Department of Mathematics
Indian Institute of Technology, Bombay
Powai, Mumbai–400 076, INDIA.

MA 203–Mathematics III

Autumn 2006

Instructors:

A. Athavale
N. Nataraj
R. Raghunathan (Instructor in-charge)
G. K. Srinivasan

Name	:	____________________
Roll No	:	____________________

Course contents of MA 203 (Mathematics III):

Ordinary differential equations of the 1st order, exactness and integrating factors, variation of parameters, Picard's iteration method. Ordinary linear differential equations of the nth order, solution of homogeneous and non-homogeneous equations. The operator method. The methods ofundetermined coefficients and variation of parameters. Systems of differential equations, Phase plane. Critical points, stability.

Infinite sequences and series of real and complex numbers. Improper integrals. The Cauchy criterion, tests of convergence, absolute and conditional convergence. Series of functions. Improper integrals depending on a parameter. Uniform convergence. Power series, radius of convergence. Power series methods for solutions of ordinary differential equations. Legendre equation and Legendre polynomials, Bessel equations and Bessel functions of first and second kind. Orthogonal sets of functions. Sturm-Liouville problems. Orthogonality of Bessel functions and Legendre polynomials. The Laplace transform. The Inverse transform. Shifting properties, convolutions, partial fractions. Fourier series, half-range expansions. Approximation by trignometric polynomials. Fourier integrals.

Texts/References

E. Kreyszig, Advanced Engineering Mathematics, 8th ed., Wiley Eastern, 1999.

Teaching Plan

[K] refers to the text book by E. Kreyszig, "Advanced Engineering Mathematics", 8th Edition, John Wiley and Sons(1999).

Policy for Attendance

Attendance in both lectures and tutorial classes is compulsory. Students who fail to attend 80% of the lectures and tutorial classes may be awarded an XX grade.

Evaluation: Figures in parentheses denote the percentage of the marks assigned to each quiz or exam.

Quiz 1 (16.67): 6:00 - 7:00 p.m., Wednesday, August 16.
Quiz 2 (16.67): 6:00 - 7:00 p.m., Wednesday, October 18
Mid-semester examination (33.33)
End-semester examination (33.33)

Topics to be covered before the mid-semester examination

No.	Topic	§in [K]	No.of Lec.
1.	Basic concepts and ideas,	1.1-1.2	
	1st order linear equations (homogeneous		
	and non homogeneous),		
	Separation of Variables,	1.3 - 1.4	3
	Exact equations, integrating factors		
	The Bernoulli Equation	1.5-1.7	
2.	Existence and Uniqueness: Picard's iteration	1.9	
	Singular solutions, enveloppes and		4
	orthogonal trajectories	1.8	
3.	Linear Differential equations:		
	Homogeneous equations with constant		
	coefficients,	2.1-2.3	4
	Existence and Uniqueness, The Wronskian,		
	Non-homogeneous equations	2.7-2.8	
4.	The Cauchy-Euler equations		
	The Method of undetermined coefficients,	2.9–2.10	4
	The method of variation of parameters	2.11-2.12	2
5.	Sequences and Series, Convergence Tests	14.1, A- 3.3	3
6.	Uniform convergence, and Power Series	14.2 - 14.5	3

Topics to be covered after the mid-semester examination

No.	Topic	§in [K]	No.of Lec.
7.	Improper integrals (p. 27 of this booklet)		3
8.	Laplace transforms and systems of equations	5.1-5.7	4
9.	Series solutions: Legendre's equation		
	and the Legendre polynomials	4.3	3
10.	Frobenius' method, the Bessel's functions	4.4–4.6	4
10.	Sturm-Liouville Problems: Eigenvalues		
	and eigenfunctions properties and more	4.7-4.8	3
11.	Fourier Series and Fourier Integrals	10.1 - 10.10	4
12.	Qual. Theory: Phase plane analysis,	3.3 - 3.5	
	Critical Points, Linearized Stability.		3

Tutorial Sheet No. 1

Q.1. Classify the following equations (order, linear or non-linear):
(i) $\frac{d^3y}{dx^3} + 4(\frac{dy}{dx})^2 = y$ (ii) $\frac{dy}{dx} + 2y = \sin x$ (iii) $y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$
(iv) $\frac{d^4y}{dx^4} + (\sin x)\frac{dy}{dx} + x^2y = 0$. (v)$(1+y^2)\frac{d^2y}{dt^2} + t\frac{d^6y}{dt^6} + y = e^t$.

Q.2. Formulate the differential equations represented by the following functions by eliminating the arbitrary constants a, b and c:

(i) $y = ax^2$ (ii) $y - a^2 = a(x-b)^2$ (iii) $x^2 + y^2 = a^2$ (iv) $(x-a)^2 + (y-b)^2 = a^2$
(v) $y = a\sin x + b\cos x + a$ (vi) $y = a(1-x^2) + bx + cx^3$ (vii) $y = cx + f(c)$.

Also state the order of the equations obtained.

Q.3. Solve the equation $x^3(\sin y)y' = 2$. Find the particular solution such that $y(x) \to \frac{\pi}{2}$ as $x \to +\infty$.

Q.4. Prove that a curve with the property that all its normals pass through a point is a circle.

Q.5. Find the values of m for which

(a) $y = e^{mx}$ is a solution of
(i) $y'' + y' - 6y = 0$ (ii) $y''' - 3y'' + 2y' = 0$.

(b) $y = x^m$ for $x > 0$ is a solution of
(i) $x^2y'' - 4xy' + 4y = 0$ (ii) $x^2y''' - xy'' + y' = 0$.

Q.6. For each of the following linear differential equations verify that the function given in brackets is a solution of the differential equation.

(i) $y'' + 4y = 5e^x + 3\sin x$ $(y = a\sin 2x + b\cos 2x + e^x + \sin x)$
(ii) $y'' - 5y' + 6y = 0$, $(y_1 = e^{3x}, y_2 = e^{2x}, c_1y_1 + c_2y_2)$
(iii) $y''' + 6y'' + 11y' + 6y = e^{-2x}$ $(y = ae^{-x} + be^{-2x} + ce^{-3x} - xe^{-2x})$
(iv) $y''' + 8y = 9e^x + 65\cos x$, $(y = ae^{-2x} + e^x(b\cos\sqrt{3}x + c\sin\sqrt{3}x) + 8\cos x - \sin x + e^x)$

Q.7. Let φ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$.

Show that $\varphi_1 + \varphi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$. Use this result to find the solutions of $y' + y = \sin x + 3\cos 2x$ passing through the origin.

Q.8. Obtain the solution of the following differential equations:

(i) $(x^2+1)dy + (y^2+4)dx = 0$; $y(1) = 0$
(ii) $y' = y\cot x$; $y(\pi/2) = 1$
(iii) $y' = y(y^2-1)$, with $y(0) = 2$ or $y(0) = 1$, or $y(0) = 0$
(iv) $(x+2)y' - xy = 0$; $y(0) = 1$
(v) $y' + \dfrac{y-x}{y+x} = 0$; $y(1) = 1$
(vi) $y' = (y-x)^2$; $y(0) = 2$
(vii) $2(y\sin 2x + \cos 2x)dx = \cos 2x dy$; $y(\pi) = 0$.
(viii) $y' = \dfrac{1}{(x+1)(x^2+1)}$

Q.9. For each of the following differential equations, find the general solution (by substituting $y = vx$)

(i) $y' = \dfrac{y^2 - xy}{x^2 + xy}$ (ii) $x^2y' = y^2 + xy + x^2$

(iii) $xy^{'} = y + x\cos^2(y/x)$ (iv) $xy' = y(\ln y - \ln x)$

Q.10. Show that the differential equation $\dfrac{dy}{dx} = \dfrac{ax + by + m}{cx + dy + n}$ where a, b, c, d, m and n are constants can be reduced to $\dfrac{dy}{dt} = \dfrac{ax + by}{cx + dy}$ if $ad - bc \neq 0$. Then find the general solution of

(i) $(1 + x - 2y) + y'(4x - 3y - 6) = 0$

(ii) $y' = \frac{y-x+1}{y-x+5}$

(iii) $(x + 2y + 3) + (2x + 4y - 1)y' = 0$.

Q.11. Solve the differential equation $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0$ with the conditions $y(0) = \dfrac{\pm 1}{2}\sqrt{3}$. Sketch the graphs of the solutions and show that they are each arcs of the same ellipse. Also show that after these arcs are removed, the remaining part of the ellipse does not satisfy the differential equation.

Q.12. The differential equation $y = xy' + f(y')$ is called a Clairaut equation (or Clairaut's equation). Show that the general solution of this equation is the family of straight lines $y = cx + f(c)$. In addition to these show that it has a special solution given by $f'(p) = -x$ where $p = y'$. This special solution which does not (in general) represent one of the straight lines $y = cx + f(c)$, is called a singular solution. Hint: Differentiate the differential equation.

Q.13. Determine the general solutions as well as the singular solutions of the following Clairaut equations. In each of the two examples, sketch the graphs of these solutions.

(i) $y = xy' + 1/y'$. (ii) $y = xy' - y'/\sqrt{1 + y'^2}$

Q.14. For the parabola $y = x^2$ find the equation of its tangent at (c, c^2) and find the ordinary differential equation for this one parameter family of tangents. Identify this as a Clairaut equation. More generally take your favourite curve and determine the ODE for the one parameter family of its tangents and verify that it is a Clairut's equation. N.B: Exercise 13 shows that the converse is true.

Q.15. In the preceeding exercises, show that in each case, the envelope of the family of straight lines is also a solution of the Clairaut equation.

Q.16. Show that the differential equation $y' - y^3 = 2x^{-3/2}$ has three distinct solutions of the form $A/\sqrt{x}$ but that only one of these is real valued.

Tutorial Sheet No. 2

Q.1. State the conditions under which the following equations are exact.

(i) $[f(x) + g(y)]dx + [h(x) + k(y)]dy = 0$

(ii) $(x^3 + xy^2)dx + (ax^2y + bxy^2)dy = 0$

(iii) $(ax^2 + 2bxy + cy^2)dx + (bx^2 + 2cxy + gy^2)dy = 0$

Q.2. Solve the following exact equations

(i) $3x(xy-2)dx+(x^3+2y)dy=0$ (ii) $(\cos x\cos y-\cot x)dx-\sin x\sin y dy=0$.

(iii) $e^x y(x+y)dx+e^x(x+2y-1)dy=0$

Q.3. Determine (by inspection suitable) Integrating Factors (IF's) so that the following equations are exact.

(i) $ydx+xdy=0$ (ii) $d(e^x\sin y)=0$

(iii) $dx+(\frac{y}{x})^2dy=0$ (iv) $ye^{x/y}dx+(y-xe^{x/y})dy=0$

(v) $(2x+e^y)dx+xe^y dy=0$, (vi) $(x^2+y^2)dx+xydy=0$

Q.4. Verify that the equation $Mdx+Ndy=0\ldots$ (1) can be expressed in the form

$$\frac{1}{2}(Mx+Ny)d(\ln xy)+\frac{1}{2}(Mx-Ny)d\ln(\frac{x}{y})=0.$$

Hence, show that (i) if $Mx+Ny=0$, then $\dfrac{1}{Mx-Ny}$ is an IF of (1) and

(ii) if $Mx-Ny=0$, then $\dfrac{1}{Mx+Ny}$ is an IF of (1).

Also show that (iii) if M and N are homogeneous of the same degree then $\dfrac{1}{Mx+Ny}$ is an IF of (1).

Q.5. If $\mu(x,y)$ is an IF of $Mdx+Ndy=0$ then prove that

$$M_y-N_x=N\frac{\partial}{\partial x}\ln|\mu|-M\frac{\partial}{\partial y}\ln|\mu|.$$

Use the relation to prove that if $\dfrac{1}{N}(M_y-N_x)=f(x)$ then there exists an IF $\mu(x)$ given by $\exp(\int_a^x f(t)dt)$ and if $\dfrac{1}{M}(M_y-N_x)=g(y)$, then there exists an IF $\mu(y)$ given by $\exp(-\int_a^y g(t)dt)$. Further if $M_y-N_x=f(x)N-g(y)M$ then $\mu(x,y)=\exp(\int_a^x f(x')dx'+\int_a^y g(y')dy')$ is an IF, where a is any constant.

Determine an IF for the following differential equations:

(i) $y(8x-9y)dx+2x(x-3y)dy=0$.

(ii) $3(x^2+y^2)dx+(x^3+3xy^2+6xy)dy=0$

(iii) $4xy+3y^2-x)dx+x(x+2y)dy=0$

Q.6. Find the general solution of the following differential equations.

(i) $(y-xy')+a(y^2+y')=0$

(ii) $[y+xf(x^2+y^2)]dx+[yf(x^2+y^2)-x]dy=0$

(iii) $(x^3+y^2\sqrt{x^2+y^2})dx-xy\sqrt{x^2+y^2}dy=0$

(iv) $(x+y)^2y'=1$

(v) $y'-x^{-1}y=x^{-1}y^2$

(vi) $x^2y'+2xy=\sinh 3x$

(viii) $y'+y\tan x=\cos^2 x$

(ix) $(3y-7x+7)dx+(7y-3x+3)dy=0$.

Q.7. Solve the following homogeneous equations.

(i) $(x^3+y^2\sqrt{x^2+y^2})dx-xy\sqrt{x^2+y^2}dy=0$,

(ii) $(x^3+y^3)dx-3xy^2dy=0$

(iii) $(x^2+6y^2)dx+4xydy=0$,

(iv) $xy'=y(\ln y-\ln x)$.

(v) $xy'=y+x\cos^2\frac{y}{x}$

Q.8. Solve the following first order linear equations.

(i) $xy' - 2y = x^4$ (iii) $y' = 1 + 3y\tan x$

(ii) $y' + 2y = e^{-2x}$ (iv) $y' = \operatorname{cosec} x + y\cot x$.

(v) $y' = \operatorname{cosec} x - y\cot x$. (vi) $y' - my = c_1 e^{mx}$

Q.9. A differential equation of the form $y' + f(x)y = g(x)y^\alpha$ is called a Bernoulli equation. Note that if $\alpha = 0$ or 1 it is linear and for other values it is nonlinear. Show that the transformation $y^{1-\alpha} = u$ converts it into a linear equation. Use this to solve the following equations.

(i) $e^y y' - e^y = 2x - x^2$ (iv)$(xy + x^3y^3)\dfrac{dy}{dx} = 1$.

(ii) $2(y+1)y' - \dfrac{2}{x}(y+1)^2 = x^4$ (v) $\dfrac{dy}{dx} = xy + x^3y^3$

(iii) $xy' = 1 - y - xy$ (vi) $xy' + y = 2x^6y^4$

(vii) $6y^2dx - x(2x^3 + y)dy = 0$ (Bernoulli in x).

Q.10. (i) Solve $(x^2 + 6y^2)dx - 4xydy = 0$ as a Bernoulli equation.

(ii) Consider the initial value problem $y' = y(1-y), y(0) = 0$. Can this be solved by the meethod of separation of variables? As a Bernoulli equation?
Put $y = 1 - u$, $u(0) = 1$ and solve the resulting equation as a Bernoulli equation.

(iii) Solve $2ydx + x(x^2\ln y - 1)dy = 0$. Hint: The equation is Bernoulli in x.

(iv) Solve $\cos y \sin 2x dx + (\cos^2 y - \cos^2 x)dy = 0$
(**Hint:** Put $z = -\cos^2 x$; resulting ODE is Bernoulli in z.)

Q.11. Find the orthogonal trajectories of the following families of curves.

(i) $x^2 - y^2 = c^2$ (ii) $y = ce^{-x^2}$ (iii) $e^x \cos y = c$ (iv) $x^2 + y^2 = c^2$

(v) $y^2 = 4(x+h)$ (vi) $y^2 = 4x^2(1-cx)$ (vii) $y^2 = x^3/(a-x)$

(viii) $y = c(\sec x + \tan x)$. (ix) $xy = c(x+y)$

(x) $x^2 + (y-c)^2 = 1 + c^2$

Q.12. Find the ODE for the family of curves $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, $(0 < b < a)$ and find the ODE for the orthogonal trajectories. Explain the anomaly.

Q.13. A differential equation of the form $y' = P(x) + Q(x)y + R(x)y^2$ is called Riccati's equation. In general, the equation cannot be solved by elementary methods. But if a particular solution $y = y_1(x)$ is known, then the general solution is given by $y(x) = y_1(x) + u(x)$ where u satisfies the Bernoulli equation

$$\frac{du}{dx} - (Q + 2Ry_1)u = Ru^2.$$

(i) Use the method to solve $y' + x^3y - x^2y^2 = 1$, given $y_1 = x$.

(ii) Use the method to solve $y' = x^3(y-x)^2 + x^{-1}y$ given $y_1 = x$.

Q.14. Consider the differential equation $y' + P(x)y = 0$, where $P(x)$ is continuous on an interval I. Show that

(i) if $y = f(x)$ is a solution and $f(x_0) = 0$ for $x_0 \in I$, then $f(x) = 0$ for all $x \in I$

(ii) if $f(x)$ and $g(x)$ are two solutions such that $f(x_0) = g(x_0)$ for some $x_0 \in I$, then $f(x) = g(x)$ for all $x \in I$.

Q.15. Determine by Picard's method, successive approximations to the solutions of the following initial value problems. Compare your results with the exact solutions.

(i) $y' = 2\sqrt{y}; \quad y(1) = 0$

(ii) $y' - xy = 1; \quad y(0) = 1$

(iii) $y' = x - y^2; \quad y(0) = 1.$

Q.16. Show that the function $f(x, y) = |\sin y| + x$ satisfies the Lipschitz's condition

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1|$$

with $M = 1$, on the whole xy plane, but f_y does not exist at $y = 0$.

Q.17. Examine whether the following functions satisfy the Lipschitz condition on the xy plane. Does $\frac{\partial f}{\partial y}$ exist ? Compute the Lipschitz constant wherever possible.

(i) $f = |x| + |y|$

(ii) $f = 2\sqrt{y}$ in $\Re : |x| \leq 1, 0 \leq y \leq 1$ or in $\Re : |x| \leq 1, \frac{1}{2} < y < 1$

(iii) $f = x^2|y|$ in $\Re : |x| \leq 1, |y| \leq 1$

(iv) $f = x^2 \cos^2 y + y \sin^2 x, |x| \leq 1, |y| < \infty$

Tutorial Sheet No. 3

Q.1. Find the curve $y(x)$ through the origin for which $y'' = y'$ and the tangent at the origin is $y = x$.

Q.2. Find the general solutions of the following differential equations.

(i) $y'' - y' - 2y = 0$ (ii) $y'' - 2y' + 5y = 0$

Q.3. Find the differential equation of the form $y'' + ay' + by = 0$, where a and b are constants for which the following functions are solutions:

(i) $e^{-2x}, 1$ (ii) $e^{-(\alpha+i\beta)x}, e^{-(\alpha-i\beta)x}$.

Q.4. Are the following statements true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false. Here Ly denotes $y'' + P(x)y' + Q(x)y$.

(i) If $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I, then they are linearly independent on any interval containing I.

(ii) If $y_1(x)$ and $y_2(x)$ are linearly dependent on an interval I, then they are linearly dependent on any subinterval of I.

(iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $L(y) = 0$ on an interval I, they are linearly independent solution of $L(y) = 0$ on any interval I contained in I.

(iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained in I.

Q.5. Are the following pairs of functions linearly independent on the given interval?

(i) $\sin 2x, \cos(2x + \frac{\pi}{2}); x > 0$ (ii) $x^3, x^2|x|; -1 < x < 1$

(iii) $x|x|, x^2; 0 \le x \le 1$ (iv) $\log x, \log x^2; x > 0$ (v) $x, x^2, \sin x; x \in \mathbb{R}$

Q.6. Solve the following:

(i) $y'' - 4y' + 3y = 0,\ y(0) = 1, y'(0) = -5;$ (ii) $y'' - 2y' = 0,\ y(0) = -1, y(\frac{1}{2}) = e - 2.$

Q.7. For what non-negative values of λ do there exist non trivial solutions φ of $\varphi'' + \lambda^2\varphi = 0$ satisfying
(i) $\varphi(0) = 0 = \varphi(\pi)$, (ii) $\varphi'(0) = 0 = \varphi'(\pi)$
(iii) $\varphi(0) = \varphi(\pi), \varphi'(0) = \varphi'(\pi)$, (iv)$\varphi(0) = -\varphi(\pi), \varphi'(0) = -\varphi'(\pi)$.

Q.8. Solve the following initial value problems.

(i) $(D^2 + 5D + 6)y = 0,\quad y(0) = 2, y'(0) = -3$ (ii) $(D + 1)^2 y = 0,\quad y(0) = 1, y'(0) = 2$

(iii) $(D^2 + 2D + 2)y = 0,\quad y(0) = 1, y'(0) = -1$

Q.9. Solve the following initial value problems.

(i) $(x^2D^2 - 4xD + 4)y = 0, y(1) = 4, y'(1) = 1$

(ii) $(4x^2D^2 + 4xD - 1)y = 0, y(4) = 2, y'(4) = -0.25$

(iii) $(x^2D^2 - 5xD + 8)y = 0, y(1) = 5, y'(1) = 18$

Q.10. Using the Method of Undetermined Coefficients, determine a particular solution of the following equations. Also find the general solutions of these equations.

(i) $y'' + 2y' + 3y = 27x$ (ii) $y'' + y' - 2y = 3e^x$
(iii) $y'' + 4y' + 4y = 18\cosh x$ (iv) $y'''' + y = 6\sin x$
(v) $y'' + 4y' + 3y = \sin x + 2\cos x$ (vi) $y'' - 2y' + 2y = 2e^x \cos x$
(vii) $y'' + y = x\cos x + \sin x$ (viii) $2y'''' + 3y'' + y = x^2 + 3\sin x$
(ix) $y''' - y' = 2x^2e^x$ (x) $y''' - 5y'' + 8y' - 4y = 2e^x \cos x$

Q.11. Solve the following initial value problems.

(i) $y'' + y' - 2y = 14 + 2x - 2x^2, y(0), y'(0) = 0.$
(ii) $y'' + y' - 2y = -6\sin 2x - 18\cos 2x;\ y(0) = 2, y'(0) = 2.$
(iii) $y'' - 4y' + 3y = 4e^{3x}, y(0) = -1, y'(0) = 3.$

Q.12. Find a solution $y = y(x)$ of the initial value problem

$$y'' + y = f(x), y(0) = 0, y'(0) = 1,\ \text{where } f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi \\ \pi e^{-x\pi} & \text{if } x > \pi \end{cases}$$

such that both y and y' are continuous functions.

Q.13. For each of the following equations, write down the form of the particular solution. Do not go further and compute the Undetermined Coefficients.

(i) $y'' + y = x^3 \sin x$ (ii) $y'' + 2y' + y = 2x^2e^{-x} + x^3e^{2x}$ (iii) $y' + 4y = x^3e^{-4x}$ (iv) $y^{(4)} + y = xe^{x/\sqrt{2}} \sin(x/\sqrt{2})$.

Q.14. Solve the Cauchy-Euler equations: (i) $x^2y'' - 2y = 0$ (ii) $x^2y'' + 2xy' - 6y = 0$. (iii) $x^2y'' + 2xy' + y/4 = 1/\sqrt{x}$

Q.15. Find the solution of $x^2y'' - xy' - 3y = 0$ satisfying $y(1) = 1$ and $y(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Q.16. Show that every solution of the constant coefficient equation $y'' + \alpha y' + \beta y = 0$ tends to zero as $x \to \infty$ if and only if the real parts of the roots of the characteristic polynomial are negative.

Tutorial Sheet No. 4

Q.1. Using the Method of Variation of Parameters, determine a particular solution for each of the following.
(i) $y'' - 5y' + 6y = 2e^x$ (ii) $y'' + y = \tan x, 0 < x < \frac{\pi}{2}$
(iii) $y'' + 4y' + 4y = x^{-2}e^{-2x}, x > 0$ (iv) $y'' + 4y = 3 \text{ cosec } 2x, 0 < x < \frac{\pi}{2}$
(v) $x^2y'' - 2xy' + 2y = 5x^3 \cos x$ (vi) $xy'' - y' = (3 + x)x^3e^x$

Q.2. Let $y_1(x)$ and $y_2(x)$ be two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, $a < x < b$, and let $W(x)$ be the Wronskian of these two solutions. Prove that $W'(x) = -p(x)W(x)$. If $W(x_0) = 0$ for some x_0 with $a < x_0 < b$, then prove that $W(x) = 0$ for each x with $a < x < b$.

Q.3. Let $y = y_1(x)$ be a solution of $y'' + p(x)y' + q(x)y = 0$. Let I be an interval where $y_1(x)$ does not vanish, and $a \in I$ be any element. Prove that the general solution is given by

$$y = y_1(x)[c_2 + c_1\psi(x)] \text{ where } \psi(x) = \int_a^x \frac{\exp[-\int_a^t p(u)du]}{y_1^2(t)}dt.$$

Q.4. For each of the following ODEs, you are given one solution. Find a second solution.

(i) $4x^2y'' + 4xy' + (4x^2 - 1)y = 0; y_1(x) = \sin x/\sqrt{x}$
(ii) $y'' - 4xy' + 4(x^2 - 2)y = 0; y_1 = e^{x^2}$
(iii) $x(x-1)y'' + 3xy' + y = 0; y_1 = x/(x-1)^2$;
(iv) $xy'' - y' + 4x^3y = 0, y_1 = \cos x^2$
(v) $x^2(1 - x^2)y'' - x^3y' - \left(\frac{3 - x^2}{4}\right)y = 0,\ y_1 = \sqrt{\frac{1 - x^2}{x}}$.
(vi) $x(1 + 3x^2)y'' + 2y' - 6xy = 0, y_1 = 1 + x^2$
(vii) $(\sin x - x \cos x)y'' - (x \sin x)y' + (\sin x)y = 0, y_1 = x$.

Q.5. Computing the Wronskian or otherwise, prove that the the functions $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$, where $r_1, r_2, \ldots, r_n$ are distinct real numbers, are linearly independent.

Q.6. Let $y_1(x), y_2(x) \ldots, y_n(x)$ be n linearly independent solutions of the nth order homogeneous linear differential equation $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y + p_n(x)y = 0$. Prove that $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + \ldots + c_n(x)y_n(x)$ is a solution of the nonhomogeneous equation $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y =$ where $c_1(x), c_2(x), \ldots, c_n(x)$ are given by $c_i(x) = \int \frac{D_i(x)}{W(x)}dx$, where $D_i(x)$ is the determinant of

the matrix obtained from the matrix defining the Wronskian $W(x)$ by replacing its i th column by
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ r(x) \end{pmatrix}$$

Q.7. Three solutions of a certain second order non-homogeneous linear differential equation are

$$y_1(x) = 1 + e^{x^2} \quad y_2(x) = 1 + xe^{x^2}, \quad y_3(x) = (1+x)e^{x^2} - 1.$$

Find the general solution of the equation.

Q.8. For the following nonhomogeneous equations, a solution y_1 of the corresponding homogeneous equation is given. Find a second solution y_2 of the corresponding homogeneous equation and the general solution of the nonhomogeneous equation using the Method of Variation of Parameters.

(i) $(1+x^2)y'' - 2xy' + 2y = x^3 + x, \quad y_1 = x$ (ii) $xy'' - y' + (1-x)y = x^2, \quad y_1 = e^x$

(iii) $(2x+1)y'' - 4(x+1)y' + 4y = e^{2x}, \quad y_1 = e^{2x}$

(iv) $(x^3 - x^2)y'' - (x^3 + 2x^2 - 2x)y' + (2x^2 + 2x - 2)y = (x^3 - 2x^2 + x)e^x, \quad y_1 = x^2$

Q.9. Reduce the order of the following equations given that $y_1 = x$ is a solution.

(i) $x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0$ (ii) $y''' + (x^2 + 1)y'' - 2x^2y' + 2xy = 0$

Q.10. Find the complementary function and particular integral for the following differential equations

(i) $y^{(4)} + 2y^{(2)} + y = \sin x$ (ii) $y^{(4)} - y^{(3)} - 3y^{(2)} + 5y' - 2y = xe^x + 3e^{-2x}$

Q.11. Solve the following Cauchy-Euler equations

(i) $x^2y'' + 2xy' + y = x^3$ (ii) $x^4y^{(4)} + 8x^3y^{(3)} + 16x^2y^{(2)} + 8xy' + y = x^3$

(iii) $x^2y'' + 2xy' + \frac{y}{4} = \frac{1}{\sqrt{x}}$

Q.12. Find a particular solution of the following inhomogeneous Cauchy-Euler equations.

(i) $x^2y'' - 6y = \ln x$ (ii) $x^2y'' + 2xy' - 6y = 10x^2$.‘

Q. 13. Find a second solution of

(i) $(x^2 - x)y'' + (x+1)y' - y = 0$ given that $(1+x)$ is a solution.

(ii) $(2x+1)y'' - 4(x+1)y' + 4y = 0$ given that e^{2x} is a solution.

Q. 14. Find a homogeneous linear differential equation on $(0, \infty)$ whose general solution is $c_1x^2e^x + c_2x^3e^x$. Does there exist a homogeneous differential equation with constant coefficients with general solution $c_1x^2e^x + c_2x^3e^x$?

Tutorial Sheet No. 5

Q.1. Are the following real sequences bounded, eventually monotone and convergent?

(i) $k, 2k^2, 3k^3 \dots (|k| < 1)$ (ii) $a, 2^2a^2, 3^2a^3, 4^2a^4, \dots (|a| < 1)$

(iii) $\frac{10^1}{1!}, \frac{10^2}{2!}, \dots \frac{10^n}{n!} \dots$. (iv) $\frac{1}{2}, \frac{1 \cdot 3}{2 \cdot 4}, \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \dots$

(v) $1, 1 + \frac{1}{1!}, 1 + \frac{1}{1!} + \frac{1}{2!}, \dots, 1 + \frac{1}{1!} + \dots + \frac{1}{n!}, \dots$

Q.2. (i) Prove that $\left(\ln\left(1 + \frac{1}{n}\right)\right)^{1/n}$ converges to 1 as $n \longrightarrow \infty$.

(ii) Show that if $\{a_n\}$ is monotone decreasing then so is the sequence $\left\{\frac{1}{n}\left(a_1 + a_2 + \dots + a_n\right)\right\}_n$

Q.3. Examine the following series for convergence.

(i) $\sum_1^\infty \frac{(n!)^2}{(2n)!}$ (ii) $\sum_1^\infty \frac{n!}{n^n}$ (iii) $\sum_{n=10}^\infty \frac{1}{\ln n}$ (iv) $\sum_{n=100}^\infty \frac{1}{\ln(\ln n)}$

(v) $\sum_{n=1}^\infty \frac{n^{1000}}{n!}$ (vi) $\sum_{n=1}^\infty \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n^4}$ (vii) $\sum_{n=0}^\infty \frac{i^n}{n^2 + i}$ (viii) $\sum_{n=1}^\infty \frac{n+i}{2^n n}$.

Q.4. Examine the series $\sum_{n=0}^\infty a_n$ for convergence, where $a_n = (\frac{1}{2})^n$, if n is even and $a_n = (\frac{1}{3})^n$, if n is odd.

Q.5. Examine the following series for absolute and conditional convergence.

(i) $\sum_{n=2}^\infty \frac{(-1)^n}{\ln n}$ (ii) $\sum_{n=1}^\infty \frac{(-1)^n n\pi^n}{e^{2n} + 1}$ (iii) $\sum_{k=0}^\infty u_k$, where $u_{2k} = \frac{1}{3^{2k}}$ and $u_{2k+1} = \frac{-1}{2^{2k+1}}$, $k = 0, 1, 2, \dots$

(iv) $\sum_{k=0}^\infty (u_k + v_k)$, where $u_k = \frac{i}{2^{3k}}$ and $v_k = \frac{1}{2^{3k+1}}$, $k = 0, 1, 2, \dots$

Q.6. For what real values of x, are the following series convergent.

(i) $\sum_1^\infty \frac{x^n}{n!}$ (ii) $\sum_1^\infty \frac{x^n}{n}$ (iii) $\sum_1^\infty (x^n + x^{-n})$ (iv) $\sum_1^\infty \frac{1}{x^n + x^{-n}}$ (v) $\sum_{n=0}^\infty \frac{x^n}{1 + x^{2n}}$

(vi) $\sum_{n=r}^\infty n(n-1)\dots(n-r+1)x^{n-r}$ (vii) $\sum_{n=1}^\infty \frac{x^{n+r}}{n(n+1)(n+2)\cdots(n+r)}$

Q.7. (**Modified Comparison Test.**) Let $\sum u_n$ and $\sum v_n$ be two series of positive terms. Let $\frac{u_{n+1}}{u_n} \le \frac{v_{n+1}}{v_n}$ for $n \ge n_0$. Show that

(i) if $\sum v_n$ is convergent, then so is $\sum u_n$. (ii) if $\sum u_n$ is divergent, then so is $\sum v_n$.

Examine the series $\sum_{n=1}^\infty \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (n+1)!}$ for convergence. (Hint: Take $v_n = \frac{1}{(n+1)(2n+1)^{1/2}}$ and apply (i)).

Q.8. Discuss the convergence of (i) $\sum_{n=1}^\infty (a^{1/n} - 1)$ (ii) $\sum_{n=1}^\infty (1 - \cos(1/n))$ (iii) $\sum_{n=3}^\infty \frac{\ln(1 + \frac{1}{n})}{x^{\ln \ln n}}$, $x > 0$

Q.9. Examine the convergence of the following improper integrals.

(i) $\displaystyle\int_0^1 \frac{dx}{x^{p+1}}$ (ii) $\displaystyle\int_1^\infty \frac{dx}{x^{p+1}}$ (iii) $\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}dx$ (iv) $\displaystyle\int_0^\infty \frac{\cos x dx}{\sqrt{1+x^3}}$ (iv) $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}dx$

(v) $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$ $(n, m \geq 0)$

Q.10. Examine the convergence of the following improper integrals and determine the value of the convergent integrals.

(i) $\displaystyle\int_0^\infty e^{-x^2}dx$, (ii) $\displaystyle\int_0^\infty x^{n-1}e^{-x}dx$, (iii) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{1+2x^2}}$ (iv) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

Q.11. Show that $\displaystyle\int_0^\infty \frac{e^{-ax^2}-e^{-bx^2}}{x^2}dx$ is convergent and find its value. Hint for the last part: Try integrating by parts.

Q.12. (i) Show that both $\displaystyle\int_0^\infty \sin x^2 dx$ and $\displaystyle\int_0^\infty \cos x^2 dx$ are convergent integrals and find their values. (Fresnel Intergals)

(ii) Show that the integrals $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$ and $\displaystyle\int_0^\infty \frac{x^2dx}{1+x^4}$ converge and find their values. Hint: Rewrite them as beta integrals and use Euler's reflection formula.

(iii) Express $\displaystyle\int_0^\infty \frac{x^{a-1}dx}{1+x}$ as a beta integral and show that it is equal to $B(a, 1-a)$ where $0 < a < 1$.

Q.13. Test for convergence and evaluate the convergent integrals.

(i) $\displaystyle\int_0^{\pi/2} \frac{dx}{\cos x}$, (ii) $\displaystyle\int_1^3 \frac{dx}{\sqrt{4x-x^2-3}}$, (iii) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+5}$ (iv) $\displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{\tan x}}dx$.

(v) $\displaystyle\int_0^{\pi/2} \sqrt{\tan x}dx$.

Q.14. Test the following improper integrals for convergence:

(i) $\displaystyle\int_0^\infty x\sin x\,dx$ (ii) $\displaystyle\int_0^\infty \frac{dx}{1+2x^2+3x^4}$ (iii) $\displaystyle\int_3^\infty \frac{dx}{\sqrt{x(1-x)(2-x)}}$

(iv) $\displaystyle\int_2^\infty \frac{\sqrt{2x^2-3}}{(x^{12}+1)^{1/5}}dx$ (v) $\displaystyle\int_0^{\pi/2} \ln\sin x\,dx$.

Q.15. Prove that the improper integral $\displaystyle\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right)dx$ is convergent and that its value is πa if $a > 0$. Hint: Integrate by parts.

Q.16. Test the following series for convergence.

(i) $\displaystyle\sum_{n=10}^\infty \frac{1}{n\ln n}$ (ii) $\displaystyle\sum_{n=10}^\infty \frac{1}{n^p\ln n}$ (iii) $\displaystyle\sum_{10}^\infty \frac{1}{n(\ln n)^p}$ (iv) $\displaystyle\sum_{n=100}^\infty \frac{1}{n\ln n(\ln\ln n)^\alpha}$.

Q.17. Evaluate $\displaystyle\int_{-\infty}^\infty \exp(-x^2)dx$.

Q.18. Discuss for convergence the following series:

(i) $\sum \frac{\log(n!)}{2^n}$ (ii) $\sum_{n=1}^{\infty} \Big(\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\Big)^p$ (iii) $\sum_{n=1}^{\infty} \frac{1}{n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)$ (iv) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)$

(v) $\sum \frac{1}{n^{n+\frac{1}{n}}}$ (vi) $\sum x^n \exp(-\frac{1}{2}\log n)$

Q.19. Show that the rearrangement

$$\Big(1+\frac{1}{3}-\frac{1}{2}\Big)+\Big(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\Big)+\Big(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\Big)$$

converges and that its value is $\frac{3}{2}\log 2$

Tutorial Sheet No. 6

Q.1. Examine each of the following sequence for uniform convergence in the given region.

(i) x^n, $0 \le x \le 1$ (ii) x^n, $0 < x \le \delta < 1$ (iii) $\frac{x}{1+nx}$, $0 < x < \infty$ (iv) $n^2 x e^{-nx}$, $0 \le x \le 1$.

(v) $x^n(1-x)^n$ on $[0,1]$ (vi) $\frac{n^2 x}{1+n^3x^2}$ on $(0,\infty)$ (vii) $\frac{n^2x^2}{1+n^3x^2}$ on $(0,\infty)$ (viii) $\frac{n^2x}{1+n^4x^2}$ on $(0,\infty)$

Q.2. Show that if $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly, then $f_n(x) \to 0$ uniformly.

Q.3. Examine the uniform convergence of each of the following series $\sum_{1}^{\infty} u_k(x)$ in the given region.

(a) (i) $u_k = \frac{x}{((k-1)x+1)(kx+1)}$, $0 < x < 1$; (ii) $u_k = x^{k-1} - x^k$, $|x| \le \alpha < 1$.

(b) $\sum_{1}^{\infty}(u_k - u_{k-1})$ when (i) $u_k = \frac{x}{1+k^2x}$ and (ii) $u_k = \frac{kx}{1+k^2x^2}$, $0 \le \alpha < x < \infty$.

Q.4. Show that $\sum_{n=0}^{\infty} \cos^n x$ converges uniformly on $\Big(\frac{\pi}{6}, \frac{\pi}{2}\Big)$. Is the convergence uniform on $\Big(0, \frac{\pi}{2}\Big)$?

Q.5. Prove that $\sum_{1}^{\infty} \frac{x^2}{(1+x^2)^{n-1}}$ is absolutely convergent but not uniformly convergent on $[0,1]$.

Q.6. Show that $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent for all real x but not absolutely convergent for all real x.

Q.7. Prove that $\sum_1^\infty \frac{(-1)^n(x^2+n)}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any real value of x.

Q.8. Show that $\sum_1^\infty \frac{x}{n(1+nx^2)}$ converges uniformly and absolutely for all real x.

Q.9. Show that $f_n(x) = \frac{n^2x}{1+n^3x}$ converges uniformly to zero on $[0,1]$. What can you say about the sequence of derivatives $f_n'(x)$?

Q.10. Show that $f_n(x) = \frac{n^2x}{1+n^3x^2}$ converges to zero pointwise. Examine if $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$.

Q.11. Show that $\lim_{n\longrightarrow\infty}\int_0^{\pi/2}\cos^{2n} x dx = \int_0^{\pi/2}\lim_{n\longrightarrow\infty}\cos^{2n} x dx$ although the sequence of functions $\cos^{2n} x$ fails to converge uniformly on $[0,\pi/2]$. Hence determine the limit of the convergent sequence given in tutorial sheet 5, Q1 (iv).

Hint: Break the integral into a sum of an integral over $[0,\epsilon]$ and an integral over $[\epsilon,\pi/2]$. On $[\epsilon,\pi/2]$ we have uniform convergence and estimate the integral over $[0,\epsilon]$.

Q.12. Prove that $\sum_1^\infty \frac{1}{n^x}, \sum_{n=10}^\infty \frac{1}{\ln n(n^x)}$ converges uniformly if $x \geq 1+\alpha > 1$.

Q.13. Find the radius of convergence of the following power series:

(i) $\sum z^n$ (ii) $\sum \frac{z^m}{m!}$ (iii) $\sum m! z^m$ (iv) $\sum_{m=k}^\infty m(m-1)\cdots(m-k+1)z^m$ (v) $\sum \frac{(2n)!}{2^{2n}(n!)^2}z^n$

(vi) $\sum_1^\infty \frac{z^m}{m(m+1)\cdots(m+k+1)}$ (vii) $\sum_1^\infty \frac{n^n}{n!}z^n$ (viii) $\sum_1^\infty \frac{(2n)!}{n^n}z^n$ (ix) $\sum_1^\infty \frac{(3n)!}{2^n(n!)^3}z^n$

Q.14. Suppose $f_n(x)$ converges uniformly to $f(x)$ on $(0,\infty)$ does it follow that $\int_0^\infty f_n(x)dx \longrightarrow \int_0^\infty f(x)dx$ as $n \longrightarrow \infty$?

Q.15. Determine the radius of convergence of $\sum n! x^{n^2}$ and $\sum x^{n!}$

Tutorial Sheet No. 7

Q.1. Apply the power series method to determine the general solution of the following differential equations.

(i) $(1-x^2)y' = y$ (ii) $y' = xy$ $y(0) = 1$ (iii) $(1-x^2)y' = 2xy$

(iv) $y' - 2xy = 1$, $y(0) = 0$. From (iv) deduce the Taylor series for $e^{x^2}\int_0^x e^{-t^2}\,dt$.

Q.2. Find the solution as a power series in powers of $(x-1)$.

(i) $y'' + y = 0$ (ii) $y'' - y = 0$

Q.3. Find the power series solutions for the following differential equations.

(i) Legendre's equation: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$.

(ii) Tchebychev's equation: $(1-x^2)y'' - xy' + p^2y = 0$.

(iii) Airy's equation: $y'' - xy = 0$.

(iv) Hermite's equation : $y'' - x^2y = 0$.

Q.4. Show that the function $(\sin^{-1} x)^2$ satisfies the IVP: $(1-x^2)y'' - xy' = 2, \quad y(0) = 0, \quad y'(0) = 0$. Hence, find the Taylor series for $(\sin^{-1} x)^2$. What is its radius of convergence ?

Q.5. Attempt a power series solution (with center at the origin) for $x^2y'' - (1+x)y = 0$. Explain why the procedure does not give any nontrivial solutions.

Q.6. Prove that if $\{f_1, \ldots, f_n, \ldots\}$ and $\{g_1, \ldots, g_n, \ldots\}$ are two sets of orthogonal vectors such that

$$\text{Lin. Span}\Big\{f_1, \ldots, f_n\Big\} = \text{Lin. span}\Big\{g_1, \ldots, g_n\Big\}, \quad \text{for all } n,$$

then $f_n = c_n g_n$ for each n.

Q.7. Prove that the polynomials $\dfrac{d^n}{dx^n}\Big(x^2 - 1\Big)^n$ for $n = 1, 2, 3, \ldots$ form an orthogonal family of polynomials. Deduce from this the Rodrigues formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$. Hint Use the preceeding exercise.

Q.8. Prove that $(1 - 2xh + h^2)^{-1/2} = \sum_0^\infty P_n(x)h^n$

Q.9. Use the formula in Q.3, tutorial sheet 4 to find the second solution of the Legendre differential equation for the cases $p = 1$ and $p = 2$.

Q.10. Show that if $f(x)$ is a polynomial with double roots at a and b then $f''(x)$ vanishes atleast twice in (a, b). Generalize this and show (using Rodrigues' formula) that $P_n(x)$ has n distinct roots in $(-1, 1)$ and in particular, $P_n(x)$ cannot have a double root.

Q.11. Establish the following recurrence relations for $P_n(x)$.

(i) $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$ (ii) $P'_{n+1} - xP'_n - (n+1)P_n = 0$

(iii) $xP'_n - P'_{n-1} - nP_n = 0$ (iv) $P'_{n+1} - P'_{n-1} - (2n+1)P_n = 0$

(v) $(x^2-1)P'_n - nxP_n + nP_{n-1} = 0$

Q.12. Prove the following relations:

(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P'_n(-x) = (-1)^{n+1}P'_n(x)$

(iii) $P_n(1) = 1; P_n(-1) = (-1)^n$ (iv) $P_{2n+1}(0) = 0; P_{2n}(0) = (-1)^n \dfrac{(2n)!}{2^{2n}(n!)^2}$

(v) $P'_n(1) = \dfrac{1}{2}n(n+1)$ (vi) $P'_n(-1) = (-1)^{n-1}\dfrac{1}{2}n(n+1)$

(vii) $P'_{2n}(0) = 0$ (viii) $P'_{2n+1}(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}$.

Q.13. Prove that $\int_{-1}^{+1} P_m(x)P_n(x)dx = 0 \;\; (m \neq n)$ and $\int_{-1}^{1} P_n(x)^2 dx = \dfrac{2}{2n+1}$.

Q.14. Let $z_1, z_2, \ldots, z_n$ be the zeros of $P_n(x)$. Show that

$$\frac{1}{P_n(x)^2(1-x^2)} = \frac{1}{2}\Big(\frac{1}{1-x} + \frac{1}{1+x}\Big) + \sum_{j=1}^{n} \frac{1}{(x-z_j)^2(1-z_j^2)(P'(z_j))^2}$$

Hence determine the second solution of the Legendre equation of order n using the formula $y_2 = y_1 \int \frac{1}{y_1^2} \exp\Big(-\int P(x)dx\Big)dx$

Q.15. Prove the following relation if $(n-m)$ is even $(m \leq n)$

(i) $\int_{-1}^{+1} P_m' P_n' dx = m(m+1)$ (ii) $\int_{0}^{1} P_m P_n dx = 0$ if $(n-m)$ is even and $n \neq m$.

(iii) $\int_{-1}^{1} x^m P_n'(x)dx = 0$ if $m \leq n$. What is the value of the integral if instead $n-m$ is odd?

Tutorial Sheet No. 8

Q.1. Locate and classify the singular points for the following differential equations:

(i) Bessel's equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$.

(ii) Laguerre's equation: $xy'' + (1-x)y' + \lambda y = 0$.

(iii) Jacobi's equation: $x(1-x)y'' + (\gamma - (\alpha+1)x)y' + n(n+\alpha)y = 0$.

(iv) The hypergeometric equation: $x(1-x)y'' + [c - (a+b+1)x)]y' - aby = 0$.

(v) $xy'' + (\cot x)y' + xy = 0$

Q.2. Attempt a Frobenius series solution $y(x) = x^\rho \sum_{n=0}^{\infty}$ for the $x^2y'' + (3x-1)y' + y = 0$, and compute the successive coefficients and the radius of convergence of the series solution. Why does the method fail?

Q.3. Find two linearly independent solutions of the following differential equations:

(i) $x(x-1)y'' + (4x-2)y' + 2y = 0$.

(ii) $2x(x+2)y'' + y' - xy = 0$.

(iii) $x^2y'' + x^3y' + (x^2-2)y = 0$.

(iv) $xy'' + 2y' + xy = 0$.

Q.4. Show that the hypergeometric equation has a regular singular point at infinity[1], but that the point of infinity is an irregular singular point for the Airy's equation.

[1] The differential equation $y'' + p(x)y' + q(x)y = 0$ has a regular singular point at infinity, if after substitution of $x = 1/t$ in the ODE, the resulting ODE has a regular singular point at the origin.

Q.5. Using the indicated substitutions, reduce the following differential equations to Bessel's equation and find the general solution in term of the Bessel functions.

(i) $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0$, $(\lambda x = z)$. (ii) $xy'' - 5y' + xy = 0$, $(y = x^3u)$.

(iii) $y'' + k^2xy = 0$, $(y = u\sqrt{x}, \ \frac{2}{3}kx^{3/2} = z)$.

(iv) $x^2y'' + (1-2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0$, $(y = x^\nu, \ x^\nu = z)$.

Q.6. (a) Prove that $[x^nJ_n]' = x^nJ_{n-1}$ and $[x^{-n}J_n]' = -x^{-n}J_{n+1}$

(b) Use (a) to prove that (i) $J_{n-1} + J_{n+1} = \dfrac{2n}{x}J_n$ (ii) $J_{n-1} - J_{n+1} = 2J_n'$

Q.7. Show that (i) $J_{1/2} = \sqrt{\dfrac{2}{\pi x}}\sin x$ (ii) $J_{-1/2} = \sqrt{\dfrac{2}{\pi x}}\cos x$ (iii) $J_{\pm 3/2} = \sqrt{\dfrac{2}{\pi x}}\Big(\dfrac{\sin x}{x} \mp \cos x\Big)$

Q.8. When n is an integer show that

(i) $J_n(x)$ is an even function if n is even (ii) $J_n(x)$ is an odd function if n is odd.

Q.9. Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$ and one zero of $J_{n-1}(x)$.

Q.10. Prove that $\exp\left(\dfrac{tx}{2} - \dfrac{x}{2t}\right) = \displaystyle\sum_{-\infty}^{\infty} J_n(x)t^n$ (This formula is due to Schlömilch). Use Schlömilch's formula to show that $J_0^2 + 2\displaystyle\sum_1^a J_n^2 = 1$. Deduce that $|J_0| \le 1; |J_n| \le \frac{1}{\sqrt{2}}$.

Q.11. Prove that (i) $\cos(x\sin\theta) = J_0(x) + 2\displaystyle\sum_1^\infty \cos 2n\theta J_{2n}(x)$ (ii) $\sin(x\sin\theta) = 2\displaystyle\sum_1^\infty \sin(2n+1)\theta J_{2n+1}(x)$.

Q.12. Show that $\dfrac{1}{2}\dfrac{d}{dx}[J_n^2 + J_{n+1}^2] = \dfrac{n}{x}J_n^2 - \dfrac{n+1}{x}J_{n+1}^2$, $\dfrac{d}{dx}[xJ_nJ_{n+1}] = x(J_n^2 - J_{n+1}^2)$, and deduce that

(i) $J_0^2 + 2\displaystyle\sum_1^\infty J_n^2 = 1$ (ii) $\displaystyle\sum_0^\infty (2n+1)J_nJ_{n+1} = \dfrac{x}{2}$. Hint for (ii): Look at $\dfrac{d}{dx}\left[x\displaystyle\sum_{n=0}^\infty (2n+1)J_nJ_{n+1}\right]$

Q.13. Prove the following.

(i) $J_3 + 3J_0' + 4J_0''' = 0$

(ii) $J_2 - J_0 = aJ_c''$ find a and c.

(iii) $\displaystyle\int J_{\nu+1}dx = \int J_{\nu-1}dx - 2J_\nu$.

Q.14. Using the identity $J_{p+1} = -J_p' + pJ_p/x$, to prove that

(i) $J_{n+\frac{1}{2}}(x) = (-1)^n\sqrt{\dfrac{2}{\pi}}x^{n+\frac{1}{2}}\Big(\dfrac{1}{x}\dfrac{d}{dx}\Big)^n\Big(\dfrac{\sin x}{x}\Big)$ (Use induction)

(ii) $\dfrac{J_{p-1}(x)}{J_p(x)} = \dfrac{2p}{x} - \cfrac{1}{\frac{2p+2}{x} - \cfrac{1}{\frac{2p+4}{x} - \dots}}$. What happens when $p = 1/2$? (proceed formally)

Q.15. Use the Schlömilch formula and the identity $e^{x(t-\frac{1}{t})/2}e^{y(t-\frac{1}{t})/2} = e^{(x+y)(t-\frac{1}{t})/2}$ to prove (for $n = 0. \pm 1, \pm 2, \ldots$) $J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y)$

Q.16. Show that $\sqrt{x}\Big[c_1 J_{1/3}(\frac{2}{3}x^{3/2}) + c_2 J_{-1/3}(\frac{2}{3}x^{3/2})\Big]$ is the general solution of the Airy's equation $y''+xy = 0$.

Q.17. Find two linearly independent series solutions of Bessel's equation with $p = 0$. (note that one of these will not be a power series). Perform a similar exercise for Laguerre's equation.

Additional Problems involving the Bessel and Legendre functions:

1. **Laplace's integral for** $P_n(x)$. The following integral representations were given by Pierre Simon Laplace in his Mécanique Céleste. Prove the following:

   (i) $P_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} (x + \sqrt{x^2-1}\cos\phi)^n d\phi$

   (ii) $P_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \frac{d\phi}{(x + \sqrt{x^2-1}\cos\phi)^{n+1}}$

2. A Frobenius series solution $(x-1)^{\rho}\sum_{n=0}^{\infty} a_n(x-1)^n$ for Legendre's differential equation $(1-x^2)y'' - 2xy' + p(p+$ is sought. Determine the indicial equation and the Frobenius exponents. Show that there is a power series solution that converges in the disc $|x-1| < 2$.

3. Show that $xJ_0(x)$ satisfies $y'' + y = -J_1(x)$ and hence deduce that $xJ_0(x) = \displaystyle\int_0^x \cos(x-t)J_1(t)dt$.

4. Prove that for a non-negative integer n, $J_k(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(x\sin\theta - k\theta)d\theta$

   Hint: Put $t = e^{i\theta}$ in Schlömilch's formula and integrate (after multiplying through by $e^{-ik\theta}$).

5. Prove the following estimate used in the proof of Schlömilch's formula: $|J_p(x)| \leq \dfrac{|x|^p}{p!}e^{|x|}$.

6. Prove the following for integral values of p:

   (i) $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} e^{itx}(1-t^2)^{-1/2}dt$

   (ii) $J_p(x) = \dfrac{x^p}{2^p\sqrt{\pi}\Gamma(p+\frac{1}{2})}\displaystyle\int_{-1}^{1} e^{itx}(1-t^2)^{p-1/2}dt$

7. Show that $xy'' + y' + xy = 0$ has a solution of the form $(\ln x)Q_1(x) + Q_2(x)$, where $Q_1(x)$ and $Q_2(x)$ are power series. Hint: One solution is a power series with non-zero constant term. Obtain other solution using the formula in Q.3, tutorial sheet 4.

8. For a function $f(x_1, x_2)$ which decays sufficiently rapidly the Fourier transform is defined as

$$\hat{f}(\xi_1, \xi_2) = \int_{\mathbb{R}^2} f(x_1, x_2)\exp(-i(x_1\xi_1 + x_2\xi_2))dx_1dx_2.$$

(i) Show that if $f(x_1, x_2)$ is radial, that is it depends only on $r = \sqrt{x_1^2 + x_2^2}$, then so is its Fourier transform (that is, show that $\hat{f}(\xi_1, \xi_2)$ depends only on $\rho = \sqrt{\xi_1^2 + \xi_2^2}$).

(ii) Let $f(x_1, x_2) = \phi(r)$ and $\hat{f}(\xi_1, \xi_2) = \psi(\rho)$. Then

$$\psi(\rho) = 2\pi \int_0^\infty r\phi(r) J_0(\rho r) dr.$$

Solution to Q1(i): The integral is obviously a polynomial since upon expanding the integrand we find that the integrals of odd powers of the cosine vanish. Call the integrals $Q_n(x)$. Then clearly

$$Q_0(x) = 1 = P_0(x), \quad Q_1(x) = x = P_1(x).$$

Our job will be over if we establish the relation

$$(n+1)Q_{n+1} - (2n+1)xQ_n(x) + nQ_{n-1}(x) = 0.$$

For convenience we shall denote $x + \sqrt{x^2-1}\cos\phi$ by A. Now writing $A^{n+1} = xA^n + A^n\sqrt{x^2-1}\cos\phi$ we get

$$Q_n(x) = xQ_{n-1}(x) + \frac{\sqrt{x^2-1}}{\pi} \int_0^\pi A^{n-1}\cos\phi d\phi$$

Upon integrating by parts once we get

$$Q_n(x) = xQ_{n-1}(x) + \frac{(n-1)(x^2-1)}{\pi} \int_0^\pi A^{n-2}\sin^2\phi d\phi \qquad (*)$$

Write $(x^2-1)\sin^2\phi$ as follows:

$$\begin{aligned}
(x^2-1)\sin^2\phi &= (x^2-1) - (\sqrt{x^2-1}\cos\phi)^2 \\
&= (x^2-1) - [(x + \sqrt{x^2-1}\cos\phi)^2 - x^2 - 2x\sqrt{x^2-1}\cos\phi] \\
&= (x^2-1) - [A^2 - x(x + \sqrt{x^2-1}\cos\phi) - x\sqrt{x^2-1}\cos\phi] \\
&= (x^2-1) - [A^2 - xA - x(A-x)] = -1 - A^2 + 2xA.
\end{aligned}$$

Substituting in (*) we get

$$Q_n(x) = xQ_{n-1}(x) + (n-1)(-Q_{n-2}(x) - Q_n(x) + 2xQ_{n-1}(x)),$$

which upon rearrangement gives

$$nQ_n(x) - (2n-1)xQ_{n-1}(x) + (n-1)Q_{n-2}(x) = 0$$

Replacing n by $n+1$ we get the relation sought. The second formula of Laplace can be derived similarly.

Tutorial Sheet No. 9

Q.1. Solve the following boundary value problems.

(i) $y'' - y = 0,\ \ y(0) = 0,\ \ y(1) = 1$ (ii) $y'' - 6y' + 25y = 0,\ \ y'(0) = 1,\ \ y(\pi/4) = 0$

(iii) $x^2y'' + 7xy' + 3y = 0,\ y(1) = 1,\ y(2) = 2$

(iv) $y'' + y' + y = x,\ y(0) + 2y'(0) = 1,\ \ y(1) - y'(1) = 8$

(v) $y'' + \pi^2 y = 0,\ \ y(-1) = y(1),\ \ y'(-1) = y'(1)$.

Q.2. Find the eigenvalues and eigenfunctions of the following boundary value problems.

(i)$y'' + \lambda y = 0,\quad y(0) = 0,\quad y'(1) = 0.$ (ii)$y'' + \lambda y = 0,\quad y(0) = 0,\quad y(\ell) = 0.$

(iii)$y'' + \lambda y = 0,\ y(0) = y'(0),\ y(1) = 0.$, (iv)$y'' + \lambda y = 0,\quad y(0) = y(2\pi),\ y'(0) = y'(2\pi).$

(v)$(e^{2x}y')' + e^{2x}(\lambda + 1)y = 0,\quad y(0) = 0,\quad y(\pi) = 0.$

Q.3. For which values of λ, does the boundary value problem

$$y'' - 2y' + (1 + \lambda)y = 0, \quad y(0) = 0,\ \ y(1) = 0$$

have a non-trivial solution ?

Q.4. Show that the eigenvalues of the boundary value problem $y'' + \lambda y = 0,\quad y(0) = 0,\quad y(1) + y'(1) = 0$ are obtained as solutions of $\tan k = -k$, where $k = \sqrt{\lambda}$. Conclude from a plot that this equation has infinitely many solutions. Show that the eigenfunctions are $y_m = \sin(k_m x)$.

Q.5. Determine the normalised eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0,\ \ y(0) = 0 = y(1)$.

Q.6. Expand the function $f(x) = x,\quad x \in [0,1]$ in terms of the normalised eigenfunctions $\phi_n(x)$ of the boundary value problem $y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y(1) + y'(1) = 0$.

Q.7. Find the eigenfunctions and the eigenvalues of the following Sturm-Liouville problems.

(i) $y'' + 2y' + (\lambda + 1)y = 0;\quad y(0) = y(\pi) = 0$ (ii) $x^2y'' + xy' + \lambda y = 0;\quad y(1) = y(\ell) = 0.$

Q.8. Verify that $J_n(\frac{kx}{a})$ satisfies $\frac{d}{dx}\left[x\frac{d}{dx}\left\{J_n(\frac{kx}{a})\right\}\right] + \left(\frac{k^2x}{a^2} - \frac{n^2}{x}\right)J_n(\frac{kx}{a}) = 0.$

Multiply by $J_n(\frac{\ell x}{a})$ and integrate by parts from 0 to a to get

$$kJ_n'(k)J_n(l) - \frac{kl}{a^2}\int_0^a xJ_n'\left(\frac{\ell x}{a}\right)J_n'\left(\frac{kx}{a}\right)\,dx + \int_0^a (\frac{k^2}{a^2}x - \frac{n^2}{x})J_n(\frac{kx}{a})J_n(\frac{lx}{a})dx = 0,$$

where prime ($\prime$) denotes differentiation with respect to the argument of J_n. Interchange k and ℓ to obtain the relation

$$\int_0^a xJ_n(\frac{kx}{a})J_n(\frac{\ell x}{a})dx = a^2\frac{\ell J_n(k)J_n'(\ell) - kJ_n(\ell)J_n'(k)}{k^2 - \ell^2}.$$

Prove that if k and ℓ are the roots of the Bessel's equation $J_n(\lambda) = 0$ then

$$\begin{aligned}\int_0^a xJ_n(\frac{kx}{a})J_n(\frac{\ell x}{a})dx &= 0 \qquad (k \neq \ell) \\ &= \frac{1}{2}a^2[J_n'(k)]^2 \qquad (k = \ell) \\ &= \frac{1}{2}a^2 J_{n+1}^2(k) \qquad (k = \ell).\end{aligned}$$

Q.9. The function $P_n(x)$ satisfies the equation $\frac{d}{dx}[(1-x^2)P_n'] + n(n+1)P_n = 0$. Proceed as indicated in Q.8 above to prove that $\int_{-1}^{+1} P_m P_n dx = 0 \qquad (m \neq n)$.

Q.10. If $x^n = \sum_{r=0}^{n} a_r P_r(x)$ prove that $a_n = \frac{2^n (n!)^2}{(2n)!}$.

Q.11. Prove that $\int_{-1}^{+1} (1-x^2)[P_n'(x)]^2\, dx = \frac{2n(n+1)}{2n+1}$.

Q.12. Represent the following functions in terms of Legendre Polynomials: (i) $5x^3+x$ (ii) $10x^3-3x^2-5x-1$

Q.13. Show that $f(x) = x^n, (0 < x < 1; n = 0, 1, 2, \ldots)$ can be represented by the Fourier-Bessel series $x^n = \sum \frac{2J_n(k_i x)}{k_i J_{n+1}(k_i)}$ where k_i's are the roots of $J_n(k) = 0$.

Q.14. Represent the following functions in a Fourier-Bessel series containing the functions $J_0(k_i x/a)$ where k_i are the roots of $J_0(k) = 0$. (i) $f(x) = a^2 - x^2$ $(0 < x < a)$ (ii) $f(x) = 1$ $(0 < x < a/2); f(x) = 0$, $(a/2 < x < a)$.

Tutorial Sheet No. 10

Q.1. Show that

$$\begin{aligned}\sum_1^\infty \frac{1}{n}\sin nx \sin^2 n\alpha &= \text{constant} \qquad (0 < x < 2\alpha) \\ &= 0 \qquad (2\alpha < x < \pi)\end{aligned}.$$

Q.2. Prove that $\frac{1}{3} + \frac{4}{\pi}\sum_1^\infty \frac{1}{m}\sin\frac{2}{3}m\pi\cos 2m\pi x = \begin{cases} 1 & (0 < x < 1/3) \\ 1 & (2/3 < x < 1) \\ -1 & (1/3 < x < 2/3) \end{cases}$.

Q.3. Show that $\frac{1}{96}\pi(\pi - 2x)(\pi^2 + 2\pi x - 2x^2) = \sum_0^\infty \cos\frac{(2n+1)x}{(2n+1)^4} \quad (0 \leq x \leq \pi)$.

Q.4. Prove that $\sum_1^\infty \frac{(-1)^{n-1}\cos nx}{n^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \quad (-\pi \leq x \leq \pi)$.

Q.5. Show that $\displaystyle\sum_{0}^{\infty}\frac{\sin(2n+1)x}{2n+1)^3}=\frac{1}{8}\pi x(\pi-x) \quad (0\le x\le\pi).$

Q.6. Prove that $\displaystyle\sum_{0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}=\frac{\pi}{8}(\pi-2x) \quad (0<x<\pi).$

Q.7. Show that $\displaystyle\frac{1}{2}+\frac{1}{4}\cos x-\frac{\cos 2x}{1.3}-\frac{\cos 3x}{2.4}-\frac{\cos 4x}{3.5}\ldots=\frac{1}{2}(\pi-x)\sin x(0\le x\le\pi).$

Q.8. (**Fourier Theorem.**) Let $f(x)$ be a periodic function of period 2π on the real axis which is piecewise continuously differentiable. Suppose further that $\displaystyle\int_{-\pi}^{\pi}|f(x)|dx<\infty$. Let a_n and b_n be defined by the relations

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt dt \quad \text{and} \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt, \qquad n=0,1,2,\ldots$$

The series $\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$ converges to $f(x)$ if $f(t)$ is continuous at $t=x$ and converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ if $f(t)$ has a finite discontinuity at $t=x$. From the Fourier expansions given in Q.1 through Q.7 and the Fourier theorem stated above deduce the following results.

(i) $\displaystyle 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+\ldots=\frac{2\pi}{3\sqrt{3}}$ (ii) $\displaystyle 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+\ldots=\frac{\pi}{3\sqrt{3}}$

(iv) $\displaystyle 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$ (v) $\displaystyle 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$

(vi) $\displaystyle 1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-+\ldots=\frac{\pi^3}{32}$ (vii) $\displaystyle 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\ldots=\frac{\pi^2}{8}$

(viii) $\displaystyle\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots=\frac{\pi}{4}-\frac{1}{2}$

Q.9. Using Parseval's identity, prove that $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots=\frac{\pi^4}{96}$.

(Hint: Use $f(x)=\begin{cases} x, & -\pi/2<x<\pi/2 \\ \pi-x, & \pi/2<x<3\pi/2. \end{cases}$

Q.10. Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π, where

(i) $f(x)=x, \quad 0<x<2\pi.$

(ii) $f(x)=\begin{cases} -x, & -\pi\le x<0 \\ x, & 0\le x<\pi \end{cases}$

(iii) $f(x)=x+|x|, \quad -\pi<x<\pi.$

Q.11. Find the Fourier series of the periodic function $f(x)$, of period $p=2$, when

$$f(x)=\begin{cases} 0, & -1<x<0 \\ x, & 0<x<1 \end{cases}$$

Q.12. State whether the given function is even or odd. Find its Fourier series

(i) $f(x) = \begin{cases} k, & -\pi/2 < x < \pi/2 \\ \\ 0, & \pi/2 < x < 3\pi/2 \end{cases}$ (ii) $f(x) = 3x(\pi^2 - x^2), \quad -\pi < x < \pi.$

Q.13. Using Fourier Integrals, show that

(i) $\displaystyle\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2} dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$ (ii) $\displaystyle\int_0^\infty \frac{\cos xw}{1+w^2} dw = \frac{\pi}{2} e^{-x} (x > 0)$

(iii) $\displaystyle\int_0^\infty \frac{w^3 \sin xw}{w^4+4} dw = \frac{\pi}{2} e^{-x} \cos x (x > 0)$ (iv) $\displaystyle\int_0^\infty \frac{\sin w \cos xw}{w} dw = \begin{cases} \pi/2 & \text{if } \ 0 \le x < 1 \\ \pi/4 & \text{if } \quad x = 1 \\ 0 & \text{if } \quad x > 1. \end{cases}$

Q.14. Let $f(x)$ be defined over R and $\displaystyle\int_{-\infty}^{\infty} |f(x)| dx < \infty$. Further, let

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv; \ B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv$$

The Fourier integral $\displaystyle\int_0^\infty (A(w)\cos wx + B(w) \sin wx) dw$ converges to $f(x)$ if $f(t)$ is continuous at $t = x$ and converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ if $f(t)$ has a finite discontinuity at $t = x$. From the Fourier integral theorem stated above deduce the following results.

(i) $\displaystyle\int_0^\infty \frac{\sin w}{w} dw = \pi/2$ (ii) $\displaystyle\int_0^\infty \frac{\cos w}{1+w^2} dw = \frac{\pi}{2e}$ (iii) $\displaystyle\int_0^\infty \frac{w \sin w}{1+w^2} dw = \frac{\pi}{2e}$ (iv) $\displaystyle\int_0^\infty \frac{w^3 \sin \frac{\pi w}{2}}{w^4+4} dw = 0$

Q.15. If $A(w) = \displaystyle\frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$, then show that

(i) $f(ax) = \displaystyle\frac{1}{a} \int_0^\infty A(\frac{w}{a}) \cos wx \, dw$. (ii) $xf(x) = -\displaystyle\int_0^\infty \frac{dA}{dw} \sin wx \, dw$. (iii) $x^2 f(x) = -\displaystyle\int_0^\infty \frac{d^2A}{dw^2} \cos wx \, dw$.

Q.16. Find the Fourier cosine integral of $f(x) = \dfrac{1}{1+x^2}$, and the Fourier sine integral of $f(x) = \dfrac{x}{1+x^2}$. Both functions are defined on the interval $[0, \infty)$.

Tutorial Sheet No. 11

Q.1. Find the Laplace Transform of the following functions.

(i) $t \cos wt$ (ii) $t \sin wt$ (iii) $e^{-t} \sin^2 t$ (iv) $t^2 e^{-at}$ (v) $(1 + te^{-t})^3$ (vi) $(5e^{2t} - 3)^2$

(vii) $te^{-2t} \sin wt$ (viii) $t^n e^{at}$ (ix) $t^2 e^{-at} \sin bt$ (xi) $\cosh at \cos at$

Q.2. Find the inverse Laplace transforms of the following functions.

(i) $\dfrac{s^2 - w^2}{(s^2+w^2)^2}$ (ii) $\dfrac{2as}{(s^2-a^2)^2}$ (iii) $\dfrac{1}{(s^2+w^2)^2}$ (iv) $\dfrac{s^3}{(s^4+4a^4)}$ (v) $\dfrac{s-2}{s^2(s+4)^2}$ (vi) $\dfrac{1}{s^4-2s^3}$ (vii) $\dfrac{1}{s^4(s^2+\pi^2)}$

(viii) $\dfrac{s^2+a^2}{(s^2-a^2)^2}$ (ix) $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$ (x) $\dfrac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^2}$

Q.3. Solve the following intial value problems using Laplace transforms and convolutions.

(i) $y'' + y = \sin 3t;\ y(0) = y'(0) = 0$ (ii) $y'' + 3y' + 2y = e^{-t};\ y(0) = y'(0) = 0$

(iii) $y'' + 2y' - 8y = 0;\ y(0) = 1; y'(0) = 8$ (iv) $y'' + 2y' + y = 2\cos t;\ y(0) = 3, y'(0) = 0$

(v) $y'' - 2y' + 5y = 8\sin t - 4\cos t;\quad y(0) = 1; y'(0) = 3$

(vi) $y'' - 2y' - 3y = 10 \sin h2t;\quad y(0) = 0; y'(0) = 4$

Q.4. Solve the following systems of differential equations using Laplace transforms.

(i) $x' = x + y,\ y' = 4x + y$ (ii) $x' = 3x + 2y,\ y' = -5x + y$

(iii) $x'' - x + y' = y = 1,\ y'' + y + x' - x = 0$ (iv) $x' = 5x + 8y + 1,\ y' = -6x - 9y + t,\ x(0) = 4,\ y(0) = -3$

(v) $y_1' + y_2 = 2\cos t;\ y_1 + y_2' = 0; y_1(0) = 0; y_2(0) = 1$

(vi) $y_1'' + y_2 = -5\cos 2t;\ y_2'' + y_1 = 5\cos 2t;\ \ y_1(0) = 1, y_1'(0) = 1, y_2(0) = -1, y_2'(0) = 1$

(vii) $2y_1' - y_2' - y_3' = 0; y_1' + y_2' = 4t + 2;\ y_2' + y_3 = t^2 + 2, y_1(0) = y_2(0) = y_3(0) = 0$

(viii) $y_1'' = y_1 + 3y_2; y_2'' = 4y_1 - 4e^t;\ y_1(0) = 2; y_1'(0) = 3, y_2(0) = 1, y_2'(0) = 2$

Q.5. Assuming that for a Power series in $\frac{1}{s}$ with no constant term the Laplace transform can be obtained term-by-term, i.e., assuming that $\mathcal{L}^{-1}[\sum_0^\infty \frac{A_k}{s^{k+1}}] = \sum_0^\infty A_k \frac{t^k}{k!}$, where $A_0, A_1 \ldots A_k \ldots$ are real numbers, prove that

(i) $\mathcal{L}^{-1}(\frac{1}{s-1}) = e^t$ (ii) $\mathcal{L}^1(\frac{1}{s^2+1}) = \sin t$

(iii) $\mathcal{L}^1(\frac{1}{s}e^{-b/s}) = J_0(2\sqrt{bt})\ \ (b > 0)$ (iv) $\mathcal{L}^{-1}(\frac{1}{\sqrt{s^2+a^2}}) = J_0(at)\ \ (a > 0)$

(v) $\mathcal{L}^{-1}(\frac{e^{-b/s}}{\sqrt{s}}) = \frac{1}{\sqrt{\pi t}}\cos(2\sqrt{bt})\ \ (b > 0)$ (vi) $\mathcal{L}^{-1}(\tan^{-1}\frac{1}{s}) = \frac{\sin t}{t}$

Q.6. Find the Laplace transform of the following periodic functions.

(i) $f(t), f(t+p) = f(t)$ for all $t > 0$ and $f(t)$ piecewise continuous

(ii) $f(t) = |\sin wt|$

(iii) $f(t) = 1 (0 < t < \pi); f(t) = -1 (\pi < t < 2\pi);\ f(t + 2\pi) = f(t)$

(iv) $f(t) = t (0 \le t \le 1), f(t) = 2 - t (1 \le t \le 2); f(t + 2) = f(t)$

(v) $f(t) = \sin t (0 \le t \le \pi), f(t) = 0 (\pi \le t \le 2\pi); f(t + 2\pi) = f(t)$

Q.7. Find the Laplace Transform of $f(t)$ where $f(t) = n,\ n - 1 \le t \le n, n = 1, 2, 3, \ldots$

Q.8. Find $f(t)$ given $\mathcal{L}[f(t)] = (e^{-s} - e^{-2s} - e^{-3s} + e^{-4s})/s^2$

Q.9. Find the Laplace Transform of (i) $f(t) = u_\pi(t)\sin t$ (ii) $f(t) = u_1(t)e^{-2t}$ where $u_\pi(u_1)$ is the Heaviside step function.

Q.10. Find (i) $\mathcal{L}^{-1}\left[\ln \frac{s^2 + 4s + 5}{s^2 + 2s + 5}\right]$

Q.11. If $\mathcal{L}[f(t)] = F(s), \mathcal{L}[g(t)] = G(s)$ prove that $\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du$. Also show that $\mathcal{L}^{-1}[\frac{F(s)}{(s+a)^2 + a^2}] = \frac{1}{a}e^{-at}\int_0^t f(u)e^{au}\sin a(t-u)du.$

Q.12. Compute the Laplace transform of a solution of $ty'' + y' + ty = 0,\ t > 0$, satisfying $y(0) = k$, $Y(1) = 1/\sqrt{2}$, where k is a real constant and Y denotes the Laplace transform of y.

Q.13. Compute the convolution of $t^{a-1}u(t)$ and $t^{b-1}u(t)$ and use the convolution theorem to prove

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$$

where $B(a,b)$ denotes the Beta function and $\Gamma(a)$ the Gamma function. Use this to find the value of $\Gamma(1/2)$ and hence of $\displaystyle\int_{-\infty}^{\infty} \exp(-x^2)dx$.

Q.14. Suppose $f(x)$ is a function of exponential type and $\mathcal{L}f = 1/\sqrt{s^2+1}$. Determine $f * f$.

Q.15. Evaluate the following integrals by computing their Laplace transforms.

(i) $f(t) = \displaystyle\int_0^\infty \frac{\sin(tx)}{x}dx$ (ii) $f(t) = \displaystyle\int_0^\infty \frac{\cos tx}{x^2+a^2}dx$ (iii) $f(t) = \displaystyle\int_0^\infty \sin(tx^a)dx, \quad a > 1$

(iv) $\displaystyle\int_0^\infty \frac{1}{x^2}(1-\cos tx)dx$ (v) $\displaystyle\int_0^\infty \frac{\sin^4 tx}{x^3}dx$ (vi) $\displaystyle\int_0^\infty \Big(\frac{x^2-b^2}{x^2+b^2}\Big)\frac{\sin tx}{x}dx$

Q.16. Solve the following integral/integro-differential equations

(i) $y(t) = 1 - \sinh t + \displaystyle\int_0^t (1+x)y(t-x)dx$ (ii) $A = \displaystyle\int_0^t \frac{y(x)dx}{\sqrt{t-x}}$, where A is a constant.

(iii) $\dfrac{dy}{dt} = 1 - \displaystyle\int_0^t y(t-\tau)d\tau$, $y(0) = 1$.

Q.17. Find a real general solution of the following nonhomogeneous linear systems.

(i) $y_1' = y_2 + e^{3t}, \quad y_2' = y_1 - 3e^{3t}$.

(ii) $y_1' = 3y_1 + y_2 - 3\sin 3t, \quad y_2' = 7y_1 - 3y_2 + 9\cos 3t - 16\sin 3t$.

(iii) $y_1' = y_2 + 6e^{2t}, \quad y_2' = y_1 - 3e^{2t}, \quad y_1(0) = 11, \quad y_2(0) = 0$.

(iv) $y_1' = 5y_2 + 23, \quad y_2' = -5y_1 + 15t, \quad y_1(0) = 1, \quad y_2(0) = -2$.

(v) $y_1' = y_2 - 5\sin t, \quad y_2' = -4y_1 + 17\cos t, \quad y_1(0) = 5, \quad y_2(0) = 2$.

(vi) $y_1' = 5y_1 + 4y_2 - 5t^2 + 6t + 25, \quad y_2' = y_1 + 2y_2 - t^2 + 2t + 4, \quad y_1(0) = 0, \quad y_2(0) = 0$.

Q.18 Prove that the Laplace transform of $(1-e^{-t})^\nu$ is $B(s, \nu+1)$ where $B(a,b)$ is the beta function.

Q.19 Show that if $f(t) = 1/(1+t^2)$ then its Laplace transform $F(s)$ satisfies the differential equation $F'' + F = 1/s$. Deduce that $F(s) = \displaystyle\int_0^\infty \frac{\sin\lambda d\lambda}{(\lambda+s)}$.

Q.20 Show that the Laplace transform of $\log t$ is $-s^{-1}\log s - Cs^{-1}$. Identify the constant C in terms of the gamma function.

Q.21 Evaluate the integral $\displaystyle\int_0^\infty \exp\Big\{-\Big(at + \frac{b}{t}\Big)\Big\}\frac{dt}{\sqrt{t}}$ where a and b are positive. Use this result to compute the Laplace transform of $\dfrac{1}{\sqrt{t}}\exp\Big(\dfrac{-b}{t}\Big)$., $b > 0$.

Tutorial Sheet No. 12

Q.1. Determine the orbit of each of the following systems:

(i)$\dot{x} = y,\ \ \dot{y} = -x$ (ii)$\dot{x} = y(1 + x + y),\ \ \dot{y} = -x(1 + x + y)$ (iii) $\dot{x} = 2xy,\ \ \dot{y} = x^2 - y^2$

(iv) $\dot{x} = xye^{-3x},\ \ \dot{y} = -2xy^2$ (v) $\dot{x} = ax - bxy,\ \ \dot{y} = cx - dxy,\ (a, b, c, d > 0)$.

Q.2. Find the equilibrium points of the following differential equations

(i) $\dot{x}_1 = 1 - x_2,\quad \dot{x}_2 = x_1^3 + x_2.$ (ii) $\dot{x}_1 = x_1^2 + x_2 - 1,\quad \dot{x}_2 = 2x_1x_2.$

(iii) $\dot{x}_1 = (x_1 - 1)(x_2 - 1),\ \dot{x}_2 = (x_1 + 1)(x_2 + 1).$

(iv) $\ddot{y} + \dot{y} - (y^3 + y^2 - 2y) = 0.$ (v) $\dot{x}_1 = x_2^2 - 5x_1 + 6,\ \dot{x}_2 = x_1 - x_2.$

Q.3. Describe the phase portrait of each of the following system and determine the nature and stability properties of the critical points.

(i) $\dot{x_1} = -x_1,\quad \dot{x_2} = -x_2.$ (ii) $\dot{x_1} = x_1,\quad \dot{x_2} = -x_1 + 2x_2.$

(iii) $\dot{x_1} = -x_2,\quad \dot{x_2} = x_1.$ (iv) $\dot{x_1} = ax_1 - x_2,\quad \dot{x_2} = x_1 + ax_2,\quad a \neq 0.$

(v) $\dot{x_1} = -x_1 - 2x_2,\quad \dot{x_2} = 4x_1 - 5x_2.$ (vi) $\dot{x_1} = -4x_1 - x_2,\quad \dot{x_2} = x_1 - 2x_2.$

Q.4. Determine whether each solution of the system of differential equations $\dot{\mathbf{X}} = A\dot{\mathbf{X}}$, where $A =$

(i) $\begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{pmatrix}$.

is stable, asymptotically stable or unstable.

Q.5. Discuss qualitative behaviour of the equilibrium points of Q.2 (i)-(v) and (viii) $(\mu < 2,\ \mu = 2,\ \mu > 2)$.

Q.6. (Competing species) Each of the following problems (i) - (vi) can be interpreted as describing the interaction of two species with population x and y. In each of these problems carry out the following steps:

(a) Find the equilibrium or the critical points.

(b) For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system. Classify each critical point and determine whether it is asymptotically stable, stable or unstable.

(c) Determine the limiting behaviour of x and y as $t \longrightarrow \infty$ and interpret the result in terms of the populations of two species.

(i) $\dot{x} = x(1 - x - y),\quad \dot{y} = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$

(ii) $\dot{x} = x(1 - x - y),\quad \dot{y} = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right).$

(iii) $\dot{x} = x(\frac{3}{2} - \frac{1}{2}x - y),\quad \dot{y} = y(2 - y - 1.125x)$

(iv) $\dot{x} = x(1 - x + \frac{1}{2}y),\quad \dot{y} = y(2.5 - 1.5y + 0.25x)$

(v) $\dot{x} = x(1.5 - 0.5y),\quad \dot{y} = y(-0.5 + x).$

Q.7. For the following problems, carry out the steps in 6(a) and (b) and also discuss the limiting behaviour of the solution.

(i) The Predator-Prey model: $\dot{x} = ax - bxy,\quad \dot{y} = -cy + dxy$;
$(a, b, c, d,$ positive real constants).

(ii) The Lanchestrian model for conventional guerilla war:
$\dot{x} = -ay, \qquad \dot{y} = -bx - cxy.$

Handout (Improper Integrals)

In Riemann integration the range of integration is finite and the integrand is bounded in that range. It is possible, however, to so extend the theory that the symbol

$$\int_a^b f(x)\,dx \tag{1}$$

may sometimes have a meaning even when either a or b or both are infinite or $f(x)$ is not bounded. It is convenient to have a local definition of boundedness of a function. A function $f : (a,b) \mapsto \mathbb{R}$ is said to be **bounded** at a point c in (a,b) if it is bounded in some interval $(c-\delta, c+\delta)$ around c which is contained in (a,b), otherwise, it is said to be **unbounded** at c. The function f is said to be unbounded at the left end point a (respectively at the right end point b), if it is unbounded in every interval $(a, a+\delta)$ (respectively, $(b-\delta, b)$) contained in (a,b). If either a or b or both are infinite and f is bounded at each point contained in (a,b) and at a or b, whenever they are finite, then the symbol (1) is called an **improper integral of the first kind**. Some examples of improper integrals of the first kind are $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$, $\displaystyle\int_0^{\infty} e^{-x}\,dx$ and $\displaystyle\int_0^{\infty} \cos x\,dx$.

If a and b are both real, but f is unbounded at some point in $[a,b]$, then (1) is called an **improper integral of the second kind**. Examples of improper integrals of the second kind are $\displaystyle\int_0^1 \frac{\sin x}{x^{5/2}}$, $\displaystyle\int_2^3 \frac{dx}{(x-2)(x-3)}$ and $\displaystyle\int_{-1}^1 \frac{dx}{x^{1/2}}$.

If the integral of a function f is to be evaluated over an unbounded interval (a,b) and also f is unbounded at some point of $[a,b]$, then (1) is called an **improper integral of the third kind**. Some examples of improper integrals of the third kind are

(i) $\displaystyle\int_0^{\infty} \frac{dx}{x^3+x^{1/3}}$ (ii) $\displaystyle\int_0^{\infty} \frac{dx}{(1-x)}$.

Convergence of improper integrals of the first kind.

Definition. Let f be Riemann integrable in $[a,R]$ for every $R > a$. [2] Then the improper integral $\displaystyle\int_a^{\infty} f(x)dx$ of first kind is said to be **convergent** if $\displaystyle\lim_{R\to\infty}\int_a^R f(x)dx$ exists and is finite. The improper integral $\displaystyle\int_a^{\infty} f(x)dx$ is said to be **divergent**, if it is not convergent.

Example 1. Discuss the convergence of $\displaystyle\int_a^{\infty} \frac{dx}{x^p}$, $a > 0$, for all real values of p.

Case I ($p < 1$) $\displaystyle\lim_{R\to\infty}\int_a^R \frac{dx}{x^p} = \lim_{R\to\infty}\frac{R^{1-p}-a^{1-p}}{1-p} = \infty$ (divergent)

Case II ($p = 1$) $\displaystyle\lim_{R\to\infty}\int_a^R \frac{dx}{x} = \lim_{R\to\infty}(\log R - \log a) = \infty$ (divergent)

Case III ($p > 1$) $\displaystyle\lim_{R\to\infty}\int_a^R \frac{dx}{x^p} = \lim_{R\to\infty}\frac{R^{(1-p)}-a^{(1-p)}}{1-p} = -\frac{a^{(1-p)}}{1-p}$ finite (convergent).

[2] In place of this hypothesis, we may simply assume that f is continuous in the interval $a \le x < \infty$.

Example 2. Show that $\displaystyle\int_0^\infty \cos x\, dx$ is divergent.

Since $\displaystyle\lim_{R\to\infty}\int_0^R \cos x\, dx = \lim_{R\to\infty} \sin R$ does not exist, the integral is divergent.

Theorem 1 (The comparison test). Assume that
(i) $f(x)$ and $g(x)$ are Riemann integrable in $[a, R]$ for every $R > a$,
(ii) $0 \le f(x) \le g(x)$, $a \le x < \infty$ and
(iii) $\displaystyle\int_a^\infty g(x)dx$ is convergent.
Then $\displaystyle\int_a^\infty f(x)dx$ is convergent.

Proof: By hypothesis (iii), $\displaystyle\lim_{R\to\infty}\int_a^R g(x)dx = B$ say. Let $F(R) = \displaystyle\int_0^R f(x)dx$. Then $F(R)$ is monotonically increasing and by hypothesis (ii) it is bounded above since

$$0 \le F(R) = \int_a^R f(x)dx \le \int_a^R g(x)dx \le B.$$

Hence, $\displaystyle\lim_{R\to\infty}\int_a^R f(x)$ exists and is finite, so the integral $\int_a^\infty f(x)\, dx$ is convergent.

Theorem 2. Assume that
(i) f and g are Riemann integrable in $[a, R]$ for every $R > a$;
(ii) $0 \le f(x) \le g(x), a \le x < \infty$;
(iii) $\displaystyle\int_a^\infty f(x)dx$ is divergent.
Then $\displaystyle\int_a^\infty g(x)dx$ is divergent.

Proof: As in Theorem 1.

Example 3. Show that $\displaystyle\int_2^\infty \frac{x^2}{\sqrt{1+x^7}}\, dx$ is convergent.

Note that on $[2, \infty)$, $0 < \dfrac{x^2}{\sqrt{x^7+1}} < \dfrac{1}{x^{3/2}}$. But since $\displaystyle\int_2^\infty \frac{dx}{x^{3/2}}$ is convergent, so is the given integral, by the comparison test.

Example 4. Show that $\displaystyle\int_2^\infty \frac{x^3}{\sqrt{1+x^7}}dx$ is divergent.

Observe that

$$\frac{x^3}{\sqrt{x^7+1}} = \frac{1}{\sqrt{x}\sqrt{1+x^{-7}}} \ge \frac{1}{\sqrt{1+2^{-7}}}\cdot\frac{1}{\sqrt{x}}, \quad 2 \le x < \infty.$$

But $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}dx$ is divergent. Hence, by Theorem 2, the given integral is divergent.

Test for Absolute Convergence.

Definition. The integral $\int_a^\infty f(x)\,dx$ is said to be **absolutely convergent** if $\int_a^\infty |f(x)|\,dx$ is convergent.

Definition. The integral $\int_a^\infty f(x)\,dx$ is said to be **conditionally convergent** if $\int_a^\infty f(x)\,dx$ is convergent, but not absolutely convergent.

Example 5. Show that $\int_1^\infty \frac{\sin x}{x^2}dx$ is absolutely convergent.
Note that

$$0 \le \frac{|\sin x|}{x^2} \le \frac{1}{x^2}, \quad 1 \le x < \infty.$$

Since $\int_1^\infty \frac{dx}{x^2}$ is convergent by Example 1, the integral $\int_1^\infty \frac{|\sin x|}{x^2}\,dx$ is convergent by Theorem 1, so the given integral is absolutely convergent.

Theorem 3. Assume that
(i) f is Riemann integrable in $[a, R]$ for every $R > a$;
(ii) $\int_a^\infty |f(x)|\,dx$ is convergent.
Then $\int_a^\infty f(x)\,dx$ is convergent.

Proof : Since $0 \le |f(x)| + f(x) \le 2|f(x)|$, on the interval (a, ∞) the integral is convergent by Theorem 1.

$$\int_a^\infty [|f(x)| + f(x)]\,dx$$

is convergent. If we subtract from this integral the convergent integral in hypothesis (ii), we obtain the required convergence. ∎

Example 6. Show that $\int_\pi^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent.

Solution. In the interval $k\pi \le x \le (k+1)\pi$ where $k = 1, 2, \ldots$ we have

$$\frac{|\sin x|}{x} \ge \frac{|\sin x|}{(k+1)\pi}. \tag{2}$$

Hence,

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \ge \frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin x|dx = \frac{2}{(k+1)\pi}.$$

If $n\pi \le R < (n+1)\pi$

$$\int_\pi^R \frac{|\sin x|}{x}dx \ge \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{k+1}. \tag{3}$$

As $R \to \infty$, so does n and so does the right hand side of the inequality (3). This proves that $\int_\pi^\infty \frac{|\sin x|}{x}\,dx$ is divergent. Hence, the integral is not absolutely convergent. However, we show that this integral is convergent. Indeed, integrating by parts, we have

$$\int_{\pi}^{R} \frac{\sin x}{x}\, dx = \frac{1}{\pi} - \frac{\cos R}{R} + \int_{\pi}^{R} \frac{\cos x}{x^2}\, dx. \tag{4}$$

Since $|\frac{\cos x}{x^2}| \leq \frac{1}{x^2}$, by Theorem 1 and Example 1, $\int_{\pi}^{\infty} \frac{\cos x}{x^2}\, dx$ converges absolutely and hence, it converges. Letting $R \mapsto \infty$ in (4), we now conclude that the integral converges.

Limit Test for convergence.

Theorem 4. Assume that

(i) $f(x)$ is Riemann integrable in $[a, R]$ for every $R > a$;

(ii) $\lim\limits_{x\to\infty} x^p f(x) = A$ for some $p > 1$.

Then

$$\int_a^{\infty} |f(x)|\, dx \text{ is convergent.} \tag{5}$$

Proof : From hypothesis (ii), we have $\lim\limits_{x\to\infty} x^p|f(x)| = |A|$. Hence there is a number b such that

$$x^p|f(x)| \leq |A| + 1, \quad b < x < \infty.$$

Since, $\displaystyle\int_a^{\infty} \frac{dx}{x^p}$, $p > 1$ is convergent, by Theorem 1, we have $\displaystyle\int_a^{\infty} |f(x)|$ is convergent. ∎

Limit test for divergence.

Theorem 5. Assume that

(i) $f(x)$ is Riemann integrable in $[a, R]$ for every $R > a$;

(ii) $\lim\limits_{x\to\infty} x^p f(x) = A \neq 0$ (or $= \pm\infty$) for some $p \leq 1$.

Then $\displaystyle\int_a^{\infty} f(x)dx$ is divergent.

The test fails if $A = 0$.

Proof : Case I : Let $A > 0$. Then a number $b > 0$ exists such that

$$x^p f(x) > \frac{A}{2}, \qquad b \leq x < \infty. \tag{6}$$

Since $\displaystyle\int_b^{\infty} \frac{dx}{x^p}$ is divergent, by Theorem 2, $\displaystyle\int_b^{\infty} f(x)\, dx$ is divergent whence the desired conclusion. If $A = +\infty$, the argument is similar.

Case II : $A < 0$ (or $A = -\infty$). In this case the integral $\displaystyle\int_a^{\infty} [-f(x)]\, dx$ may be treated by Case I.

Next we show that the test fails when $A = 0$. Consider the integrals

$$\int_1^{\infty} \frac{dx}{x^2} \quad \text{and} \quad \int_2^{\infty} \frac{dx}{x \log x}.$$

In each case, for $p = 1$, we obtain $A = 0$. Hence, the first integral is convergent and the second integral is divergent. ∎

Illustration of Theorem 4 and Theorem 5.

Example 7. Show that $\int_0^\infty e^{-x^2}\, dx$ is convergent.

Note that

$$\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 e^{-x^2} = 0.$$

Hence, the result follows.

Example 8. $\int_0^\infty \frac{\cos x}{\sqrt{1+x^3}}\, dx$ is absolutely convergent.

Observe that

$$\lim_{x\to\infty} x^{5/4} f(x) = \lim_{x\to\infty} \frac{\cos x}{x^{1/4}\sqrt{1+x^{-3}}} = 0.$$

Therefore, the conclusion follows from Theorem 4.

Example 9. $\int_0^\infty \frac{dx}{\sqrt{1+2x^2}}$ is divergent.

Note that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{1+2x^2}} = \frac{1}{\sqrt{2}} \neq 0.$$

Therefore, the result follows from Theorem 5.

Improper Integrals of the Second Kind.

Let $f(x)$ be unbounded at a and let it be Riemann integrable in $[a+\epsilon, b]$ for all $\epsilon > 0$ such that $0 < \epsilon < b-a$. Let $F(\epsilon) = \int_{a+\epsilon}^b f(x)\, dx$.

Definition. The improper integral $\int_a^b f(x)\, dx$ is said to be **convergent**, if $\lim_{\epsilon\to 0^+} F(\epsilon) = A$, for some $A \in \mathbb{R}$. In this case A is called the value of the integral. The integral (7) is said to be **divergent**, if it is not convergent.
Likewise, if $f(x)$ is unbounded at b and it is Riemann integrable in $[a, b-\epsilon]$ for all $\epsilon > 0$ such that $0 < \epsilon < b-a$, let $F(\epsilon) = \int_a^{b-\epsilon} f(x)\, dx$.
Definition. The improper integral (7) is said to be **convergent**, if $\lim_{\epsilon\to 0^+} F(\epsilon) = A$, for some $A \in \mathbb{R}$. In this case again, A is called the value of the integral, and the integral (7) is said to be **divergent**, if it is not convergent.
If the function f is unbounded at a point c in (a, b), then the integral (7) is said to be convergent if both the integrals $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ converge according to the above definitions.

Remark. In the above definitions and the results to follow, in place of hypothesis of Riemann integrability of f, one may simply assume f to be continuous in the corresponding interval. For instance, if f is unbounded at a, we may assume f to be continuous in the interval $(a, b]$.

Example 10. Discuss the convergence of $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$, where p is real.

Case I: $-\infty < p \le 0$, the integral is proper.

Case II: $p = 1$.

$$\lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b \frac{dx}{x-a} = \lim_{\epsilon\to 0^+} [\log(b-a) - \log\epsilon] = \infty.$$

Hence, the integral is divergent.
Case III: $1 < p < \infty$

$$\lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b \frac{dx}{(x-a)^p} = \lim_{\epsilon\to 0^+} \left[\frac{(b-a)^{1-p}}{1-p} - \frac{\epsilon^{1-p}}{1-p}\right]$$

does not exist. Hence, the integral is divergent.
Case IV: $0 < p < 1$.

$$\lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b \frac{dx}{(x-a)^p} = \frac{(b-a)^{1-p}}{1-p}.$$

Hence, the integral is convergent.

Thus the integral $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$ is convergent for $-\infty < p < 1$ and divergent for $p \ge 1$.

Theorem 6. **(Comparison Test)** Assume that

(i) $f(x)$ and $g(x)$ are Riemann integrable in $[a+\epsilon, b]$ for every $\epsilon > 0$ with $0 < \epsilon < b-a$;

(ii) $0 \le f(x) \le g(x), \quad a < x \le b$;

(iii) $\displaystyle\int_a^b g(x)\,dx$ is convergent.

Then $\displaystyle\int_a^b f(x)\,dx$ is convergent.

Proof : For $\epsilon > 0$, $\displaystyle\int_{a+\epsilon}^b f(x)\,dx \le \int_{a+\epsilon}^b g(x)dx \le \int_a^b g(x)\,dx$. As $\epsilon \to 0$ the integral on the left is increasing, but remains bounded above. Consequently, it approaches a limit. ∎

Theorem 7. Assume that

(i) $f(x)$ and $g(x)$ satisfy condition (i) of Theorem 6,

(ii) $0 \le g(x) \le f(x), \quad a < x \le b$ and that

(iii) $\displaystyle\int_a^b g(x)\,dx$ is divergent

Then $\displaystyle\int_a^b f(x)\,dx$ is divergent.

Proof : This is similar, to that of Theorem 2, except for a change in the limits of integration. ∎

Theorem 8. Assume that

(i) $f(x)$ is Riemann integrable in $[a+\epsilon, b]$ for every $\epsilon > 0$ with $0 < \epsilon < b-a$;

(ii) $\displaystyle\int_a^b |f(x)|\,dx$ is convergent.

Then $\displaystyle\int_a^b f(x)\,dx$ is convergent.

Proof : It is same as in Theorem 3 except for a change in the limits of integration. ∎

Limit Tests.

Theorem 9. Assume

(i) $f(x)$ satisfies condition (i) of Theorem 8 and

(ii) $\displaystyle\lim_{x\to a_+} (x-a)^p f(x) = A, \qquad \text{for some } p, \; 0 < p < 1.$

Then $\int_a^b |f(x)|\,dx$ is convergent.

Proof : By hypothesis (ii), there is a c such that $(x-a)^p|f(x)| \leq |A|+1, \quad a < x \leq c < b$. Since $\displaystyle\int_a^b \frac{1}{(x-a)^p}dx$ is convergent for $0 < p < 1$, by Theorem 7, the integral $\displaystyle\int_a^b |f(x)|\,dx$ is convergent. ∎

Theorem 10. Assume that

(i) $f(x)$ satisfies the condition (i) of Theorem 1;

(ii) $\displaystyle\lim_{x\to a_+} (x-a)^p f(x) = A \neq 0$ (or $\pm\infty$), for some $p \geq 1$.

Then $\displaystyle\int_a^b f(x)\,dx$ is divergent. The test fails if $A = 0$

Proof : Proceed exactly as in Theorem 5. To show that the test fails when $A = 0$, consider the integrals

$$\text{(a)} \int_0^1 \frac{dx}{\sqrt{x}}dx \qquad \text{(b)} \int_0^{1/2} \frac{dx}{x\log(\frac{1}{x})}.$$

For $p = 1$, $A = 0$ in both the integrals, but (a) is convergent and (b) is divergent. ∎

Example 11. Discuss the convergence of $\displaystyle\int_0^{1/2} (\log\frac{1}{x})^\alpha\,dx$.

For $\alpha \leq 0$ the integral is proper and, for $\alpha > 0$, $\displaystyle\lim_{x\to 0^+} \sqrt{x}f(x) = 0$. Hence, by the limit test the integral is convergent.

Example 12. Discuss the convergence of $\displaystyle\int_0^1 t^{x-1}e^{-t}dt$.

Let $f(t) = t^{x-1}e^{-t}$. There are four cases to consider.

Case I $(x > 1)$ $\lim\limits_{t\to 0^+} f(t) = \lim\limits_{t\to 0^+} t^{x-1}e^{-t} = 0$, and hence the integral is proper.
Case II $(x = 1)$ $\lim\limits_{t\to 0^+} f(t) = \lim\limits_{t\to 0^+} e^{-t} = 1$, hence the integral is proper.
Case III $(0 < x < 1)$ $\lim\limits_{t\to 0+} t^{1-x}f(t) = 1$.
Case IV $(x \leq 0)$ $\lim\limits_{t\to 0+} tf(t) = 1 \neq 0$ for $x = 0$ and the limit is $+\infty$ for $x < 0$. Hence, the integral is convergent for $0 < x < 1$, proper for $x \geq 1$ and divergent for $x \leq 0$.

Example 13. Show that the **Beta Integral**

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \left(= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$$

exists for $m, n > 0$.

The integral is proper for $m \geq 1$, and $n \geq 1$. The integrand is unbounded at 0 if $m < 1$ and it is unbounded at 1 if $n < 1$.

Let $m < 1$ and $n < 1$. We take any number, say $\frac{1}{2}$, between 0 and 1 and examine the convergence of the improper integrals

$$\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx \quad \text{and} \quad \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx.$$

at 0 and 1 respectively.

Convergence at 0:

$$\lim_{x\to 0^+} x^{1-m}x^{m-1}(1-x)^{n-1} = 1.$$

Since $\displaystyle\int_0^{1/2} \frac{1}{x^{1-m}}dx$ is convergent for $1 - m < 1$ or $m > 0$, by the comparison test $\displaystyle\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at $x = 0$, for $m > 0$. The convergence of

$$\int_{1/2}^1 x^{m-1}(1-x)^{n-1}dx$$

at 1 is equivalent to the convergence of

$$-\int_{1/2}^0 (1-t)^{m-1}(t)^{n-1}dt \quad \text{(using the substitution } 1 - x = t\text{)}$$

at 0. Since,

$$\int_0^{1/2} t^{n-1}(1-t)^{m-1}dt \quad \text{is convergent if } n > 0,$$

the given integral exists if $m > 0$ and $n > 0$.

Example 14. (The Gamma integral) Show that the gamma integral $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ converges if $a > 0$.

Here the integrand is unbounded at 0, if $a < 1$. Thus, we have to examine the convergence at ∞ as well as at 0. Consider any positive number, say 1, and examine the convergence of

$$\int_0^1 f(x)dx \quad \text{and} \quad \int_1^\infty f(x)dx$$

where $f(x) = x^{a-1}e^{-x}$ at 0 and ∞, respectively.

(i) Convergence at 0: Let $a < 1$. Note that $\lim_{x\to 0} x^{1-a} \cdot x^{a-1}e^{-x} = 1$ and $\int_0^1 \frac{1}{x^{1-a}}dx$ is convergent if $1-a < 1$ or $a > 0$.

(ii) Convergence at ∞: We know that

$$e^x > x^{a+1} \quad \text{for all } a.$$

Thus $x^{a-1}e^{-x} < \frac{1}{x^2}$. But $\int_1^\infty \frac{1}{x^2}dx$ is convergent. Hence, by the comparison test, $\int_1^\infty x^{a-1}e^{-x}dx$ is convergent for all a. Thus, the integral $\int_0^\infty x^{a-1}e^{-x}dx$ is convergent for $a > 0$.

Example 15. Let us discuss the convergence of the following integrals

$$\text{(i)} \int_0^1 \frac{dx}{x^2 + x^{1/2}} \qquad (ii) \int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

The integrand in (i) is unbounded at 0 and the one in (ii) is unbounded at 1. Since $\frac{1}{x^2+x^{1/2}} \le \frac{1}{x^{1/2}}$ and

$$\lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{1}{x^{1/2}}\, dx = \lim_{\epsilon\to 0^+} \left(2 - 2\sqrt{\epsilon}\right) = 2,$$

the integral in (i) is convergent. Since

$$\lim_{\epsilon\to 0^+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \lim_{\epsilon\to 0^+} \left[\sin^{-1}(1-\epsilon) - \sin^{-1} 0\right] = \sin^{-1} 1 = \frac{\pi}{2},$$

the integral in (ii) is convergent.

Example 16. Discuss the convergence of the improper integral

$$\text{(i)} \int_0^2 \frac{x}{1-x}\, dx \qquad (ii) \int_0^\infty \frac{dx}{x^2 + \sqrt{x}} \qquad (iii) \int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

(i)Let $f(x) = \frac{x}{1-x}$. The function f is unbounded at $x = 1$. We write

$$\int_0^2 \frac{xdx}{1-x} = \int_0^1 \frac{xdx}{1-x} + \int_1^2 \frac{xdx}{1-x}.$$

Since $f(x) = \frac{1}{1-x} - 1$, the integral becomes

$$\int_0^2 f(x)dx = \int_0^1 \frac{dx}{1-x} + \int_1^2 \frac{dx}{1-x} - 2.$$

Also,
$$\lim_{\epsilon\mapsto 0^+}\int_0^{1-\epsilon}\frac{dx}{1-x}=\lim_{\epsilon\mapsto 0^+}(-\ln(\epsilon))=\infty,\quad \lim_{\epsilon\mapsto 0^+}\int_{1+\epsilon}^{2}\frac{dx}{1-x}=\lim_{\epsilon\mapsto 0^+}\ln(\epsilon)=-\infty.$$
Hence, the two integrals $\int_0^1\frac{dx}{1-x}$, $\int_1^2\frac{dx}{1-x}$ are both divergent . Therefore, the integral $\int_0^2 f(x)dx$ is also divergent.

(ii) Here the integral is an improper integral of the third kind. This can be expressed as a sum of improper integrals of first kind and second kind.
$$\int_0^\infty\frac{dx}{x^2+\sqrt{x}}=\int_0^1\frac{dx}{x^2+\sqrt{x}}+\int_1^\infty\frac{dx}{x^2+\sqrt{x}}=I_1+I_2\quad\text{(say)}$$
For $0\le x\le 1$, $\frac{1}{x^2+\sqrt{x}}\le\frac{1}{\sqrt{x}}$ and so by Theorem 9, I_1 is convergent. For $1\le x$, $\frac{1}{x^2+\sqrt{x}}\le\frac{1}{x^2}$, and again by Theorem 1 and Example 1, I_2 is convergent. Thus the given integral is convergent.

(iii) We formally write
$$\int_{-\infty}^\infty\frac{dx}{1+x^2}=\int_{-\infty}^0\frac{dx}{1+x^2}+\int_0^\infty\frac{dx}{1+x^2}.$$
Now
$$\int_0^\infty\frac{dx}{1+x^2}=\lim_{R\mapsto\infty}\int_0^R\frac{dx}{1+x^2}=\lim_{R\mapsto\infty}\left[\tan^{-1}(R)\right]=\frac{\pi}{2}.$$
Likewise, $\displaystyle\int_{-\infty}^0\frac{dx}{1+x^2}=\frac{\pi}{2}$. Hence, $\displaystyle\int_{-\infty}^\infty\frac{dx}{1+x^2}$ is convergent with value equal to π.

The integral test.

Example 17. Assume that

(i) $f(x)$ is a non-increasing function for $x\ge 1$,

(ii) $f(x)\ge 0$ for $x\ge 1$ and

(iii) f is Riemann integrable in $[1,R]$ for every $R>1$.

Then the series $\sum_{n=1}^\infty f(n)$ and the integral $\int_1^\infty f(x)dx$ are convergent or divergent together.

Proof : For $n\in\mathbb{N}$, by hypothesis 1, if $n\le x\le n+1$ then $f(n)\ge f(x)\ge f(n+1)$.Integrating from n to $n+1$, we obtain
$$\int_n^{n+1}f(n)\,dx\ge\int_n^{n+1}f(x)\,dx\ge\int_n^{n+1}f(n+1)\,dx,$$
which gives
$$f(n)\ge\int_n^{n+1}f(x)\,dx\ge f(n+1).$$
Thus, for $N\in\mathbb{N}$, We have
$$\sum_{n=1}^{N-1}f(n)\ge\int_1^N f\,dx\ge\sum_{n=1}^{N-1}f(n+1)=\sum_{k=2}^{N}f(k).$$

If $\int_1^\infty f dx$ converges to A, then by the second part of above inequality $\sum_{k=2}^N f(k) \leq \int_1^\infty f dx \leq A$, which implies that the series $\sum_{k=2}^\infty f(k)$ is convergent. On the other hand, if the series $\sum_{n=1}^\infty f(n)$ is convergent with sum S , then fix $R > 1$ and pick $N \in \mathbb{N}$ such that $N > R$. By the first part of the above inequality,

$$\int_1^R f\,dx \leq \int_1^N f\,dx \leq \sum_{n=1}^{N-1} f\,dx \leq S.$$

This proves that the integral $\int_1^\infty f(x)dx$ is convergent. The proof of the statement about divergence is similar and it is left to reader. ∎

Example 18. Discuss the convergence of the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^\alpha}$, where $\alpha > 0$.

Let $f(x) = \frac{1}{x(\ln x)^\alpha}$. Since $\alpha > 0, f(x)$ is non-negative and non-increasing for $x \geq 2$. By the integral test, the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^\alpha} = \sum_{n=2}^\infty f(n)$ and the integral $\int_2^\infty f\,dx$ converge or diverge together. Now

$$\int_2^R f\,dx = \int_2^R \frac{1}{x(\ln x)^\alpha} = \frac{1}{1-\alpha}\,(\ln x)^{1-\alpha}\,|_2^R, \quad \text{if } \alpha \neq 1.$$

For $\alpha = 1$, $\int_2^R f dx = \ln(\ln x)|_2^R$. Hence, $\lim_{R\mapsto\infty} \int_2^R f dx = \infty$, if $\alpha \leq 1$ and is equal to $= \frac{(\ln 2)^{1-\alpha}}{\alpha-1}$, if $\alpha > 1$. Thus the integral $\int_2^\infty f dx$ converges if $\alpha > 1$ and diverges if $0 \leq \alpha \leq 1$. By the integral test, the given series converges if $\alpha > 1$ and diverges if $0 \leq \alpha \leq 1$.