow (A, B)$ where $\{U, V\}$ and $\{A, B\}$ are open coverings of topological spaces, the following diagram commutes where the vertical maps are induced by f.

$$\begin{array}{ccccccccc} \longrightarrow & H_n(U\cap V) & \longrightarrow & H_n(U)\oplus H_n(V) & \longrightarrow & H_n(U\cup V) & \xrightarrow{\delta_n} & H_{n-1}(U\cap V) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & H_n(A\cap B) & \longrightarrow & H_n(A)\oplus H_n(B) & \longrightarrow & H_n(A\cup B)) & \longrightarrow & H_{n-1}(A\cap B) & \longrightarrow \end{array} \tag{34.13}$$

Proof: The proof is left for the reader. The non-trivial part concerns only the squares involving the connecting homomorphism for which (ii) of the previous theorem may be employed. □

Exercises

1. Show that the map defined by (34.1) is the restriction to Δ_p of an affine map. Note: An affine map is the composition of a linear map and a translation.

2. Suppose $T : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{m+1}$ is an affine map such that $T(\Delta_n) \subset \Delta_m$, then $T_\sharp$ maps the subgroup $A_p(\Delta_n)$ into $A_p(\Delta_m)$ and is a chain map from the complex $\{A_p(\Delta_n)\}$ to $\{A_p(\Delta_m)\}$. Further prove the following:

   (i) If $\mathbf{b} \in \Delta_n$ and $\sigma \in A_p(\Delta_n)$ then $T_\sharp(K_{\mathbf{b}}\sigma) = K_{T\mathbf{b}}(T_\sharp\sigma)$.

   (ii) If $\mathbf{b}$ is the barycenter of σ then $\mathbf{b}$ is the barycenter of $T_\sharp\sigma$.

   What happens if we consider a *degenerate* two simplex where the points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are not affinely independent? Discuss the case of the two simplex $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2]$.

3. Examine what happens if the term referred to as junk in equation (34.7) is retained.

4. Complete the details of the proof of theorem (34.4).

5. Show that $\mathcal{B}^k$ is chain homotopic to the identity map. What is the chain homotopy?

6. Suppose that the maps g and h in the exact sequence

$$A \longrightarrow B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow E$$

   are replaced by the composites

$$\tilde{g} : B \xrightarrow{g} C \xrightarrow{\lambda} X, \qquad \tilde{h} : X \xrightarrow{\lambda^{-1}} C \xrightarrow{h} D$$

   the result is again an exact sequence.

7. Fill in the details in the proof of theorem (34.8). See exercise 6 of lecture 29.

Lecture - XXXV The Mayer Vietoris sequence and its applications

The proof of Mayer Vietoris sequence is reminiscent of the Seifert Van Kampen theorem. While the Seifert Van Kampen theorem enables us to relate the fundamental group of a union $U \cup V$ in terms of the fundamental groups of U, V and $U \cap V$, the situation here is slightly more involved. The precise relationship between the homologies of $U, V, U \cap V$ and $U \cup V$ is described in terms of the long exact sequence of theorem (34.7).

As in the Seifert Van Kampen theorem we obtain from a push-out diagram of topological spaces a push out diagram of chain complexes which turns into a short exact sequence of complexes. The corresponding long-exact sequence gives, after an application of the excision theorem of the last lecture, the Mayer Vietoris sequence. It is one of the most efficient tools available for the computation of homology groups. We restate here the theorem for convenience.

Theorem 35.1: Suppose U and V are subsets of a topological space such that $\text{Int}\, U \cup \text{Int}\, V = X$. Then there is a long exact sequence

$$\longrightarrow H_n(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_n(U) \oplus H_n(V) \xrightarrow{q_n} H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow$$

Interpretation of the connecting homomorphism: We use equation (29.18) to describe explicitly the connecting homomorphism in the Mayer Vietoris sequence. Take a representative cycle ζ in $H_n(U \cup V)$. Theorem (34.6) implies that an arbitrary element of $H_n(U \cup V)$ can be represented as a sum of chains

$$\zeta = \zeta_1 + \zeta_2$$

where $\zeta_1 \in S_n(U)$ and $\zeta_2 \in S_n(V)$. Note that we are resorting to an abuse notation in writing ζ_1 instead of $i_\sharp(\zeta_1)$. We conclude that $\partial\zeta_1 = -\partial\zeta_2$. Thus $\partial\zeta_1$ and $\partial\zeta_2$ are both cycles in $U \cap V$. According to (29.18), the homomorphism δ_n is given by

$$\delta_n(\overline{\zeta}) = \overline{\partial\zeta_1}$$

Corollary 35.2: The homology groups of the spheres S^n $(n \geq 1)$ are given by

$$H_m(S^n) = \begin{cases} 0 & \text{if } m \neq 0,\ m \neq n \\ \mathbb{Z} & \text{if } m = 0,\ m = n \end{cases}$$

Proof: We take $U = S^n - \{\mathbf{e}_{n+1}\}$ and $V = S^n - \{-\mathbf{e}_{n+1}\}$ and note that $U \cap V$ deformation retracts to S^{n-1}. Consider the portion of the Mayer Vietoris sequence

$$\longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow H_{n-1}(U) \oplus H_{n-1}(V) \longrightarrow$$

Since U and V are contractible spaces, we get for the case $n \geq 2$,

$$0 \longrightarrow H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow 0,$$

and hence $H_n(S^n) \cong H_{n-1}(S^{n-1})$ $(n \geq 2)$. By induction the result would follow as soon as we prove it for the case $n = 1$. For this case let us take a look at the end of the Mayer Vietoris sequence:

$$0 \longrightarrow H_1(S^1) \xrightarrow{\delta_1} H_0(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_0(U) \oplus H_0(V) \longrightarrow H_0(U \cup V) \longrightarrow$$

Since δ_1 is injective,

$$H_1(S^1) \cong \text{im } \delta_1 = \ker (\kappa', -\kappa'').$$

To understand the map $(\kappa', -\kappa'')$ we take a basis of $H_0(U \cap V)$ consisting of a pair of points $a \in U$ and $b \in V$. The singleton $\{a\}$ generates $H_0(U)$ and

$$k'(ma + nb) = ma + nb = n(b - a) + (m + n)a$$

which is a boundary in $H_0(U)$ if and only if $m + n = 0$. Likewise $k''(ma + nb) = 0$ in $H_0(V)$ if and only if $m + n = 0$. Thus the kernel of (κ', κ'') is the infinite cyclic group generated by the zero chain $a - b$. Hence we get

$$H_1(S^1) \cong \mathbb{Z}.$$

To calculate $H_n(S^1)$ for $n \geq 2$ we look at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow$$

and observe that since $H_n(U) = H_n(V) = H_{n-1}(U \cap V) = 0$ when $n \geq 2$,

$$H_n(S^1) = \{0\}, \quad n \geq 2.$$

Corollary 35.3: For $m, n \in \mathbb{N}$ with $m < n$, the spheres S^m and S^n are non-homeomorphic. Also $\mathbb{R}^m$ and $\mathbb{R}^n$ are non-homeomorphic.

Proof: The first part follows from the fact that the homology groups $H_n(S^m)$ and $H_n(S^n)$ are non-isomorphic. If $\mathbb{R}^m$ and $\mathbb{R}^m$ were homeomorphic then their one-point compactifications would also be homeomorphic which means S^n and S^m would be homeomorphic leading to a contradiction.

Homology groups of adjunction spaces: We shall now consider the space $Y = X \sqcup_f E^k$ obtained by attaching a $k-$cell E^n to X via an attaching map

$$f : S^{k-1} \longrightarrow X.$$

We shall closely follow the method used in lecture 26 to compute the fundamental groups of the projective plane and Klein's bottle. We do not have to keep track of base points and use the Mayer Vietoris sequence instead of the Seifert Van Kampen theorem. We shall use the same notations and denote by p the center of E^k, the interior of E^k by U and the space $Y - \{p\}$ by V. The space $U \cap V$ deformation retracts to a space homeomorphic to S^{k-1}. Since V deformation retracts to X, the spaces V and X have the same homology groups and $H_n(U) = \{0\}$ when $n \geq 1$. We are ready to prove the following result:

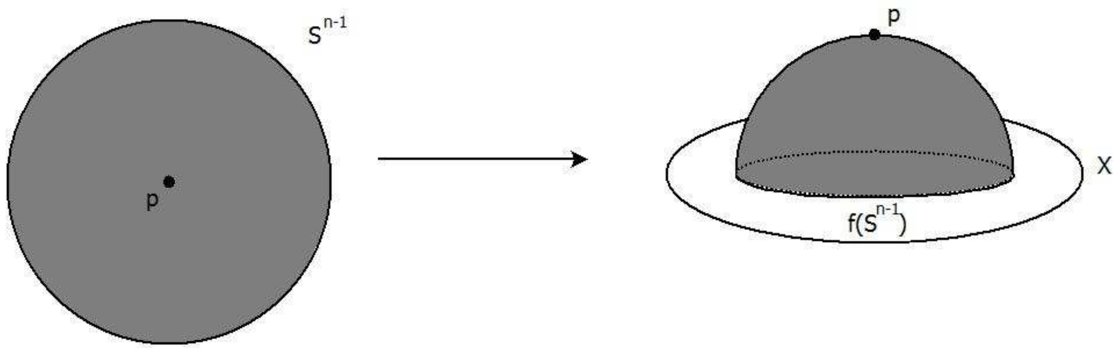


Figure 26: Adjucntion space

Theorem 35.4 $H_n(X \sqcup_f E^k) = H_n(X)$ if $n \neq k, k-1$.

Proof: Looking at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(Y) \longrightarrow H_{n-1}(U \cap V) \longrightarrow$$

we get the result directly when $n \geq 2$. If $n = 1$ then necessarily $k \geq 3$ and we look at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(Y) \longrightarrow H_0(U \cap V) \xrightarrow{\cong} H_0(U) \oplus H_0(V).$$

Observe that $H_1(U \cap V) = \{0\}$ and we get the exact sequence

$$0 \longrightarrow H_1(V) \longrightarrow H_1(Y) \longrightarrow 0,$$

establishing the result when $n = 1$. □

The cases $n = k, k-1$ are more technical and we shall merely state the relevant results.

Theorem 35.5: With notations as in theorem (35.4),

$$H_{k-1}(X \sqcup_f E^k) = H_{k-1}(X)/\text{im } H_{k-1}(f), \quad H_k(X \sqcup_f E^k) = H_k(X) \oplus \ker H_{k-1}(f)$$

Corollary 35.6 (Homology groups of $\mathbb{R}P^2$): $H_0(\mathbb{R}P^2) = \mathbb{Z}$, $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$. All other homology groups vanish.

Proof: Recall example (25.4) that $\mathbb{R}P^2$ arises from S^1 by attaching a two cell using the attaching map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^2$. Since $H_1(f) : \mathbb{Z} \longrightarrow \mathbb{Z}$ is given by $n \mapsto 2n$ the result immediately follows from theorems (35.4)-(35.5).

Exercises

1. Prove that a homeomorphism E^n onto itself maps each boundary point of E^n to a boundary point.

2. Determine the homology groups of the Klein's bottle.

3. Determine the homology groups of the double torus.

4. Establish the isomorphism $H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V)$ in the proof of theorem (35.4)

5. Let C_k be the disjoint union of k copies of S^1 in $\mathbb{R}^3$. Determine the homology groups of the complement $\mathbb{R}^3 - C_k$.

6. Determine the homology groups of $\mathbb{R}P^3$. Try computing the homology groups of $\mathbb{R}P^4$.

7. Determine the homology groups of $S^n \vee S^m$. Use exercise 4 of lecture 25. to calculate the homology groups of $S^2 \times S^4$.

Lecture - XXXVI Maps of spheres

We are now in a position to prove the general Brouwer's fixed point theorem as well as a few other surprising results concerning maps of spheres. As demonstrated in lecture 10, these higher dimensional analogues were inaccessible via the theory of the fundamental group. We shall introduce the notion of the degree of a map of spheres generalizing the notion introduced in lectures (12-13).

Theorem 36.1 (No retraction theorem): The sphere S^{n-1} is not a retract of the closed unit ball E^n.

Proof: Assume $n \geq 2$. A retraction $r : E^n \longrightarrow S^{n-1}$ would imply that $H_{n-1}(r) : E^n \longrightarrow S^{n-1}$ is surjective which is plainly false since $H_{n-1}(E^n) = \{0\}$ whereas $H_{n-1}(S^{n-1}) = \mathbb{Z}$. The case $n = 1$ is left to the reader.

Corollary 36.2 (Brouwer's fixed point theorem): Every continuous map $f : E^n \longrightarrow E^n$ has a fixed point.

Proof: The proof is similar to the one given in lecture 10 for the case $n = 2$.

Degree of a map: We now generalize the notion of the degree of a map $f : S^1 \longrightarrow S^1$ that we have defined earlier in lectures 12-13. We shall show later that for each $n \in \mathbb{N}$, there is a continuous function having degree n.

Definition 36.1: For $n \geq 1$, the degree of a continuous map $f : S^n \longrightarrow S^n$ is defined to be the integer m such that

$$H_n(f)(\overline{\eta}) = m\,\overline{\eta} \tag{36.1}$$

where $\overline{\eta}$ is a generator for the infinite cyclic group $H_n(S^n)$. Since $H_n(f) : H_n(S^n) \longrightarrow H_n(S^n)$ is a group homomorphism the choice of either of the two generators would yield the same result.

Theorem 36.3: Suppose that $f : S^1 \longrightarrow S^1$ is a continuous map such that $f(1) = 1$ then the degree of f as defined above agrees with the notion of degree as defined in lectures (12-13). Moreover the generator for the group $H_1(S^1)$ is the homology class of the cycle

$$\eta : t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1. \tag{36.2}$$

Note that we have tacitly identified the standard one simplex Δ_1 with $[0, 1]$.

Proof: Since $\pi_1(S^1,1)$ is abelian, the abelianization $\Pi : \pi_1(S^1,1) \longrightarrow H_1(S^1)$ is an isomorphism and hence maps a generator of $\pi_1(S^1,1)$ to a generator of $H_1(S^1)$. Since (36.2) represents a generator for $\pi_1(S^1,1)$ we infer that the cycle (36.2) is a generator for $H_1(S^1)$. We deduce from the diagram (32.1) that

$$f_* = \Pi^{-1} \circ H_1(f) \circ \Pi. \tag{36.3}$$

From (12.6) and (36.3) we see that

$$H_1(f)\Pi[\eta] = (\Pi \circ f_*)[\eta] = \Pi((\deg f)[\eta]) = (\deg f)\Pi[\eta].$$

Appealing to the definition (36.1) we see that $m = \deg f$.

Corollary 36.4: The map $f : S^1 \longrightarrow S^1$ given by $f(x,y) = (x,-y)$ has degree -1.

Proof: The induced map $f_* : \pi_1(S^1,1) \longrightarrow \pi_1(S^1,1)$ is multiplication by -1.

Theorem 36.5: The degree satisfies the following properties.

(i) The degree of the identity map $S^n \longrightarrow S^n$ is $+1$.

(ii) If f and g are two continuous maps from S^n to itself then $\deg(g \circ f) = (\deg g)(\deg f)$.

(iii) Homotopic maps from S^n into itself have the same degree.

(iv) If $f : S^n \longrightarrow S^n$ is a homotopy equivalence then degree of f is ± 1.

(v) Any map homotopic to the constant map has degree zero.

(vi) Any two maps $S^n \longrightarrow S^n$ having the same degree are homotopic (Theorem of H. Hopf).

Proof: The first five are easy exercises for the reader. We shall not prove (vi).

The anti-podal map and its properties: Let us now calculate the degree of the anti-podal map $A : S^n \longrightarrow S^n$ given by $A(\mathbf{x}) = -\mathbf{x}$. The anti-podal map is the composite of reflections in the coordinate hyperplanes and so it suffices to compute the degree of one of them say

$$R_n : (x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_n, x_{n+1}). \tag{36.4}$$

Theorem 36.6: The degree of the map (36.4) is -1 and hence the degree of the antipodal map $A : S^n \longrightarrow S^n$ is $(-1)^{n+1}$.

Proof: From corollary (36.4) the case $n = 1$ follows (exercise 1). The general case is done by induction. Let us consider the covering $\{U, V\}$ where $U = S^n - \{\mathbf{e}_{n+1}\}$, and $V = S^n - \{-\mathbf{e}_{n+1}\}$. The map R_n fixes U and V but when restricted to the equator S^{n-1} gives R_{n-1}. The naturality of the Mayer Vietoris sequence gives us the commutative diagram

$$\begin{array}{ccccc}
H_n(U \cup V) & \xrightarrow{\delta_n} & H_{n-1}(U \cap V) & \xrightarrow{H_{n-1}(r)} & H_{n-1}(S^{n-1}) \\
\Big\downarrow{\scriptstyle H_n(R_n)} & & \Big\downarrow{\scriptstyle H_n(R_n)} & & \Big\downarrow{\scriptstyle H_{n-1}(R_{n-1})} \\
H_n(U \cup V) & \xrightarrow{\delta_n} & H_{n-1}(U \cap V) & \xrightarrow{H_{n-1}(r)} & H_{n-1}(S^{n-1}).
\end{array}$$

The map $H_{n-1}(r)$ is isomorphism induced by the retraction of $U \cap V$ onto the equator S^{n-1}. The connecting homomorphisms δ_n are isomorphisms as we have seen in the last lecture. Since the map $H_{n-1}(R_{n-1}) : \mathbb{Z} \longrightarrow \mathbb{Z}$ on the extreme right is given by multiplication by -1, the same is the case with the map $H_n(R_n)$ on the extreme left whereby we conclude that R_n has degree -1.

Corollary 36.7: The antipodal map $A : S^n \longrightarrow S^n$ is homotopic to the identity map if and only if n is odd.

Proof: If n is even then the identity map and the anti-podal map have different degrees and so cannot be homotopic. The converse is done in exercise 3.

Theorem 36.8: If f and g are a pair of continuous maps from S^n to itself such that $f(\mathbf{x}) \neq g(\mathbf{x})$ for every $\mathbf{x} \in S^n$. Then g is homotopic to $A \circ f$.

Proof: Since $f(\mathbf{x}) \neq g(\mathbf{x})$ the reader may verify that $tAf(\mathbf{x}) + (1-t)g(\mathbf{x}) \neq 0$ for any $t \in [0,1]$. Normalizing we get the desired homotopy:

$$F : (t, \mathbf{x}) \mapsto \frac{tAf(\mathbf{x}) + (1-t)g(\mathbf{x})}{\|tAf(\mathbf{x}) + (1-t)g(\mathbf{x})\|}, \quad t \in [0,1], \ \ \mathbf{x} \in S^n.$$

Corollary 36.9: If n is odd then any continuous map $f : S^n \longrightarrow S^n$ has a fixed point or sends a point to its antipode. Hence the pair $\{f(\mathbf{x}), \mathbf{x}\}$ cannot be linearly independent for every $\mathbf{x} \in S^n$.

Proof: Suppose $f(\mathbf{x}) \neq \mathbf{x}$ for any $\mathbf{x} \in S^n$, we see by theorem (36.8) that f is homotopic to the antipodal map. Further, if also $f(\mathbf{x}) \neq -\mathbf{x}$ for every $\mathbf{x} \in S^n$, theorem (36.8) implies f is homotopic to the identity map. This contradicts corollary (36.7).

Corollary 36.10 (Hairy ball theorem): If n is even, any continuous tangent vector field on S^n must have a zero.

Proof: A continuous, non-vanishing tangent vector field upon normalization yields a continuous map $f : S^n \longrightarrow S^n$ such that the pair of vectors $\{f(\mathbf{x}),\ \mathbf{x}\}$ is every where orthonormal which contradicts corollary (36.9).

Suspension: Given a topological space X, the suspension of X denoted by ΣX, is obtained from $X \times [0,1]$ by passing to a quotient (see the figure that follows equation (36.6)):

$$\Sigma X = (X \times [0,1])/(X \times \{0\} \cup X \times \{1\})$$

Using polar coordinates we can see that $\Sigma S^{n-1} \cong S^n$ via the homeomorphism $\phi : S^{n-1} \times [0,1] \longrightarrow S^n$

$$(\omega, t) \mapsto ((\sin \pi t)\, \omega, \cos \pi t), \quad t \in [0,1], \ \ \omega \in S^{n-1} \subset \mathbb{R}^n. \tag{36.5}$$

With this identification, given $f : S^{n-1} \longrightarrow S^{n-1}$ continuous we define $\Sigma f : S^n \longrightarrow S^n$ by

$$(\Sigma f)((\sin \pi t)\, \omega, \cos \pi t) = ((\sin \pi t) f(\omega), \cos \pi t) \tag{36.6}$$

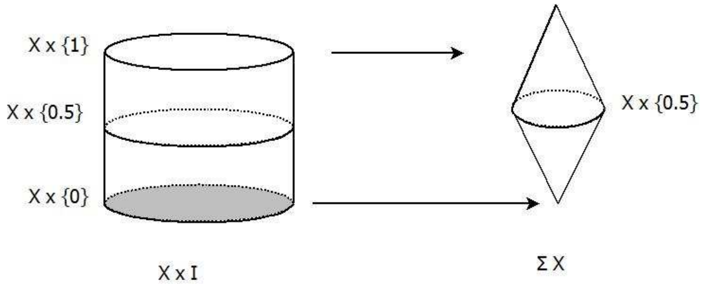


Figure 27: Suspension

Theorem 36.11: Given a continuous map $f : S^n \longrightarrow S^n$, the degree of Σf equals deg f. For every $m \in \mathbb{Z}$ there is a continuous map $f : S^n \longrightarrow S^n$ with degree m.

Proof: The argument parallels the one used to prove theorem (36.6) and is left for the reader.

Exercises

1. Show that if R' and R'' are two reflections (each with respect to a coordinate plane) then they are conjugate by a homeomorphism. Deduce that both R' and R'' have degree -1.
2. Show that if a continuous map $f : S^n \longrightarrow S^n$ misses a point of S^n then f is homotopic to the constant map and so has degree zero.
3. Show that if n is odd then the antipodal map of S^n is homotopic to the identity map. Hint: Do it first for the case $n = 1$ and show that the homotopy may be achieved via a continuous rotation. The general case follows along similar lines by working with pairs of coordinates.
4. Show that $\mathbb{R}P^{2n}$ has the fixed point property.
5. Let $\eta : S^{2n} \longrightarrow \mathbb{R}P^{2n}$ be the covering projection. Show that $H_{2n}(\eta)$ is the zero map.
6. Show that the map (36.5) is a homeomorphism and (36.6) defines a continuous map. More generally given a continuous map $f : X \longrightarrow Y$ show that the composite

$$X \times [0,1] \xrightarrow{f \times \mathrm{id}} Y \times [0,1] \longrightarrow \Sigma Y$$

   induces a map $\Sigma f : \Sigma X \longrightarrow \Sigma Y$. Imitate the computation in theorem [//] of lecture [//] to show that $H_{n+1}(\Sigma X) = H_n(X)$ when $n \geq 1$. What happens when $n = 0$?
7. Prove theorem (36.11). Note that the map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^m$ has degree m.
8. Determine the degree of a polynomial as a map from S^2 to itself. Reprove the fundamental theorem of algebra.

Lecture - XXXVII Relative homology

The homology groups $H_n(X)$ we have hitherto been studying are called the *absolute* homology groups. The relative homology groups $H_n(X, A)$ that we define here provide us a tool for understanding the geometry of a space X in relation with its subspace A. This is facilitated by a long exact sequence in homology for the pair (X, A). For instance if A is a retract of X, this sequence breaks off into a bunch of short exact sequences each of which splits. The groups $H_n(X, A)$ are related to the absolute homology groups $H_n(X/A)$ for sufficiently well behaved pairs (X, A) but we shall not get into this discussion here (see [16], p. 50).

Recall that if A is a subspace of X and z is a non-trivial $n-$cycle in A then it may be a boundary when viewed as a cycle in X. In other words, the inclusion map $i : A \longrightarrow X$ need not induce an injective map in homology. The relative homology group measures $H_{n+1}(X, A)$ the deviation from injectivity of the map $H_n(i)$.

Definition 37.1: (i) Given a topological space X and a subspace A, $S_n(A)$ may be regarded as a subgroup of $S_n(X)$ and the group $S_n(X, A)$ of relative $n-$chains is the quotient group $S_n(X)/S_n(A)$.

(ii) For each $n = 1, 2, \dots,$ we define the boundary maps $\overline{\partial}_n : S_n(X, A) \longrightarrow S_{n-1}(X, A)$ as

$$\overline{\partial}_n \overline{c} = \overline{\partial_n c} \tag{37.1}$$

It is readily verified that $\overline{\partial}_{n-1} \circ \overline{\partial}_n = 0$ leading to the quotient complex $S(X)/S(A)$ consisting of the sequence of groups $\{S_n(X, A)\}$ and the boundary maps (37.1).

(iii) The homology groups of the quotient complex $S(X)/S(A)$ are called the relative homology groups and are denoted by the symbol $H_n(X, A)$.

For a slightly more explicit description of these groups we introduce the group $Z_n(X, A)$ of relative $n-$cycles and the group $B_n(X, A)$ of relative boundaries. The group $Z_n(X, A)$ is the subgroup of $S_n(X)$ consisting of chains $c \in S_n(X)$ such that the boundary $\partial_n c$ is a chain in A. That is,

$$Z_n(X, A) = \{c \in S_n(X)/\partial_n c \in S_{n-1}(A)\}. \tag{37.2}$$

In keeping with the convention that $S_{-1}(A) = \{0\}$ (see definition (29.5)), $Z_0(X, A) = S_0(X)$. We see that $c \in Z_n(X, A)$ if and only if $\overline{c}$ is in the kernel of $\overline{\partial}_n$. Likewise the group $B_n(X, A)$ of relative boundaries is defined to be the subgroup of $S_n(X)$ consisting of chains $c \in S_n(X)$ such that

$$c = \partial_{n+1} c' \text{ mod}(S_n(A)),$$

for some $c' \in S_{n+1}(X)$. In other words there exists $c' \in S_{n+1}(X)$ and $a \in S_n(A)$ such that

$$c - \partial_{n+1} c' = a.$$

Obviously $c \in B_n(X, A)$ if and only if $\overline{c}$ belongs to the image of $\overline{\partial}_{n+1}$ whereby we conclude

Theorem 37.1: $H_n(X,A) = Z_n(X,A)/B_n(X,A)$.

We now consider the short exact sequence of complexes induced by the inclusion i and p denoting the projection onto the quotient:

$$0 \longrightarrow S(A) \xrightarrow{i_\sharp} S(X) \xrightarrow{p} S(X,A) \longrightarrow 0.$$

Equation (37.1) states that p is a chain map and exactness of this sequence is an easy exercise. Theorem (29.6) now gives

Theorem 37.2: For a pair (X,A) of topological spaces there is a long exact sequence in homology:

$$\longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(p)} H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \qquad (37.3)$$

We remark that the connecting homomorphism has a simple geometrical description in this case. If we take a relative $n-$cycle namely an element $c \in Z_n(X,A)$ then $\partial_n c$ is an element of $S_{n-1}(A)$ and $i^{-1}(\partial_n c)$ is simply $\partial_n c$ viewed as a chain in A. We summarize this observation as a lemma:

Lemma 37.3: For a pair (X,A) of spaces the connecting homomorphism $\delta_n : H_n(X,A) \longrightarrow H_{n-1}(A)$ is given by

$$\delta_n \overline{c} = \overline{\partial_n c}, \quad c \in Z_n(X,A). \qquad (37.4)$$

Despite the notation, $\partial_n c$ in (37.4) is not a boundary in $S_{n-1}(A)$ since c is not a chain in $S_n(A)$ but a chain in $S_n(X)$. If ζ is a cycle in A then for sure, it is a cycle in X as well but then it may be actually be a boundary X, in other words $H_n(i)\overline{\zeta} = 0$. This happens precisely when $\overline{\zeta}$ is in the image of δ_{n+1} by exactness of (37.3). Figure below depicts a cycle in A (annulus) which is a boundary in X (the polygonal region).

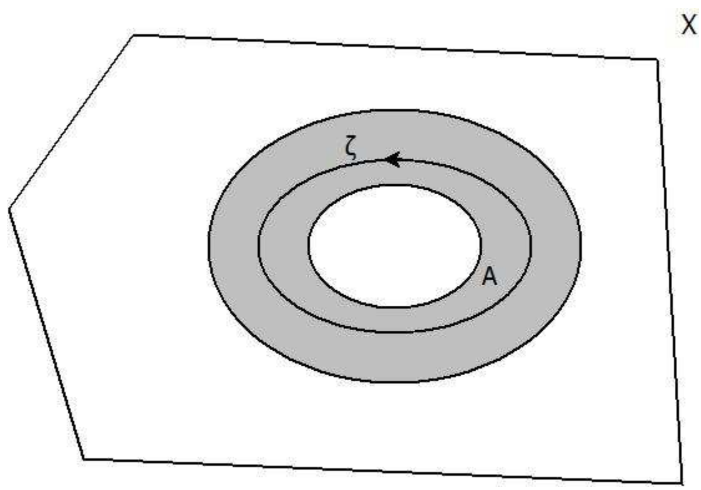


Figure 28:

The long exact sequence in the preceding theorem is *natural* in the following sense.

Theorem 37.4 (Naturality): Given a map of pairs $f : (X,A) \longrightarrow (Y,B)$ the following diagram commutes

$$\begin{array}{ccccccccccc}
\longrightarrow & H_n(A) & \xrightarrow{i} & H_n(X) & \xrightarrow{p} & H_n(X,A) & \xrightarrow{\delta_n} & H_{n-1}(A) & \longrightarrow \\
& \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_{n-1}(f)} & \\
\longrightarrow & H_n(B) & \xrightarrow{i} & H_n(Y) & \xrightarrow{p} & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow
\end{array} \tag{37.5}$$

Proof: From theorem (29.6) or the specific description of δ_n given above, (37.5) follows immediately.

Retraction: We shall now define the notion of a split exact sequence and show that whenever A is a retract of X, the long exact sequence (37.3) breaks off into a bunch of short exact sequences each of which splits.

Definition 37.2: A short exact sequence of abelian groups/chain complexes

$$0 \longrightarrow L \xrightarrow{f} G \xrightarrow{g} K \longrightarrow 0, \tag{37.6}$$

splits on the right if there exists a group homomorphism (respectively a chain map) $\phi : K \longrightarrow G$ such that $g \circ \phi = \mathrm{id}_K$. The short exact sequence (37.6) *splits on the left* if there exists group homomorphism (respectively a chain map) $\theta : G \longrightarrow L$ such that $\theta \circ f = \mathrm{id}_L$.

Lemma 37.5: Given a short exact sequence (37.6), the following are equivalent:

(i) The sequence splits on the left.

(ii) The sequence splits on the right.

(iii) G is isomorphic to $\mathrm{im} f \oplus K$.

Proof: We begin by proving (ii) implies (iii). Note that ϕ is injective and so im ϕ is isomorphic to K. Let $x \in G$ be arbitrary and observe that

$$x - \phi \circ g(x)$$

lies in the kernel of g and hence in the image of f. Thus,

$$x = (x - \phi \circ g(x)) + \phi \circ g(x) \in \mathrm{im} f + \mathrm{im}\, \phi.$$

We leave it to the reader to check that the sum $(\mathrm{im} f + \mathrm{im}\, \phi)$ is direct. It is easy to show that (iii) implies (i). We now show that (i) implies (ii). Let $k \in K$ and choose any $x \in G$ such that $k = g(x)$. Define $\phi(k) = x - (f \circ \theta)(x)$. To check that this is well defined, suppose that $k = g(x') = g(x'')$ for a pair of elements $x', x'' \in G$. There exists $y \in L$ such that $x' - x'' = f(y)$. Applying $f \circ \theta$ to this equation we get

$$(f \circ \theta)(x') - (f \circ \theta)(x'') = f \circ \theta \circ f(y) = f(y) = x' - x'',$$

from which we see that $(f \circ \theta)(x') - x' = (f \circ \theta)(x'') - x''$. It is trivial to see that the map ϕ that we have defined is a group homomorphism and satisfies the requirement $g \circ \phi = \mathrm{id}_K$. □

Theorem 37.6: A retraction $r : X \longrightarrow A$ gives for each $n = 0, 1, 2, \ldots$ a short exact sequence

$$0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(p)} H_n(X, A) \longrightarrow 0. \tag{37.7}$$

Each of these short exact sequences splits. Thus

$$H_n(X) = H_n(A) \oplus H_n(X, A), \quad n = 0, 1, 2, \ldots$$

Proof: We show that $\delta_n = 0$ for every n which would give us the sequences (37.7). For $c \in Z_n(X, A)$ we have the chains $r_\sharp(c) \in S_n(A)$ and $\partial_n c \in S_{n-1}(A)$. Now,

$$\partial_n r_\sharp(c) = r_\sharp(\partial_n c) = (r_\sharp \circ i_\sharp)(\partial_n c) = (r \circ i)_\sharp(\partial_n c) = \partial_n c$$

Hence $\partial_n c$ is the boundary of the chain $r_\sharp(c) \in S_n(A)$ and so represents the zero element in $H_{n-1}(A)$. From lemma (37.3) we conclude that δ_n is the zero map. The short exact sequence (37.7) splits on the left since $H_n(r) \circ H_n(i)$ is the identity map on $H_n(A)$. $\square$

Example 37.1 Let us calculate the relative homology groups $H_n(X, A)$ where X is the Möbius band and A is its boundary. Since the central circle is a deformation retract of X, we see that $H_n(X) = H_n(A) = 0$ when $n \geq 2$ and we infer from (37.3) that $H_n(X, A) = 0$ when $n \geq 3$. We now recall that the map $i_* : \pi_1(A) \longrightarrow \pi_1(X)$ induced by inclusion is the group homomorphism of $\mathbb{Z}$ into itself given by $x \mapsto 2x$. Since the fundamental groups are abelian the map $H_1(i) = i_*$ and so the kernel of $H_1(i)$ is trivial. The portion of the exact sequence (37.3) with $n = 2$ gives $H_2(M, A) = 0$. Finally since $H_0(i) : H_0(A) \longrightarrow H_0(M)$ is an isomorphism (why?), we conclude from (37.3) (with $n = 0$) that the map $H_1(X) \longrightarrow H_1(X, A)$ is surjective with kernel $2\mathbb{Z}$. Hence $H_1(X, A) = \mathbb{Z}_2$.

Exercises

1. Verify that the diagram (37.5) commutes.
2. Determine $H_n(X, A)$ when $A = \emptyset$, and when A is a singleton and $n \geq 1$. What happens if $n = 0$?
3. Compute $H_n(S^1 \times S^1, S^1 \vee S^1)$ and compare it with the absolute homology $H_n((S^1 \times S^1)/(S^1 \vee S^1))$.
4. Compute $H_k(E^n, S^{n-1})$ and compare it with $H_k(E^n/S^{n-1})$.
5. In example (35.1), prove that X/A is homeomorphic to $\mathbb{R}P^2$. Compare the groups $H_n(X, A)$ with the groups $H_n(X/A)$. Hint: To set up the homeomorphism note that $(x, y) \mapsto (x\sqrt{1 - y^2}, y)$ maps each $[-1, 1] \times \{y\}$ homeomorphically onto the chord at height y.

Lecture - XXXVIII Excision theorem

In this lecture we prove the most important theorem homology theory known as the excision theorem. We shall conclude the lecture with the definition of local homology groups that play an important role in the theory of orientability of topological manifolds. We begin with the ubiquitous five lemma.

Theorem 38.1 (The five lemma): Assume given a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
{\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_2} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5
\end{array}$$

If ψ_1, ψ_2, ϕ_1 and ϕ_2 are isomorphisms then so is η.

Proof: We shall prove that η is injective, leaving surjectivity as an exercise. Assume $\eta(a_3) = 0$ for some $a_3 \in A_3$ so that $\beta_3 \circ \eta(a_3) = 0$. The commutativity of the third square now gives $\phi_1 \circ \alpha_3(a_3) = 0$. Using injectivity of ϕ_1 and exactness of the top row we arrive at

$$a_3 = \alpha_2(a_2) \tag{38.1}$$

for some $a_2 \in A_2$. Again, $0 = \eta(a_3) = \eta \circ \alpha_2(a_2) = \beta_2 \circ \psi_2(a_2)$ showing that $\psi_2(a_2) \in \text{im } \beta_1$. Thus $\psi_2(a_2) = \beta_1(b_1)$ for some $b_1 \in B_1$ and using the surjectivity of ψ_1 we get for some $a_1 \in A_1$,

$$\psi_2(a_2) = \beta_1 \circ \psi_1(a_1) = \psi_2 \circ \alpha_1(a_1).$$

Injectivity of ψ_2 now gives $a_2 - \alpha_1(a_1) = 0$. Substituting into (38.1) we conclude $a_3 = 0$.

Theorem 38.2 (Excision): Let (X, A) be a pair and U be a subset of A such that the closure of U is contained in the interior of A. Then, the homomorphism

$$H_n(i) \; : \; H_n(X - U, \; A - U) \longrightarrow H_n(X, A)$$

induced by inclusion $i : (X - U, A - U) \longrightarrow (X, A)$ is an isomorphism for every $n = 0, 1, 2, \ldots$. In other words the set U may be excised from the pair (X, A) without affecting the homology groups of the pair.

Proof: The hypothesis implies that the pair $\mathcal{U} = \{\text{int } A,\ X - \overline{U}\}$ is an open cover of X, where $\overline{U}$ denotes the closure of U. Likewise $\mathcal{V} = \{\text{int } A,\ A - \overline{U}\}$ is an open cover of A and $S^{\mathcal{V}}(A)$ is a subcomplex of $S^{\mathcal{U}}(X)$. By theorem (29.6) the short exact sequence of complexes

$$0 \longrightarrow S^{\mathcal{V}}(A) \longrightarrow S^{\mathcal{U}}(X) \longrightarrow S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A) \longrightarrow 0$$

gives rise to a long exact sequence in homology:

$$\longrightarrow H_n^{\mathcal{V}}(A) \longrightarrow H_n^{\mathcal{U}}(X) \longrightarrow H_n(S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A)) \longrightarrow H_{n-1}^{\mathcal{V}}(A) \longrightarrow$$

On the other hand there is an obvious map of complexes induced by the inclusion maps namely

$$j\ :\ S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A) \longrightarrow S(X)/S(A),$$

resulting in a commutative diagram of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{\mathcal{V}}(A) & \longrightarrow & S^{\mathcal{U}}(X) & \longrightarrow & S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & S(A) & \longrightarrow & S(X) & \longrightarrow & S(X)/S(A) & \longrightarrow & 0
\end{array}$$

Since the long exact sequence in homology is natural (exercise 6 of lecture 29), we get the commutative diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & H_n^{\mathcal{V}}(A) & \stackrel{i}{\longrightarrow} & H_n^{\mathcal{U}}(X) & \stackrel{p}{\longrightarrow} & H_n(S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A)) & \longrightarrow & H_{n-1}^{\mathcal{V}}(A) & \longrightarrow & H_{n-1}^{\mathcal{U}}(X) & \longrightarrow \\
 & {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & \\
\longrightarrow & H_n(A) & \stackrel{i}{\longrightarrow} & H_n(X) & \stackrel{p}{\longrightarrow} & H_n(S(X)/S(A)) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow
\end{array}$$

where we the subscript star indicates the map induced in homology. The five lemma enables us to conclude that

$$j_*\ :\ H_n(S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A)) \longrightarrow H_n(S(X)/S(A)) = H_n(X, A).$$

is an isomorphism. Note the inclusion

$$k\ :\ S(X - U) \longrightarrow S^{\mathcal{U}}(X)$$

maps $S(A - U)$ into $S^{\mathcal{V}}(A)$ whereby we get an isomorphism (exercise 2)

$$\overline{k}\ :\ S(X - U)/S(A - U) = S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A). \qquad (38.2)$$

The composite $j \circ \overline{k}$ is also induced by the inclusion map $(X - U,\ A - U) \longrightarrow (X, A)$ and we have the desired isomorphism

$$(j \circ \overline{k})_*\ :\ H_n(X - U, A - U) \longrightarrow H_n(X, A), \quad n = 0, 1, 2, \dots$$

Example 38.1: Let $X = S^n$ and $A = S^n - \mathbf{e}_{n+1}$. We take U to be the complement of the *polar ice cap* namely the set of all $\mathbf{x} \in S^n$ such that $x_{n+1} \leq 2/3$ (reader is invited to draw a picture). Applying the excision theorem, and denoting the polar ice cap by D,

$$H_n(S^n, S^n - \mathbf{e}_{n+1}) \cong H_n(S^n - U,\ D - \mathbf{e}_{n+1}) = H_n(D,\ D - \mathbf{e}_{n+1}).$$

Theorem (37.2) gives $H_n(S^n, S^n - \mathbf{e}_{n+1}) \cong H_n(S^n)$ and $H_n(D, D - \mathbf{e}_{n+1}) \cong H_{n-1}(D - \mathbf{e}_{n+1})$. Since the polar ice cap is homeomorphic to an open ball,

$$H_n(S^n) \cong H_{n-1}(S^{n-1}), \quad n \geq 2.$$

Using theorem (32.1) we conclude that $H_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.

Example 38.2: In general, given a pair of open sets $\{U, V\}$ of a topological space X, let $W = (U \cup V) - V$. Then, the closure of W in $U \cap V$ is contained in U and since

$$(X - W,\ U - W) = (V, U \cap V),$$

the excision theorem gives

$$H_n(U \cup V,\ U) \cong H_n(V,\ U \cap V), \quad n = 0, 1, 2, \ldots$$

Lemma 38.3 (Barrett and Whitehead): Given a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
\longrightarrow & A_n & \xrightarrow{p_n} & B_n & \xrightarrow{q_n} & C_n & \xrightarrow{r_n} & A_{n-1} & \longrightarrow \\
 & \alpha_n \downarrow & & \downarrow \beta_n & & \downarrow \gamma_n & & \downarrow \alpha_{n-1} & \\
\longrightarrow & A'_n & \xrightarrow{p'_n} & B'_n & \xrightarrow{q'_n} & C'_n & \xrightarrow{r'_n} & A'_{n-1} & \longrightarrow
\end{array}$$

If each of the maps $\gamma_n : C_n \longrightarrow C'_n$ is an isomorphism, then the sequence

$$\longrightarrow A_n \xrightarrow{\lambda_n} B_n \oplus A'_n \xrightarrow{\mu_n} B'_n \xrightarrow{\delta_n} A_{n-1} \longrightarrow$$

is exact where, the maps are given by

$$\lambda_n = (p_n, -\alpha_n), \quad \mu_n = \beta_n + p'_n, \quad \delta_n = r_n \circ \gamma_n^{-1} \circ q'_n.$$

Corollary 38.4 (Mayer Vietoris): If $\{U, V\}$ are open subsets of a topological space,

$$\longrightarrow H_n(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_n(U) \oplus H_n(V) \xrightarrow{q_n} H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow$$

is an exact sequence.

Proof: The long exact sequence for a pair and its naturality gives a commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
\longrightarrow & H_n(U \cap U) & \longrightarrow & H_n(V) & \longrightarrow & H_n(V,\ U \cap V) & \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_n(U) & \longrightarrow & H_n(U \cup V) & \longrightarrow & H_n(U \cap V,\ U) & \longrightarrow & H_{n-1}(U) & \longrightarrow
\end{array}$$

Applying the excision theorem to the inclusion $(U \cap V,\ V) \longrightarrow (U \cup V,\ U)$, we see that the third arrow is an isomorphism in the displayed diagram. The result now follows from lemma (38.3).

Definition 38.2 (Local homology groups): Given a topological space X and $p \in X$, the local homology groups of X at p are the groups $H_n(X, X - \{p\})$ $(n = 0, 1, 2, \ldots)$.

Theorem 38.5: $H_n(X, X - \{p\}) = H_n(V, V - \{p\})$ for any open neighborhood of p.

Proof: This follows immediately from the excision theorem by taking $U = X - V$.

Some applications of the local homology groups are indicated in the exercises below.

Exercises

1. Prove that the map η in the five lemma is surjective.

2. Show that the map (38.2) is indeed an isomorphism. To prove that it is surjective use the decompositions $S^{\mathcal{U}}(X) = S(X-U) + S(\text{int}A)$ and $S^{\mathcal{V}}(A) = S(A-U) + S(\text{int}A)$.

3. Prove the Barrett-Whitehead lemma.

4. Calculate the local homology groups $H_2(X, X - \{p\})$ in the following cases:

   (i) The space X is the cylinder $S^1 \times [0,1]$ and p a point on its boundary.

   (ii) The space X is the Möbius band and p is a point on its boundary.

   Deduce that the cylinder and the Möbius band are not homeomorphic.

5. A topological manifold is a Hausdorff space in which each point has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$. Show that if p is a point on a topological manifold M,

$$H_n(M,\ M - \{p\}) \cong \mathbb{Z}.$$

Lecture - XXXIX (Test - V)

1. Calculate the homology groups of the double torus.
2. Show that any homeomorphism of E^n onto itself must preserve the boundary.
3. Show that $\mathbb{R}P^n$ is not a retract of $\mathbb{R}P^{n+1}$. Use the lifting criterion.
4. Show that $\mathbb{R}P^{2n}$ has the fixed point property. Does $\mathbb{R}P^3$ have the fixed point property?

Solutions to Test - V

1. Let X denote the double torus. Since X is path connected, $H_0(X) = \mathbb{Z}$. In exercise 7 of lecture 32, the student was asked to compute the abelianization of $\pi_1(X)$. Observe that $\pi_1(X)$ is the quotient of the free group on four elements by the normal subgroup H generated by

$$abcda^{-1}b^{-1}c^{-1}d^{-1}.$$

The above element can be written as

$$\big(aba^{-1}b^{-1}\big)bacd(ba)^{-1}(cd)^{-1}cdc^{-1}d^{-1}$$

which is a product of commutators and so H is contained in the commutator subgroup. Thus the abelianization of $\pi_1(X)$ is the free abelian group of rank four or in other words $H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. For computing the higher homology groups we use the Mayer Vietoris sequence with U and V being open sets each homeomorphic to a torus from which a tiny *closed* disc is removed (thus, each of the two pieces in figure 5 would have a little collar around boundary) and $U \cap V$ is homeomorphic to an open cylinder. We look at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(U \cap V) \xrightarrow{\partial_2} H_1(U \cap V) \longrightarrow$$

The connecting homomorphism ∂_2 must be injective since each of the two pieces U and V deformation retracts to $S^1 \vee S^1$ (how?) whereby $H_2(U) = H_2(V) = 0$. We now show that ∂_2 is surjective using the information already obtained that $H_1(X)$ is free abelian of rank four. Looking to the right of the displayed sequence,

$$H_2(U \cap V) \xrightarrow{\partial_2} H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \xrightarrow{q} H_1(U \cup V) \xrightarrow{\partial_1}$$

The homomorphism ∂_1 is zero since the subsequent map

$$H_0(U \cap V) \xrightarrow{p} H_0(U) \oplus H_0(V)$$

is given by $m \mapsto (m, m)$ (why?). Thus q is surjective and since the groups $H_1(U) \oplus H_1(V)$ and $H_1(U) \cup V)$ are both free abelian of rank four, we see that q is injective as well. Finally we conclude

$$H_2(U \cup V) = H_1(U \cap V) = \mathbb{Z}.$$

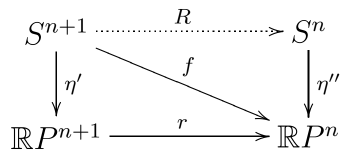
For $n \geq 3$ the Mayer Vietoris sequence immediately yields $H_n(X) = 0$.

2. Suppose that $\phi : E^n \longrightarrow E^n$ is a homeomorphism that takes a boundary point p to a point q with $\|q\| < 1$. Then the restriction $\phi : E^n - \{p\} \longrightarrow E^n - \{q\}$ is also a homeomorphism but

$$H_{n-1}(E^n - \{p\}) = 0, \quad H_{n-1}(E^n - \{q\}) = \mathbb{Z}$$

and that gives a contradiction.

3. We regard S^n as the equator S^{n+1} and since the antipodal map preserves the equator we see that $\mathbb{R}P^n$ is a subspace of $\mathbb{R}P^{n+1}$ in a natural way. Suppose that $r : \mathbb{R}P^{n+1} \longrightarrow \mathbb{R}P^n$ is a retraction. Let $\eta' : S^{n+1} \longrightarrow \mathbb{R}P^{n+1}$ be the quotient map and $f = r \circ \eta'$. We then get a commutative diagram

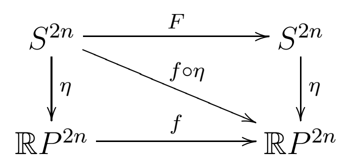


where R is the lift of f with respect to the covering map η''. Such a lift exists since S^{n+1} is simply connected when $n \geq 1$. Now let x be an arbitrary point on the equator of S^{n+1}. Then

$$\eta''(Rx) = r(\eta'(x)) = \overline{x},$$

where the second equality is due to the fact that r is a retraction. Thus $Rx = \pm x$ for each $x \in S^n$ By continuity either $Rx = x$ for every $x \in S^n$ or else $Rx = -x$ for every $x \in S^n$. In the first case R is a retraction from S^{n+1} onto S^n. In the second case $A \circ R$ is a retraction from S^{n+1} onto S^n, where A is the anti-podal map. In any case we conclude that S^n is a retract of S^{n+1} and that is a contradiction (why?).

4. Let $f : \mathbb{R}P^{2n} \longrightarrow \mathbb{R}P^{2n}$ be a continuous map. Consider the following commutative diagram where the vertical maps are the standard quotient maps and F is the lift of $f \circ \eta$:



Note that by corollary (36.9), a continuous map from S^{2n} to itself either has a fixed point or else sends a point to its antipode. Applying this fact to F and using the commutativity of the diagram it follows that f has a fixed point. Since $\mathbb{R}P^3$ is homeomorphic to $SO(3, \mathbb{R})$ and the latter does not have the fixed point property we deduce that $\mathbb{R}P^3$ does not have the fixed point property.

Lecture - XL Inductive limits

We have frequently encountered situations where a certain space X is canonically embedded in a larger space Y. A familiar example the sequence of orthogonal groups and the canonical inclusions

$$SO(2,\mathbb{R}) \longrightarrow SO(3,\mathbb{R}) \longrightarrow SO(4,\mathbb{R}) \longrightarrow \ldots \tag{40.1}$$

where, the inclusion map $SO(n,\mathbb{R}) \longrightarrow SO(n+1,\mathbb{R})$ is given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \in SO(n,\mathbb{R})$$

The inductive limit of a sequence such (40.1) is a space which contains each individual member of the sequence, and is the *smallest* such space. The precise meaning of the adjective *smallest* would be clear from the formal definition that we shall presently give.

Let us look at a situation in the category of abelian groups. For a fixed prime p let C_{p^k} denote the cyclic group of order p^k. Then for each $j \leq k$, the group C_{p^k} contains a (unique) cyclic group of order p^j giving us a sequence of groups

$$C_p \longrightarrow C_{p^2} \longrightarrow C_{p^3} \longrightarrow \ldots, \tag{40.2}$$

in which the arrows inclusion maps. All these groups may be regarded as subgroups of $\mathbb{C} - \{0\}$ or as subgroups of the smaller group S^1. However there is a *smallest* group containing a copy of each the groups C_{p^k} namely, the group

$$\left\{ \exp\left(\frac{2\pi i l}{p^k}\right) \;/\; l, k \in \mathbb{Z} \right\} \tag{40.3}$$

consisting of all p^k-th roots of unity ($k = 1, 2, \ldots$). This group (known as the Prüfer group) would then be the inductive limit of the family of cyclic groups C_{p^k} ($k = 1, 2, \ldots$).

We now proceed to the formal definitions and prove the existence and uniqueness (upto isomorphism) of the inductive limit of a family of groups. We recall the notion of a directed set.

Definition 40.1 (Directed systems): (i) A directed set is a set Λ with a partial order $\leq$ such that for any pair $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

(ii) A directed system of abelian groups is a family $\{G_\alpha \;/\; \alpha \in \Lambda\}$ of abelian groups indexed by a directed set Λ together with a family of group homomorphisms $\{f_{\alpha\beta} : G_\alpha \longrightarrow G_\beta \;/\; \alpha \leq \beta\}$ satisfying the two conditions

(a) $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ for any three $\alpha, \beta, \gamma \in \Lambda$ such that $\alpha \leq \beta \leq \gamma$.

(b) $f_{\alpha\alpha} = \mathrm{id}_{G_\alpha}$ for each $\alpha \in \Lambda$.

(iii) By dropping the adjective *abelian* from (ii) we obtain a directed system of groups.
(iv) A directed system of topological spaces is a family $\{X_\alpha \ / \ \alpha \in \Lambda\}$ of topological spaces indexed by a directed set Λ together with a family of continuous maps $\{f_{\alpha\beta} : X_\alpha \longrightarrow X_\beta \ / \ \alpha \leq \beta\}$ satisfying the two conditions (a) and (b) in (ii).

So we shall speak of a directed system in the categories **Gr**, **AbGr** or **Top**.

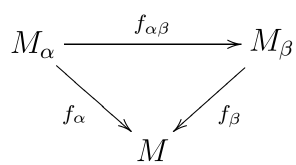
Example 40.1: The most important example of a directed set is of course $\mathbb{N}$ with its usual order and (40.1)-(40.2) furnish examples of directed systems of topological spaces and abelian groups indexed by $\mathbb{N}$, where the maps $f_{\alpha\beta}$ are inclusions.

We record a lemma whose proof is left for the student to verify

Lemma 40.1: Suppose that $\{M_\alpha \ / \ \alpha \in \Lambda\}$ is directed system in one of the categories **Gr**, **AbGr** or **Top**, and for some pair $x_\alpha \in M_\alpha$ and $x_\beta \in M_\beta$ there exists $\gamma \in \Lambda$ such that $f_{\alpha\gamma}(x_\alpha) = f_{\beta\gamma}(x_\beta)$, then for every $\delta \geq \gamma$, $f_{\alpha\delta}(x_\alpha) = f_{\beta\delta}(x_\beta)$.

Definition 40.2 (Inductive limit): Given a directed system $\{M_\alpha \ / \ \alpha \in \Lambda\}$ in one of the categories **Gr**, **AbGr** or **Top** and a family of morphisms $\{f_{\alpha\beta} : M_\alpha \longrightarrow M_\beta \ / \ \alpha \leq \beta\}$ in the same category satisfying the conditions in definition (40.1), an inductive limit is an object M together with a family of morphisms $\{f_\alpha : M_\alpha \longrightarrow M\}$ such that the following two conditions hold:

(1) For every pair $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$, $f_\beta \circ f_{\alpha\beta} = f_\alpha$, summarized as a commutative diagram:



(2) Universal property: Given an object L and a family of morphisms $g_\alpha : M_\alpha \longrightarrow L$ satisfying

$$g_\beta \circ f_{\alpha\beta} = g_\alpha, \quad \alpha, \beta \in \Lambda, \alpha \leq \beta,$$

there exists a unique morphism $\psi : M \longrightarrow L$ such that

$$\psi \circ f_\alpha = g_\alpha.$$

Notation: The inductive limit M of the system $\{M_\alpha \ /\alpha \in \Lambda\}$ will be denoted by $\varinjlim_{\alpha} M_\alpha$.

Theorem 40.2: (i) Every directed system of groups or abelian groups has an inductive limit which is unique upto isomorphism.

(ii) With the notations as in the definition (40.2), assume that $f_\alpha(x) = 0$ for some $x \in G_\alpha$. There exists $\beta \geq \alpha$ such that $f_{\alpha\beta}(x) = 0$.

Proof: (i) Let $\tilde{G}$ be the coproduct (direct sum) of the abelian groups $\{G_\alpha\}$ and we regard (for simplifying notations) the groups G_α as being subgroups of $\tilde{G}$ and $i_\alpha : G_\alpha \longrightarrow G$ the inclusion maps. Declare $x_\alpha \in G_\alpha$ and $x_\beta \in G_\beta$ as being equivalent if there exists $\gamma \in \Lambda$ such that $\gamma \geq \alpha$, $\gamma \geq \beta$ and

$$f_{\alpha\gamma}(x_\alpha) = f_{\beta\gamma}(x_\beta). \tag{40.4}$$

Lemma (40.1) states that this is a well defined equivalence relation. We denote by $\sim$ the equivalence relation just defined and define N to be the subgroup generated by

$$\{x_\alpha - x_\beta \ / \ x_\alpha \sim x_\beta\}.$$

Finally let $G = \tilde{G}/N$ and $\eta : \tilde{G} \longrightarrow G$ be the quotient map. We claim that G is the inductive limit with respect to the maps f_α given by the composition

$$G_\alpha \xrightarrow{i_\alpha} \tilde{G} \xrightarrow{\eta} \tilde{G}/N. \tag{40.5}$$

We now check the conditions (1) and (2) in definition (40.2). For $\alpha \leq \beta$ we derive from

$$(f_{\beta\beta} \circ f_{\alpha\beta})(x_\alpha) = f_{\alpha\beta}(x_\alpha).$$

the useful piece of information

$$f_{\alpha\beta}(x_\alpha) \sim x_\alpha, \quad x_\alpha \in G_\alpha. \tag{40.6}$$

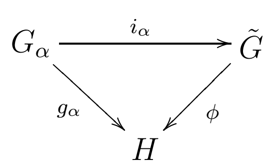
Hence $f_{\alpha\beta}(x_\alpha) - x_\alpha \in N$ whereby we conclude

$$\eta(f_{\alpha\beta}(x_\alpha)) = \eta(x_\alpha),$$

which in turn implies $f_\beta \circ f_{\alpha\beta} = f_\alpha$. Turning to the universal property (2) assume given an abelian group H and a family of group homomorphisms $g_\alpha : G_\alpha \longrightarrow H$ such that

$$g_\beta \circ f_{\alpha\beta} = g_\alpha, \quad \alpha \leq \beta. \tag{40.7}$$

We first use the defining property of the coproduct to get a group homomorphism $\phi : \tilde{G} \longrightarrow H$ such that the following diagram commutes:



That is $\phi \circ i_\alpha = g_\alpha$. From (40.4) and (40.7) we get $g_\alpha(x_\alpha) = g_\beta(x_\beta)$, or in view of the fact that we have identified G_α as a subgroup of $\tilde{G}$, $\phi(x_\alpha) = \phi(x_\beta)$. Hence there is a group homomorphism $\psi : \tilde{G}/N \longrightarrow H$ such that

$$\psi \circ \eta = \phi. \tag{40.8}$$

Upon applying this to an arbitrary $x_\alpha \in G_\alpha$ we get using (40.5) that $\psi \circ f_\alpha = g_\alpha$ for every $\alpha \in \Lambda$. The homomorphism satisfying (40.8) is unique since the elements $\{f_\alpha(x_\alpha)/\alpha \in \Lambda \text{ and } x_\alpha \in G_\alpha\}$ generate the group $\tilde{G}/N$.

We now prove (ii) which we shall use in the next lecture. Since $x \in N$, there exists a finite set of indices $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$ such that

$$x = \sum_{j=1}^{k} (x_{\alpha_j} - x_{\beta_j}) \tag{40.9}$$

where $x_{\alpha_j} \sim x_{\beta_j}$ for each j. Thus for each j there is a γ_j exceeding both α_j and β_j such that $f_{\alpha_j\gamma_j}(x_{\alpha_j}) = f_{\beta_j\gamma_j}(x_{\beta_j})$. Since (40.9) spells out a relation in the direct sum of the groups G_β, it decomposes into a bunch of equations namely

$$\begin{aligned} x &= \sum_{\alpha_i=\alpha} x_{\alpha_i} - \sum_{\beta_i=\alpha} x_{\beta_i} \\ 0 &= \sum_{\alpha_i=\lambda} x_{\alpha_i} - \sum_{\beta_i=\lambda} x_{\beta_i}, \quad \lambda \neq \alpha \end{aligned}$$

The index λ runs through a finite subset of $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$. Taking δ to be sufficiently large and applying $f_{\alpha\delta}$ to the first and $f_{\lambda\delta}$ to the second of the above displayed equations and adding we get

$$f_{\alpha\delta}(x) = \sum_{j=1}^{k} (f_{\alpha_i\delta}(x_{\alpha_j}) - f_{\beta_i\delta}(x_{\beta_j})) \tag{40.10}$$

Using lemma (40.1) we see that if δ is sufficiently large each of the summands on the right hand side of (40.10) is in N and so $f_{\alpha\delta}(x) = 0$ as asserted.

Remarks: (1) The construction can be carried out in exactly the same manner in the categories **Gr** and **Top**. In the Category **Gr**, the coproduct $\tilde{G}$ of the groups G_α is the free product with the group operation written multiplicatively and the candidate for N is the *normal subgroup* generated by

$$\{x_\alpha x_\beta^{-1} / x_\alpha \sim x_\beta\},$$

where as before we regard each G_α to be a subgroup of $\tilde{G}$ to simplify notations.

(2) In the category **Top** we proceed analogously by taking the coproduct, the disjoint union of the spaces, and defining the equivalence relation (40.4) on it and passing on to the quotient space. In applications one uses the defining properties (1) and (2) of definition (40.2) and not these details involved in the actual construction.

Exercises:

1. Prove lemma (40.1)

2. Show that the Prüfer group (40.3) is the inductive limit of the sequence of multiplicative cyclic groups C_{p^k} of order p^k, where p is a prime number.

3. Discuss the existence of inductive limits of directed systems in the categories **Gr** and **Top**.

4. Suppose that $\{G_\alpha / \alpha \in \Lambda\}$ is a directed system of groups with inductive limit G and associated maps $f_\alpha : G_\alpha \longrightarrow G$, show that G is the set theoretic union of the images $f_\alpha(G_\alpha)$, $\alpha \in \Lambda$.

Lecture - XLI The Jordan-Brouwer separation theorem

We conclude the course with a proof of the Jordan Brouwer theorem, a far reaching generalization of the Jordan curve theorem (theorem 1.1). The most transparent and clear proof of the Jordan Brouwer theorem uses the notion of inductive limits developed in the previous lecture. We shall follow closely the treatment in [16] demonstrating the power of the Mayer Vietoris sequence.

Theorem 41.1: Let X be a topological space and $\{X_\alpha \;/\; \alpha \in \Lambda\}$ be a directed system of open subsets of X such that every compact subset of X lies in some X_α. For a pair of indices $\alpha \leq \beta$, the map $f_{\alpha\beta} : H_n(X_\alpha) \longrightarrow H_n(X_\beta)$ is the homomorphism induced by inclusion $X_\alpha \longrightarrow X_\beta$. Then, the family $\{H_n(X_\alpha) \;/\alpha \in \Lambda\}$ together with the maps $f_{\alpha\beta}$ forms an inductive system of abelian groups and

$$\varinjlim_{\alpha} H_n(X_\alpha) = H_n(X) = H_n\big(\varinjlim_{\alpha} X_\alpha\big) \tag{41.1}$$

Proof: The fact that $\{H_n(X_\alpha) \;/\alpha \in \Lambda\}$ is an inductive system is clear. Let A denote the inductive limit of this system in **AbGr** and $f_\alpha : H_n(X_\alpha) \longrightarrow A$ denote the associated homomorphisms described in definition (40.2). The inclusion maps $X_\alpha \subset X$ induce homomorphisms $\iota_\alpha : H_n(X_\alpha) \longrightarrow H_n(X)$. To simplify notations, we shall suppress the bar and use the same symbol ζ to denote a cycle as well as the homology class it represents. The proof of (41.1) hinges on two simple facts:

(i) If ζ' is an $n-$chain in X then there exists an $\alpha \in \Lambda$ such that the images of the constituent simplicies in ζ' are all contained in X_α. We shall say that the chain ζ' *is supported in* X_α. Thus ζ' may be viewed as a singular chain in X_α and the latter will be provisionally denoted by ζ in the proof. Further if ζ' is a cycle in X then ζ is a cycle in X_α and $\zeta' = \iota_\alpha(\zeta)$.

(ii) If ζ' is a boundary of a chain ω' in X then there exists a $\beta \in \Lambda$ such that ζ' and ω' are both supported in X_β and the relation $\zeta = \partial\omega$ holds in X_β. In other words,

$$\iota_\alpha(\zeta) = 0 \quad \text{implies} \quad f_{\alpha\beta}(\zeta) = 0 \quad \text{for some} \quad \beta \geq \alpha. \tag{41.2}$$

To prove these note that the image of each singular simplex is a compact subset of X and each chain is a finite linear combination of singular simplicies.

Property (2) of definition (40.2) may now be applied to the family of homomorphisms ι_α. There exists a group homomorphism $\phi : A \longrightarrow H_n(X)$ such that

$$\phi \circ f_\alpha = \iota_\alpha, \quad \alpha \in \Lambda \tag{41.3}$$

To show that ϕ is surjective, by (i) above, an arbitrary cycle ζ' in X with support in X_α representing an element of $H_n(X)$ may be expressed as $\iota_\alpha(\zeta)$ where ζ is a cycle in X_α. By (41.3) we see that

$\zeta' \in \text{im}\,\phi$. To show that ϕ is injective, let $\zeta' \in A$ be such that $\phi(\zeta') = 0$ in X. By exercise 4 of lecture 40, we can write

$$\zeta' = \sum f_\alpha(\zeta_\alpha) \tag{41.4}$$

where the sum is finite and each ζ_α is a cycle in X_α. Choose a β exceeding all the indices in (41.4) and for each index α in (41.4), $f_\alpha(\zeta_\alpha) = f_\beta \circ f_{\alpha\beta}(\zeta_\alpha)$ and so using (41.3),

$$0 = \phi(\zeta') = (\phi \circ f_\beta)\big(\sum f_{\alpha\beta}(\zeta_\alpha)\big) = \iota_\beta\big(\sum f_{\alpha\beta}(\zeta_\alpha)\big)$$

Invoking (41.2) we arrive at $\sum f_{\alpha\beta}(\zeta_\alpha) = 0$ (perhaps with a larger β). Applying f_β we see that $\zeta' = 0$ as desired. □

Theorem 41.2: Let K be a subset of S^n that is homeomorphic to I^k for some k in the range $0 \le k \le n$. Then

$$H_j(S^n - K) = \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Proof: If $k = 0$ then K is a point and $S^n - K$ is homeomorphic to $\mathbb{R}^n$ and the result is true in this case. The proof now proceeds by induction on k. Assume that the result has been proved for $0 \le k \le m-1$ and let $h : K \longrightarrow I^m$ be a homeomorphism. Define the halves I^+ and I^- as

$$I^+ = \{(x_1, x_2, \dots, x_m) \in I^m \;/\; x_n \ge 1/2\}, \quad I^- = \{(x_1, x_2, \dots, x_m) \in I^m \;/\; x_n \le 1/2\}$$

and note that $I^+ \cap I^-$ is homeomorphic to the cube I^{m-1}. We construct the sets $K^+ = h^{-1}(I^+)$ and $K^- = h^{-1}(I^-)$, and use the Mayer Vietoris sequence to the following open cover of $S^n - (K^+ \cap K^{-1})$:

$$\{S^n - K^+,\; S^n - K^-\}.$$

Since $K^+ \cap K^-$ is homeomorphic to I^{m-1}, by induction hypothesis the end terms of the portion

$$H_{j+1}(S^n - K^+ \cap K^-) \longrightarrow H_j(S^n - K) \xrightarrow{(\kappa', -\kappa'')} H_j(S^n - K^+) \oplus H_j(S^n - K^-) \longrightarrow$$
$$\xrightarrow{q_j} H_j(S^n - K^+ \cap K^-)$$

are zero if $j > 0$ whereas the left most group is zero if $j = 0$. In any case $(\kappa', -\kappa'')$ is injective. Assume that for some $j > 0$, $H_j(S^n - K) \ne 0$. We choose $\zeta \in H_j(S^n - K)$, $\zeta \ne 0$ and it follows $\kappa'(\zeta) \ne 0$ or $\kappa''(\zeta) \ne 0$. Let us assume that $\kappa'(\zeta) \ne 0$. Since K^+ is homeomorphic to I^m, the process can be repeated subdividing K^+ into two pieces whose intersection is homeomorphic to I^{m-1}. Thus we construct a nested sequence of subsets

$$K = K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$$

such that for each p, the map $\kappa_p : H_j(S^n - K) \longrightarrow H_j(S^n - K_p)$ induced by inclusion, maps ζ to a non zero element. Composing with $f_p : H_j(S^n - K_p) \longrightarrow \varinjlim H_j(S^n - K_p)$ one checks that for $p, q \in \mathbb{N}$,

$$f_p \circ \kappa_p = f_q \circ \kappa_q,$$

thereby providing a map

$$f : H_j(S^n - K) \longrightarrow \varinjlim H_j(S^n - K_p).$$

Since the intersection $\bigcap K_i$ is homeomorphic to I^{m-1}, by induction hypothesis, $\varinjlim H_j(S^n - K_p) = \{0\}$. Hence $f_p(\kappa_p(\zeta)) = 0$ for every p and hence by theorem (40.2) (ii), for some $q \in \mathbb{N}$, $\kappa_q(\zeta) = 0$ which is a contradiction.

Turning to the case $j = 0$, assume that rank of $H_0(S^n - K)$ is atleast two. If we select points x and y lying in distinct path components of $S^n - K$, the cycle $\zeta = x - y$ in $S^n - K$ is not a boundary. As before we construct a nested sequence of compact sets $\{K_p\}$ with $\kappa_p(\zeta) \neq 0$ for each $p \in \mathbb{N}$. But since $S^n - \bigcap K_p$ has only one path component, $\iota_p \circ \kappa_p(\zeta)$ is a boundary where ι_p is the map induced by the inclusion $S^n - K_p \longrightarrow S^n - \bigcap K_p$ whence

$$f_p(\kappa_p(\zeta)) = 0$$

by (41.3). This in turn forces $\kappa_p(\zeta) = 0$ by theorem (40.2) (ii) and we have a contradiction.

Corollary 41.3: Suppose A is a subset of S^n homeomorphic to S^k for some k, $0 \leq k \leq n-1$, then

$$H_j(S^n - A) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } j = 0 \text{ and } k = n-1 \\ \mathbb{Z} & \text{if } j = 0 \text{ and } k \leq n-2 \\ \mathbb{Z} & \text{if } j = n-k-1 \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{41.5}$$

Proof: The result is clear if $k = 0$. We proceed by induction on k and assume the result with $k-1$ in place of k. Let $A = A^+ \cup A^-$ where A^+ and A^- are each homeomorphic to S^{k-1} and $A^+ \cap A^-$ is homeomorphic to S^{k-1}. The Mayer Vietoris sequence may be applied to the open cover $\{S^n - A^+,\ S^n - A^-\}$ of $S^n - A$ and the reader ought to verify that

$$H_{j+1}(S^n - A^+ \cap A^-) \cong H_j(S^n - A), \quad j > 0.$$

By induction hypothesis we get (41.5) for the case $j > 0$. Let us now consider the case $j = 0$. The tail end of the Mayer Vietoris sequence gives

$$0 \longrightarrow H_1(S^n - S^{k-1}) \longrightarrow H_0(S^n - S^k) \stackrel{r}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z} \stackrel{q}{\longrightarrow} \text{img } q \longrightarrow 0$$

Since the image of q is isomorphic to $\mathbb{Z}$, we see that the kernel of q must also be isomorphic to $\mathbb{Z}$ giving a short exact sequence

$$0 \longrightarrow H_1(S^n - S^{k-1}) \longrightarrow H_0(S^n - S^k) \stackrel{r}{\longrightarrow} \text{img } r \longrightarrow 0. \tag{41.6}$$

Since the image of r is free of rank one, (41.6) splits and we have

$$H_0(S^n - S^k) = H_1(S^n - S^{k-1}) \oplus \mathbb{Z}.$$

If $k = n-1$ then $1 = n-(k-1)-1$ and so the induction hypothesis gives $H_1(S^n - S^{k-1}) = \mathbb{Z}$ whereas if $k \leq n-2$ then $H_1(S^n - S^{k-1}) = 0$. □

Corollary 41.4: Suppose $A \subset S^n$ and A is homeomorphic to S^{n-1}, then $S^n - A$ is disconnected and has precisely two components.

Proof: Equation (41.5) shows that $S^n - A$ has two path components. However since $S^n - A$ an open set, $S^n - A$ is locally path connected and so the path components are the same as components. Let these components be C_1 and C_2.

Corollary 41.5 (Invariance of domain): Suppose U and V are homeomorphic subsets of $\mathbb{R}^n$. Then U is open if and only if V is open. In particular if $h : A \longrightarrow B$ is a homeomorphism between subsets of $\mathbb{R}^n$ then h maps interior points of A to interior points of B.

Proof: Let h be the homeomorphism between U and V and $p \in U$. We have to show that $h(p)$ is an interior point of V. Let K be a closed ball centered at p and contained in U so that $K' = h(K)$ is a compact subset of V containing $q = h(p)$. Let B be the (topological) boundary of K and $B' = h(B)$. We regard U and V as subsets of S^n. By theorem (41.2), $S^n - K'$ is path connected and $S^n - B'$ has two path components. However since the union

$$S^n - B' = (S^n - K') \cup (K' - B')$$

is a disjoint union of connected sets, the pieces $S^n - K'$ and $K' - B'$ are the components of $S^n - B'$. Hence they are both open in $S^n - B'$ (why?) and hence are open in S^n. The piece $K' - B'$ is then an open subset of S^n containing q and since $K' \subset V$ we see that q is an interior point of V. $\square$

Corollary 41.6 (Jordan Curve theorem): The complement of a simple closed curve C in $\mathbb{R}^2$ consists of two disjoint connected components precisely one of which is unbounded. $\square$

Exercises

1. Prove the second equality in equation (41.1).
2. Prove corollary (41.6).
3. Prove that there is no injective continuous mapping from S^n into $\mathbb{R}^n$. ([11], p. 217)
4. Show that no proper subset of S^n can be homeomorphic to S^n. ([11], p. 217)
5. Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}^n$ be an injective continuous map. Show that f is a homeomorphism onto its image. ([11], p. 217)

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Mid-Semester Examination

I (i) Prove that if G is a topological group and H is a subgroup such that H and the space of cosets G/H are connected then G is connected.

(ii) Prove that if V is a vector subspace of $\mathbb{R}^n$ of dimension less than or equal to $n-2$ then $\mathbb{R}^n - V$ is connected.

II (i) State and prove the reparametrization theorem.

(ii) Prove that the loop $\gamma : [0,1] \longrightarrow S^2$ given by

$$t \mapsto (\cos(2\pi i t), \sin(2\pi i t), 0), \quad 0 \leq t5 \leq 2\pi$$

is null-homotopic by exhibiting a homotopy.

III (i) State the no-retraction theorem and deduce from it the Brouwer's fixed point theorem.

(ii) Show that if Y is a retract of X and X has the fixed point property then so does Y.

IV (i) Show that $S^1 \vee S^1$ is the retract of $S^1 \times S^1$ minus a point but is not a retract of $S^1 \times S^1$.

(ii) Calculate the degree of the map $f : S^1 \longrightarrow S^1$ given by

$$f(z) = \frac{z^2 - z + \frac{3}{2}}{|z^2 - z + \frac{3}{2}|}$$

V (i) Provide three different statements each of which is equivalent to the statement that $p : \tilde{X} \longrightarrow X$ is a regular covering space.

(ii) Determine the deck transformations of the covering $p : \mathbb{C} - \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} - \{\pm 2\}$

$$p(z) = z^3 - 3z.$$

VI (i) Suppose G and $\tilde{G}$ are topological groups and $p : \tilde{G} \longrightarrow G$ is a covering projection which is also a group homomorphism then the kernel of p is the group of deck transformations.

(ii) If p is a prime what can you say about the group of deck transformations of a $p-$sheeted covering?

Mid-Semester Examination

I (i) Prove that the one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.

(ii) Regard S^n as the one point compactification of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Provide necessary and sufficient conditions for T to extend continuously as a map from S^n to itself.

II (i) Show that if G is a topological group and N is a discrete normal subgroup then N is contained in the center.

(ii) Show that the fundamental group of a topological group is abelian.

III (i) Define the term *star shaped domain* and show that the fundamental group of a star shaped domain is trivial.

(ii) Prove the fundamental theorem of algebra using the notion of degree of a map and homotopy.

V (i) Define the *lens spaces* and calculate their fundamental groups.

(ii) Determine the fundamental group of the Klein's bottle.

VI (i) Define the term *adjunction space* and explain how $\mathbb{R}P^2$ arises as the adjunction space by attaching a copy of E^2 to S^1. Specify clearly the attaching map.

(ii) Calculate the fundamental group of

$$\mathbb{C}^2 - \{(z_1, z_2)\ :\ z_1 z_2 = 0\}.$$

VII (i) State and prove the Barrett–Whitehead lemma.

(ii) State the relative version of the excision theorem and derive from it the relative Mayer Vietoris sequence.

VIII (i) Prove that the inductive limit of the sequence of inclusion maps

$$\longrightarrow C_p \longrightarrow C_{p^2} \longrightarrow C_{p^3} \longrightarrow$$

where C_{P^k} is the cyclic group of p^k-th roots of unity is the multiplicative group

$$\{z \in \mathbb{C}\ :\ |z| = 1,\ z^{p^k} = 1 \text{ for some } k \in \mathbb{N}\}.$$

(ii) State the Jordan Brouwer separation theorem and use it to show that there is no injective continuous map from S^n into $\mathbb{R}^n$.

Key words

Lecture - I Introduction
global analysis simple closed curve Jordan curve theorem Poincarè Bendixon theorem orientability Stokes's theorem
Lecture - II Preliminaries from general topology
compact space Lebesgue number for a cover Heine Borel theorem locally comapct space one point compactification stereo-graphic projection proper map
Lecture - III More preliminaries from general topology
connected space path connected space locally path connected space local homeomorphism Tietze's extension theorem

Lecture - IV Further preliminaries from general topology
quotient space quotient topology closed map open map universal property of quotients identification space Möbius band Klein's bottle real projective space torus Hausdorff quotients
Lecture - V Topological groups
topological groups group of orthogonal matrices group of unitary matrices discrete normal subgroup
Lecture - VII Paths, homotopies and the fundamental group
homotopy of paths reparametrization theorem juxtaposition of paths inverse path constant path loops based on x_0 base point fundamental group convex and star-shaped domains simply connected space

Lecture - VIII Categories and functors
category covariant functor contravariant functor abelianization commutator subgroup category of pairs semi-direct product coproducts products
Lecture - IX Functorial properties of the fundamental group
category of pointed topological spaces retraction no retraction theorem Brouwer's fixed point theorem fundamental theorem of algebra
Lecture - X Brouwer's theorem and its applications
Fixed point theorems Banach's fixed point theorem Fixed point property Perron Frobenius's theorem fundamental groups of spheres

Lecture - XI Homotopies of maps. Deformation retracts
Homotopies of maps Homotopy equivalence Deformation retract Fundamental group of the punctured plane
Lecture - XII-XIII Fundamental group of the circle
Lemma on mutually associative binary operations Lifting lemma Degree of a loop Degree of a map from S^1 into itself Generators for $\pi_1(S^1, 1)$ Borsuk-Ulam theorem Fundamental theorem of algebra Generators for $\pi_1(S^1 \times S^1, (1, 1))$
Lecture - XV Covering projections
Covering projections Evenly covered neighborhoods Fiber over a point Sheets over an evenly covered neighborhood Lifting problem Uniqueness of lifts
Lecture - XVI Lifting of paths and homotopies
Path lifting lemma Covering homotopy property Simple chain Chain connectedness Toral knot

Lecture - XVII Action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$
Right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ Transitive Stabilizer Conjugacy class of subgroups Regular covering Fundamental group of $\mathbb{R}P^n$
Lecture - XVIII The Lifting Criterion
Lift of maps Uniqueness of lifts Uniqueness of simply connected covers Little Picard theorem
Lecture - XIX Deck transformations
Deck transformations Deck transformations and regular coverings Homomorphism between coverings Category of coverings
Lecture - XX Orbit spaces
Properly discontinuous group actions Free group action Orbit space Lens spaces Generalized lens spaces

Lecture - XXII Fundamental group of $SO(3, \mathbb{R})$ and $SO(4, \mathbb{R})$
Quaternion Pure quaternion Conjugate of a quaternion Inverse function theorem Topological stucture of $SO(4, \mathbb{R})$
Lectures - XXIII and XXIV Coproducts and Pushouts
Coproduct of groups Free product of groups Coproductof abelian groups or direct sum Free groups Generators and relations Presenation of groups Pushouts Free product with amalgamation
Lectures - XXV Adjunction Spaces
Adjunction space Torus and Klein's bottle as adjunction spaces Projective plane as adjunction space Real projective spaces as adjunction spaces Cone over a space
Lectures - XXVI Seifert Van Kampen theorem
Seifert Van Kampen - version I Seifert Van Kampen - version II Fundamental group of a wedge of two circles Fundamental group of adjunction spaces

Lectures - XXVIII Introductory remarks on homology theory
Spectral sequences Simplicial theory Vector calculus Gauss divergence theorem
Lectures - XXIX and XXX The singular chain complex and homology groups
Standard simplex Face maps Singular chains Boundary of a singular simplex Category of chain complexes Differential chain complex Boundary operator Chain map Homology group Long exact homology sequence Connecting homomorphism
Lecture - XXXI The homology groups and their functoriality
Singular chain complex Singular cycles Singular boundaries Augmentation map Barycentric coordinates
Lecture - XXXII The abelianization of the fundamental group
The homomorphism Π_X Surjectivity of Π_X Commutator subgroup of $\pi_1(X, x_{x_0})$ Natural transformation

Lecture - XXXVII Relative homology
Group of relative cycles Group of relative boundaries Relative homology groups Long exact sequence for a pair Split exact sequence
Lecture - XXXVIII Excision theorem
Five lemma Excision theorem Barrett Whitehead theorem Mayer Vietoris sequence Local homology groups
Lecture - XL Inductive limits
Prüfer group Directed systems Inductive limit
Lecture - XLI Jordan Brouwer separation theorem
Jordan Brouwer theorem Brouwer's theorem on invariance of domain Jordan curve theorem