# *A Little Linear Algebra*

Read this before coming to class on Monday.

1. Let $A$ be an $n \times n$ real symmetric matrix. We say $A$ is positive definite if

   $$\mathbf{v}^T A\mathbf{v} > 0$$

   for each vector $\mathbf{v} \neq 0$. The following are equivalent:

   (i) $A$ is positive definite

   (ii) All eigen-values of $A$ are positive.

2. Show that if $A$ is a real symmetric matrix

   $$\inf_{\|\mathbf{v}\|=1} (\mathbf{v}^T A\mathbf{v})$$

   equals the smallest eigen-value of $A$ and the infimum is attained at an eigen-vector. Hint: Call $f(\mathbf{v}) = \mathbf{v}^t A\mathbf{v}$ over the unit sphere. The infimum exists and is attained (by compactness) at say $\mathbf{v}_0$. Then, look at the function

   $$\phi(t) = \Big(\frac{\mathbf{v}_0 + t\mathbf{e}_j}{\|\mathbf{v}_0 + t\mathbf{e}_j\|}\Big)^T A \Big(\frac{\mathbf{v}_0 + t\mathbf{e}_j}{\|\mathbf{v}_0 + t\mathbf{e}_j\|}\Big)$$

   and compute its derivative at $t = 0$.

3. Two square matrices $A$ and $B$ of size $n \times n$ are congruent if there exists a non-singular matrix $P$ of size $n \times n$ such that

   $$P^T AP = B.$$

   Show that congruence is an equivalence relation. Show that congruent matrices have the same nullity and the same rank.

4. If $A$ is a real symmetric matrix, the triple $(p, q, \nu)$ where $p$ is the number of positive eigen-values, $q$ is the number of negative eigen-values and $\nu$ is the number of zero eigen-values is called the inertia of $A$. Sylvester's Law of inertia asserts that Congruent Real Symmetric Matrices have the same inertia. So if one of them is positive definite then all matrices congruent to it also are positive definite.

5. Suppose you perform an elementary row operation and the "same" column operation on a matrix. For example on the matrix $A$ we add to first row twice the second row and after that add to the first column twice the second column to obtain $B$. Explain why $B$ is congruent to $A$. Prove that every real symmetric matrix is congruent to a diagonal matrix.

6. Discuss how to compute determine the inertia of a real symmetric matrix by performing only row and column operations (which means only rational operations and not having to solve higher degree equations or using intermediate value theorem).

## *Spectral theorem and such*

| Type of Matrix over $\mathbb{R}$ | Type of Matrix over $\mathbb{C}$ |
|---|---|
| Real Symmetric $A = A^T$ | Hermitian $A = A^*$ |
| Orthogonal $AA^T = I$ | Unitary $AA^* = I$ |
| $AA^T = A^T A$ (no special name) | $AA^* = A^* A$ (Normal Matrix) |

**Spectral Theorem (Real):** A commuting family of real symmetric matrices can be simultaneously orthogonally diagonalized.

**Spectral Theorem (Complex):** A commuting family of hermitian matrices can be simultaneously unitarily diagonalized.

**Structure of Normal Matrices:** Suppose $N$ is a $n \times n$ complex matrix, the following are equivalent:

1. $N$ is normal
2. There exists Hermitian matrices $A$ and $B$ such that $N = A + iB$ and $AB = BA$. Note that $A$ and $B$ need not be real.
3. There exists a unitary matrix $U$ such that

$$U^* N U = \text{Diagonal Matrix}$$

**Proof:** To prove (1) implies (2), simply take

$$A = \frac{1}{2}(N + N^*), \; B = \frac{1}{2i}(N - N^*)$$

and verify that $A$ and $B$ are hermitian and do the job. Note that $A$ and $B$ are NOT real !

To prove (2) implies (3) use the spectral theorem. Proof of (3) implies (1) is an easy exercise.

**Note:** The class of normal matrices contain all the others listed above and also skew symmetric and skew hermitian matrices.

We shall use these ideas to prove that the orthogonal groups $SO(n, \mathbb{R})$ are path connected in the discussion sessions.

# *Tensor Product of Vector Spaces and their basic properties*

In this course we shall only work with tensor products of finite dimensional vector spaces over the real numbers. The tensor product defined here is NOT the standard definition but for finite dimensional vector spaces (over any field) this is easily seen to be equivalent to the standard definition.

**Definition:** Let $V$ and $W$ be finite dimensional real vector spaces. The tensor product of $V$ and $W$ denoted by $V \otimes W$ is the set of all bilinear maps

$$V^* \times W^* \longrightarrow \mathbb{R}$$

**Notations:** Given vector spaces $V, W, Z$, it is convenient to have a separate notation for the set of all bilinear maps

$$V \times W \longrightarrow Z$$

and we use the notation $B(V, W; Z)$.

Further if $V = W$ and $Z = \mathbb{R}$ we shall simply write $\mathcal{F}^2(V^*)$ in place of $B(V, V, \mathbb{R})$. While we are on the subject, we denote by $\mathcal{F}^k(V^*)$ the set of all $k-$linear real valued maps $V \times V \times \cdots \times V \longrightarrow \mathbb{R}$. The notation is such that $\mathcal{F}^1(V^*) = V^*$.

$$V \otimes W = B(V^*, W^*; \mathbb{R}) \text{ and } V \otimes V = \mathcal{F}^2(V)$$

Replacing $V$ by $V^*$ and $W$ by $W^*$ we get

$$V^* \otimes W^* = B(V, W; \mathbb{R}) \text{ and } V^* \otimes V^* = \mathcal{F}^2(V^*).$$

The following result is now clear from the definition.

**Theorem:** $\text{Dim}(V \otimes W) = \text{Dim}(V^* \otimes W^*) = (\text{Dim } V)(\text{Dim } W)$.

In this course we shall mostly work with $V^* \otimes W^*$ and rarely use $V \otimes W$. The space $V \otimes W$ will be used heavily in the course on Representation theory of finite groups. Further, the spaces $V$ and $W$ would both be equal to the tangent space $T_pM$ of a manifold so that $V^* \otimes V^*$ would be the tensor product of two copies of the cotangent space at $p$.

Let us now find a basis for $V^* \otimes W^*$ in terms of bases for the individual spaces $V^*$ and $W^*$. The student must be clear about the notion of a basis for a vector space and the dual basis. The student must be clear that the double dual $V^{**}$ is naturally identified with $V$.

**Notation:** Let $V$ and $W$ be finite dimensional vector spaces and $f \in V^*$, $g \in W^*$. Then $f \otimes g$ is the bilinear map

$$f \otimes g : V \times W \longrightarrow \mathbb{R}, \quad (f \otimes g)(v, w) = f(v)g(w).$$

It is now natural to take a basis $f_1, f_2, \ldots, f_n$ of $V^*$ and a basis $g_1, g_2, \ldots, g_m$ of $W^*$ and construct the set of $mn$ elements

$$S = \{f_i \otimes g_j \ : \ 1 \leq i \leq n, \ 1 \leq j \leq m\}$$

**Theorem:** the set $S$ is linearly independent and hence forms a basis of $V^* \otimes W^*$.

**Proof:** Assume that for a set of $mn$ scalars $c_{ij}$, $1 \le i \le n,\ 1 \le j \le m$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} f_i \otimes g_j = 0 \tag{1}$$

Fix $i_0$ and $j_0$. Choose a vector $v \in V$ and a vector $w \in W$ such that

$$f_{i_0}(v) = 1, \quad g_{j_0}(w) = 1,$$

and $f_i(v) = 0$ if $i \neq i_0$, $g_j(w) = 0$ if $j \neq j_0$. Evaluate both sides of (1) at $(v, w)$ and we see that the LHS gives $c_{i_0 j_0}$ whereas the RHS gives the values 0. Thus

$$c_{i_0 j_0} = 0$$

and this must hold for each $1 \le i_0 \le n,\ 1 \le j_0 \le m$. The proof is complete.

**Exercises:** Prove that if $f, g, h \in V^*$,

$$\begin{aligned} f \otimes (g + h) &= f \otimes g + f \otimes h \\ (g + h) \otimes f &= g \otimes f + h \otimes f \end{aligned}$$

**Theorem (Associativity of the Tensor Product):** $V^* \otimes (V^* \otimes V^*)$ and $(V^* \otimes V^*) \otimes V^*$ are both naturally isomorphic to $\mathcal{F}^3(V^*)$.

**Proof:** For $f, g, h \in V^*$, let $\phi(f, g, h)$ be the trilinear map

$$(v_1, v_2, v_3) \mapsto f(v_1) g(v_2) h(v_3).$$

Consider the canonical assignment

$$f \otimes (g \otimes h) \mapsto \phi(f, g, h).$$

This will do the job. We need to check that is the map is well defined and injective. Well-definedness means that whenever

$$\sum_{i,j,k} c_{ijk} f_i \otimes (g_j \otimes h_k) = 0,$$

we must have

$$\sum_{i,j,k} c_{ijk} f_i(v_1) g_j(v_2) h_k(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in V.$$

Exercise: Do this using a basis of $V^*$ and explain why this does not violate our basic requirement that the isomorphism be canonical.

Under this identification both $f \otimes (g \otimes h)$ and $(f \otimes g) \otimes h$ represent the same tri-linear map $\phi(f, g, h)$. We see that the tensor product $V^* \otimes V^* \otimes V^*$ is unambiguously defined as the space $\mathcal{F}^3(V^*)$.

**Definition (Higher Tensor powers):** The $k-$fold tensor product $V^* \otimes V^* \otimes \cdots \otimes V^*$ is the vector space $\mathcal{F}^k(V^*)$.

## Symmetric and skew-symmetric tensors:

The set of all symmetric $k-$linear maps $V \times V \times \cdots \times V \longrightarrow \mathbb{R}$ is denoted by $\mathcal{S}^k(V^*)$ and the set of all skew-symmetric $k-$linear maps $V \times V \times \cdots \times V \longrightarrow \mathbb{R}$ is denoted by $\Lambda^k(V^*)$

**Exercises:**

1. Determine the dimensions of the spaces $\mathcal{S}^2(V^*)$ and $\Lambda^2(V^*)$.

2. Show that
$$\mathcal{F}^k(V^*) = \mathcal{S}^k(V^*) \oplus \Lambda^k(V^*)$$
provided $k = 2$ and explain why this fails when $k \geq 3$?

3. Let $K'$ be the linear span of $f \otimes g + g \otimes f$ and $K''$ be the linear span of $f \otimes g - g \otimes f$. Prove that
$$\mathcal{F}^k(V^*)/K' = \Lambda^2(V^*), \quad \mathcal{F}^k(V^*)/K'' = \mathcal{S}^2(V^*)$$
The image of $f \otimes g$ in the quotient $\mathcal{F}^k(V^*)/K'$ is denoted by $f \wedge g$. Show that $f \wedge g$ is the bilinear map
$$(v, w) \mapsto f(v)g(w) - f(w)g(v).$$
Find a basis of $\Lambda^2(V^*)$ in terms of a basis of $V^*$.

4. Suppose $f_1, f_2, f_3 \in V^*$. Define $f_1 \wedge f_2 \wedge f_3$ to be the trilinear map
$$(u_1, u_2, u_3) \mapsto \mathrm{Det}(f_i(u_j))$$
Show that the space $\Lambda^3(V^*)$ is the linear span of elements of the form $f \wedge g \wedge h$ where $f, g, h$ vary over $V^*$. Generalize this and state the corresponding theorem for $\Lambda^k(V^*)$ and specify a basis for it in terms of a basis for $V^*$.

5. Let $\phi_1, \phi_2, \ldots, \phi_n$ be a basis of $V^*$. Call an $k-$tuple $(i_1, i_2, \ldots, i_k)$ is said to be standard if $i_1 < i_2 < \cdots < i_k$ and the corresponding
$$\phi_{i_1} \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_k}$$
a standard monomial. How many such distinct monomials are there? Are they linearly independent?

6. Suppose $\theta \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ then explain how to define $\theta \wedge \eta$. First define them when $\theta$ and $\eta$ are standard monomials. At first one would obtain a non-standard monomial that needs to be "straightened out" through a certain number of transpositions of adjacent factors. For example let us consider
$$\begin{aligned}(f_2 \wedge f_5 \wedge f_6) \wedge (f_1 \wedge f_3 \wedge f_4) &= f_2 \wedge f_5 \wedge f_6 \wedge f_1 \wedge f_3 \wedge f_4 \\ &= (-1)^3 f_1 \wedge f_2 \wedge f_5 \wedge f_6 \wedge f_3 \wedge f_4\end{aligned}$$
Note that the factor $f_1$ had to move left through three transpositions. Next we need to bring $f_3$ next to $f_2$ through two transpositions. The process is referred to as the "straightening out"

process leading to a standard monomial or a monomial in which two factors are identical in which case it is zero. Extend it linearly in $\theta$ and $\eta$. Prove that

$$\theta \wedge \eta = (-1)^{kl} \eta \wedge \theta$$

We call it the wedge product of $\theta$ and $\eta$. It is by construction distributive over addition.

(a) Compute $\theta \wedge \theta$ for a simple case $\theta = f_1 \wedge f_2 + f_3 \wedge f_4$ where $f_1, f_2, f_3, f_4$ are elements of $V^*$.

(b) Consider a $2n-$dimensional space $V^*$ and let $\{p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n\}$ be a given ordered basis. Let $\eta$ be given by

$$\eta = p_1 \wedge q_1 + p_2 \wedge q_2 + \cdots + p_n \wedge q_n$$

Compute $\eta \wedge \eta \wedge \cdots \wedge \eta$ ($n$-fold wedge product). The $\eta$ above is called a *symplectic two-form.*

It is clearly of interest to know whether the wegde product is associative:

$$\theta \wedge (\eta \wedge \zeta) = (\theta \wedge \eta) \wedge \zeta.$$

The proof of this is not entirely trivial due to the straightening out involved in it.

7. Prove that a subset $\{f_1, f_2, \dots, f_k\} \subset V^*$ is linearly independent if and only if $f_1 \wedge f_2 \wedge \cdots \wedge f_k \neq 0$ as an element of $\Lambda^k(V^*)$.

8. **Effect of linear transformation:** Let $\{f_1, f_2, \dots, f_n\}$ and $\{g_1, g_2, \dots, g_n\}$ be bases of $V^*$.

(a) Let

$$\sum_{i,j} a_{ij} f_i \otimes f_j = \sum_{s,t} b_{st} g_s \otimes g_t$$

be the same element of $V^* \otimes V^*$ expressed in the two bases. Express the coefficients $b_{st}$ in terms of the coefficients $a_{ij}$.

(b) For expressing a tensor in $\Lambda^2(V^*)$ we have two options:

$$\sum_{i,j} a_{ij} f_i \wedge f_j, \qquad \text{where } a_{ij} = -a_{ji}$$

or else

$$\sum_{i<j} a_{ij} f_i \wedge f_j.$$

How do the coefficients transform under a non-singular linear transformation in each of the two cases?

(c) Discuss the situation with a symmetric tensor $f \odot g \in \mathcal{S}^2(V^*)$.

(d) Take the special case $V = T_p(M)$, $V^* = T_p^*(M)$ and the pair of bases

$$dx_1(p), \dots, dx_n(p), \quad dy_1(p), \dots, dy_n(p)$$

where $(U, x)$ and $(V, y)$ are two overlapping coordinate charts containing $p$. Write down the formulas for the transformation of components of a two tensor.

(e) Discuss the transformation of coefficients under non-singular linear transformations for elements of $\mathcal{F}^k(V^*)$, $\Lambda^k(V^*)$ and $\mathcal{S}^k(V^*)$.

9. In classical mechanics, in the context of spinning tops, one sometimes hears of the Inertia tensor which is actually a $3 \times 3$ real symmetric matrix $I$. The rotational kinetic energy is the quadratic form
$$\frac{1}{2}(\omega^T I \omega)$$
where $\omega$ is the "angular velocity vector". Is this a misleading terminology? If $V$ is a vector space, is there a way to interpret a quadratic form as a symmetric tensor of order two?

10. Let $V$ be a vector space of dimension $n$ and $\{v_1, v_2, \ldots, v_n\}$, $\{w_1, w_2, \ldots, w_n\}$ be ordered bases for $V$. Define
$$\{v_1, v_2, \ldots, v_n\} \sim \{w_1, w_2, \ldots, w_n\}$$
if the matrix transforming $\{v_1, v_2, \ldots, v_n\}$ to $\{w_1, w_2, \ldots, w_n\}$ has positive determinant. Show that this is an equivalence relation and there are two equivalence classes. A choice of one of these equivalence classes is called an orientation of $V$ and if an equivalence class is selected, we say that $V$ is an oriented vector space.

11. Prove that a choice of orientation of $V$ is equivalent to a choice of a non-zero element of $\Lambda^n(V^*)$.

**Remark:** Now let $M$ be a smooth manifold. At each point $p \in M$ we assign the vector space $\Lambda^k(T_p^*M)$. The disjoint union of all these vector spaces is denoted by
$$\Lambda^k(T^*M) = \bigcup_{p \in M} \Lambda^k(T_p^*M)$$
and we have the obvious projection map
$$\pi : \Lambda^k(T^*M) \longrightarrow M.$$
Exactly as was done for the tangent and cotangent bundle, one can topologize $\Lambda^k(T^*M)$ and infact make it a smooth manifold. A smooth section of $\pi$ is called a smooth differential $k-$form. Smoothness can also be defined directly and we can also define a smooth $k-$ form to be a family of smooth maps indexed by charts satisfying certain consistency conditions. We shall take these things up in detail in the next chapter. But we state here the theorem that is to come.

**Theorem:** For a smooth connected manifold $M$ of dimension $n$ the following are equivalent:

(1) The complement
$$\Lambda^n(T^*M) - \text{Image of the zero section}$$
is dis-connected

(2) There exists a nowhere vanishing smooth $n$-form on $M$.

(3) There is a compatible atlas $\mathcal{A}$ such that whenever $(U, \phi)$ and $(V, \psi)$ are overlapping charts, the derivative of the transition map $D(\psi \circ \phi^{-1})$ has positive determinant.

All books discuss the equivalence of conditions (2) and (3) above but most do not mention condition (1).

We shall see a couple more equivalent conditions in the algebraic topology course. A manifold that satisfies one and hence all the above conditions is said to be orientable. There are also one sided conditions namely conditions that are necessary but not sufficient and conditions that are sufficient but not necessary.

12. Let $V$ and $W$ be two vector spaces and $T : V \longrightarrow W$ be a linear transformation. Then, remembering that $\Lambda^1(V^*) = V^*$, we have the linear map

$$T^* : \Lambda^1(W^*) \longrightarrow \Lambda^1(V^*)$$

given by $f \mapsto f \circ T$ for $f \in W^*$. Take $V = W$ and $A$ be the matrix of $T$ with respect to a basis of $V$. What is the matrix of $T^*$ with respect to the dual basis? What is the relation between trace of $T$ and trace of $T^*$? Suppose we change basis of $V$, how does the matrix of $T^*$ change? What would be a basis of eigen-vectors of $T^*$ if we have a basis of eigen-vectors of $T$

13. Continuing with the notation as above, we define $T^* \otimes T^* : V^* \otimes V^* \longrightarrow V^* \otimes V^*$ to be the map given by

$$(T^* \otimes T^*)(f \otimes g) = (T^* f) \otimes (T^* g)$$

What can you say about the trace of $T^* \otimes T^*$? Generalize to $k-$fold tensor products.

14. Continuing from the previous exercises if $T : V \longrightarrow W$, define $\Lambda^2(T^*) : \Lambda^2(W^*) \longrightarrow \Lambda^2(V^*)$ via the prescription

$$\Lambda^2(T^*)(f \wedge g) = T^* f \wedge T^* g.$$

what can you say about the trace of $\Lambda^2(T^*)$? Generalize to $k-$fold tensor products.

15. Suppose $V$ is an $n-$ dimensional vector space and $\omega$ is an alternating $n-$form that is, an element of $\Lambda^n(V^*)$ and $T : V \longrightarrow V$ be a diagonalizable linear transformation. Determine $T^*(\omega)$ in terms of $\omega$ by fixing an appropriate basis for $V^*$.

16. Suppose $f, g \in V^*$ then show that $f \odot g : f \otimes g + g \otimes f \in \mathcal{S}^2(V^*)$. Take a basis for $V^*$ and write down a basis for $\mathcal{S}^2(V^*)$. If $T : V \longrightarrow V$ is a linear transformation, discuss the induced map on $\mathcal{S}^2(V^*)$.

# *Orientation of a manifold. Theorems characterizing orientability*

We begin with some point set topology. Recall from MA 403 that if $C$ is a connected subset of a topological space then the closure of $C$ is also connected. If $X$ is a topological space then the maximal connected subsets are called the connected components of $X$. This can also be obtained as follows. For $x, y \in X$, define $x \sim y$ if there is a connected subset of $X$ containing both $x$ and $y$. This is easily seen to be an equivalence relation and the equivalence classes are the connected components of $X$ (Exercise).

Since a component $C$ is a maximal connected subset and $\bar{C}$ is connected and contains $C$ we infer that

$$C = \bar{C},$$

which means the components are closed subsets of $X$. However they need not be open.

**Examples:** (i) If $X$ is the set of rationals with the usual topology then the components of $X$ are singletons.

(ii) If we take $X = \mathbb{R} - \{0\}$ then $X$ has two connected components.

**Definition (Path connected space):** A topological space $X$ is said to be path connected if given any two points $p, q \in X$ there is a path in $X$ joining $p$ and $q$. That is to say there is a continuous map $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Recall that a topological space is locally compact if each point has a neighborhood base consisting of compact neighborhoods.

**Definitions:** A topological space is said to be locally connected if each point has a neighborhood base consisting of connected neighborhoods. A topological space is said to be locally path connected if each point has a neighborhood base consisting of connected neighborhoods.

**Exercise:** Show that a connected, locally path connected space is path connected.

The set of rationals with the usual topology is neither locally compact nor locally connected. An open set in $\mathbb{R}^n$ is locally connected and locally compact. A manifold is both locally compact as well as locally connected

**Theorem:** If $X$ is locally connected then the connected components of $X$ are open.

**Proof:** Let $C$ be a connected component and $p \in C$. Then there is a neighborhood $N$ of $p$ which is connected and so $C \cup N$ is connected. Now since $C$ is a maximal connected subset of $X$ we infer that

$$C \cup N \subset C$$

which means $N \subset C$ and the proof is complete.

**Theorem:** Assume that $M$ is connected and locally path connected. Given any two points $p, q \in M$ one can construct a finite sequence of connected open sets $C_1, C_2, \ldots, C_k$ such that

(i) $p \in C_1$, $q \in C_k$

(ii) $C_j \cap C_{j+1} \neq \emptyset$, $j = 1, 2, \ldots k-1$

Figuratively speaking, we can find a chain from $p$ to $q$ with finitely many links.

Recall that we have defined an orientation of a vector space $V$ to be an equivalence class of an ordered basis. The equivalence relation was defined earlier.

**Definition (Orientatable manifold):** An orientation of a manifold is

(i) an assignment
$$p \mapsto \mu_p$$
assigning to each $p \in M$ an orientation $\mu_p$ (an equivalence class of ordered basis) of $T_pM$. The map $p \mapsto \mu_p$ must satisfy the following *coherence condition*

(ii) For each $p \in M$ there is a chart $U$ and $n-$smooth linearly independent vector fields $X_1, X_2, \ldots, X_n$ (where $n$ is the dimension of $M$) such that
$$\mu_p = [X_1(p), X_2(p), \ldots, X_n(p)], \quad \text{for all } p \in U \tag{1}$$

**Remark:** An ordered set of $n$ linearly independent vector fields on $U$ is called a local-frame on $U$.

**Theorem 1:** Let $M$ be a manifold. Then $M$ is orientable if and only if there is a compatible atlas $\mathcal{A}_0$ such that whenever $(U, \phi)$ and $(V, \psi)$ are overlapping charts in $\mathcal{A}_0$,
$$\mathrm{Det}(D(\psi \circ \phi^{-1})) > 0, \quad \text{on } \ U \cap V. \tag{2}$$

An atlas such as $\mathcal{A}_0$ is called an *orientation atlas.*

**Proof:** Let $M$ be orientable and $p \in M$. Pick a connected chart $U$ containing $p$ and a local frame $X_1(p), X_2(p), \ldots, X_n(p)$ satisfying (1). Consider the local frame
$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}.$$

If
$$\mu_p = [X_1(p), X_2(p), \ldots, X_n(p)] = \Big[\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\Big] \tag{3}$$
then retain $U$ as it is. If not then interchange the first two coordinates of $U$ and continue to denote the resulting chart $U$ and we refer to it as a *corrected chart.* Thus after "correction" equation (3) holds. We claim that The family of all such corrected charts is an orientation atlas. Well, suppose that $U$ and $V$ are overlapping corrected charts then for $p \in U \cap V$,
$$\Big[\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\Big] = \Big[\frac{\partial}{\partial y_1}\Big|_p, \frac{\partial}{\partial y_2}\Big|_p, \ldots, \frac{\partial}{\partial y_n}\Big|_p\Big]$$

both being $\mu_p$. So the matrix of transition between these two bases must have positive determinant and (2) holds.

Conversely suppose there is an orientation atlas $\mathcal{A}_0$. On $U \in \mathcal{A}_0$ we have the local frame

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$$

and declare $\mu_p$ to be the equivalence class of the above at $p$. By construction the coherence condition holds but we must check that the map $p \mapsto \mu_p$ is well-defined namely if $p$ lies in two overlapping charts $U$ and $V$ in $\mathcal{A}_0$ then

$$\left[\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\right] = \left[\frac{\partial}{\partial y_1}\Big|_p, \frac{\partial}{\partial y_2}\Big|_p, \dots, \frac{\partial}{\partial y_n}\Big|_p\right]$$

This holds because of condition (2) and the theorem is proved.

**Theorem 2:** Let $M$ be a connected smooth manifold. The following are equivalent:

(i) $M$ is orientable

(ii) The set
$$\Lambda^n(T^*M) - \text{Image of the zero section}$$
is disconnected.

**Proof:** (ii) implies (i) Suppose that

$$N = \Lambda^n(T^*M) - \text{Image of the zero section}$$

is dis-connected. Let $G_0$ be a connected component of $N$. Then since a manifold is locally connected, $G_0$ is open. The map

$$\omega \mapsto -\omega$$

is a homeomorphism of $N$ onto itself and let $G_0'$ be the image of $G_0$ under this homeomorphism. Then $G_0'$ is also a connected component whereby

$$G_0 = G_0' \quad \text{or} \quad G_0 \cap G_0' = \emptyset.$$

We shall rule out the first case by showing that it implies $G_0 = N$ and so $N$ would be connected which is false.

Take a point $\omega \in G_0$ and let $\pi(\omega) = p$. Take a connected chart $U$ in $M$ containing $p$ and $\xi_U(\omega) = (p, a)$ say and assume $a > 0$. Observe that since $\xi_U$ is a homeomorphism,

$$\xi_U^{-1}(U \times \mathbb{R}^+)$$

is connected and contains the point $\omega \in G_0$ whereby

$$G_0 \cup \xi_U^{-1}(U \times \mathbb{R}^+)$$

is connected and in view of the maximality of $G_0$,

$$\xi_U^{-1}(U \times \mathbb{R}^+) \subset G_0.$$

If $G_0 = G_0'$ then $G_0$ also contains $-\omega$ which means we also have

$$\xi_U^{-1}(U \times \mathbb{R}^-) \subset G_0,$$

and so

$$\pi^{-1}(U) \cap N = \xi_U^{-1}(U \times (\mathbb{R} - \{0\})) \subset G_0.$$

If $V$ is another connected chart of $M$ overlapping with $U$ then $\xi_V^{-1}(V \times \mathbb{R}^{\pm})$ are each connected, have non-empty intersections with $\xi_U^{-1}(U \times (\mathbb{R} - \{0\}))$ and hence non-empty intersection with $G_0$ which means

$$\pi^{-1}(V) \cap N = \xi_V^{-1}(V \times (\mathbb{R} - \{0\})) \subset G_0$$

**Exercise:** Explain how it follows that $G_0 = N$. Hint: Given ANY two charts $U$ and $W$ we can "chain up" $U$ and $V$ through finitely many links.

Next, $\xi_U$ must map $\pi^{-1}(U) \cap G_0$ into $U \times \mathbb{R}^+$ or $U \times \mathbb{R}^-$ but not both - otherwise the connected set $G_0$ meets both the connected sets $\xi_U^{-1}(U \times \mathbb{R}^{\pm})$ and so

$$\pi^{-1}(U) \cap N = \xi^{-1}(U \times \mathbb{R}^+) \cup \xi_U^{-1}(U \times \mathbb{R}^-) \subset G_0$$

So $G_0$ contains $\omega$ as well as $-\omega$ contradiction.

To construct the orientation atlas $\mathcal{A}_0$, let $U$ be a chart. If $\xi_U$ maps $\pi^{-1}(U) \cap G_0$ into $U \times \mathbb{R}^+$ then we retain $U$ as it is. In the opposite case we permute the first two coordinates of $U$ and continue to denote the resulting chart as $U$ and call it a *corrected chart*. The family $\mathcal{A}_0$ of all corrected charts covers $M$ and is the atlas we seek. To see this let $U$ and $V$ be overlapping charts in $\mathcal{A}_0$ then pick $\omega \in G_0$ such that $\omega(p) \in U\ capV$. Expressed in coordinates of $U$ and $V$,

$$\omega(p) = adx_1(p) \wedge x_2(p) \wedge \cdots \wedge x_n(p) = bdy_1(p) \wedge dy_2(p) \wedge \cdots \wedge dy_n(p).$$

Here $a > 0$ and $b > 0$ whereby

$$\mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big) > 0.$$

The proof is complete. Turning to the converse assume that the manifold is orientable and $\mathcal{A}_0$ is an orientation atlas. Let $\omega \in N$. Declare $\omega > 0$ if $\pi(\omega) = p$ and with respect to a chart $U$ containing $p$,

$$\omega = adx_1(p) \wedge dx_2(p) \wedge \cdots \wedge dx_n(p)$$

with $a > 0$. We see that there is no ambiguity in this definition since choosing a different chart containing $p$ would simply

$$\omega = a\mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big) dy_1(p) \wedge dy_2(p) \wedge \cdots \wedge dy_n(p)$$

and the coefficient is again positive. Declare $\omega < 0$ if $-\omega > 0$. Now consider

$$G_0 = \{\omega \in N \ :\ \omega > 0\}, \quad G_0' = \{\omega \in N \ :\ \omega < 0\}.$$

then $G_0$ is open (why?) and have the disconnection.

$$N = G_0 \cup G_0'.$$

The proof is complete.

**Theorem 3:** Suppose $M$ is a manifold of dimension $n$. The following are equivalent:

(i) $M$ is orientable

(ii) There is a smooth no-where vanishing $n$-form on $M$.

**Proof:** Suppose $\omega$ is a smooth nowhere vanishing $n$ form on $M$. Let $U$ be a connected chart. If

$$\omega(p)\Big(\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\Big) > 0 \tag{4}$$

then retain $U$ as it is. Otherwise interchange the first two coordinates of $U$ and continue to denote the chart as $U$ and refer to it as the *corrected chart.* We claim that the family $\mathcal{A}_0$ of such corrected charts is an orientation atlas. Well, suppose $U$ and $V$ are overlapping charts in $\mathcal{A}_0$. Then in addition to (4) we also have

$$\omega(p)\Big(\frac{\partial}{\partial y_1}\Big|_p, \frac{\partial}{\partial y_2}\Big|_p, \dots, \frac{\partial}{\partial y_n}\Big|_p\Big) > 0 \tag{5}$$

But we have

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_k}\frac{\partial}{\partial x_j} \tag{6}$$

Substituting (6) in (5), using the alternating property of $\omega$ and the positivity conditions (4) and (5) we infer

$$\mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big) > 0$$

and the proof is complete.

Turning to the converse assume that the manifold is orientable and $\mathcal{A}_0$ is an orientation atlas. We must construct a smooth nowhere vanishing $n$ form using a smooth partition of unity subordinate to $\mathcal{A}_0$. So select a family of smooth functions $\{\psi_\alpha \,:\, \alpha \in S\}$ ($S$ is some indexing set) such that

(i) $0 \leq \psi_\alpha \leq 1$ for each $\alpha \in S$

(ii) $\sum \psi_\alpha = 1$ (sum is over $S$).

(iii) The family $\{\mathrm{supp}(\psi_\alpha) \,:\, \alpha \in S\}$ is locally finite.

(iv) For each $\alpha$, the support of $\psi_\alpha$ is compact and contained in one of the open sets in $\mathcal{A}_0$.

Now pick a chart $U \in \mathcal{A}_0$ and look at

$$\omega_U = \Big({\sum_\alpha}' \psi_\alpha\Big) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = \psi_U dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \tag{7}$$

where the prime over the summation symbol indicates that the sum is over all those indices $\alpha$ such that support of $\psi_\alpha$ is contained in $U$. The sum is locally finite and the differential form as such is defined on $U$ and has support in $U$. Define it to be zero outside $U$ and we have a differential form $\omega_U$ with support in $U$. We do this for each $U \in \mathcal{A}_0$ obtaining a family of differential $n$ forms on $M$ that we shall sum and look at

$$\omega = \sum_{U \in \mathcal{A}_0} \omega_U \tag{8}$$

We show that this is nowhere vanishing.

Observe that $\psi_U$ in (7) is non-negative and remains non-negative in any overlapping coordinate chart since the condition

$$\mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big) > 0$$

holds. Also if the coefficient is positive then it remains so in any other coordinate system as well.

Let $p \in M$ be arbitrary. There is at least one $\alpha$ for which $\psi_\alpha(p) > 0$ and at least one chart $U$ containing $p$ so that $\psi_U$ in (7) is strictly positive. However in general $p$ would belong to other charts from $\mathcal{A}_0$ as well and in order to compute (8) at $p$ we must express all the relevant summands in terms of the coordinates of $U$. Thus if $V$ is one of them we must express $\omega_V(p)$ in terms of the coordinates of $U$ namely

$$\omega_V = \mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big)\omega_U = \mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big)\psi_V dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

Adding over all such terms ($V = U$ being one of them)

$$\omega(p) = (\sum_{p\in V} \mathrm{Det}\Big(\frac{\partial x_i}{\partial y_k}\Big)\psi_V)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

We see that the coefficient is strictly positive. The proof is complete.

1. Show that if $M$ and $N$ are orientable manifolds then $M \times N$ is orientable. Thus the torus $S^1 \times S^1$ is orientable.

2. Suppose $M$ is a smooth manifold with a compatible atlas consisting of two charts $U, V$ such that $U \cap V$ is connected. Show that $M$ is orientable.

3. Consider the circle $S^1$ given by $x^2 + y^2 = 1$. Take the atlas consisting of two charts
$$U = S^1 - \{(0, -1)\}, \quad V = S^1 - \{(0, 1)\}$$
On $U$ we have the smooth function $f(z) = \mathrm{Arg}(z)$ and on $V$ we have the argument function $g(z)$ taking values in $(0, 2\pi)$. At each point $p \in U$ we have the elements of $T_p^*(S^1)$ given by the equivalence class of (the germ of)
$$f(z) - f(p)$$
in the space $I/I^2$. Similarly on $V$ we have the class of (the germ of) $g(z) - g(p)$ in $I/I^2$. Does this assignment define a smooth one form on $S^1$?

4. Express the smooth one form of the previous example in terms of the charts $(x, y) \mapsto x$, $(x, y) \mapsto y$ (two each making four charts in all).

5. In the previous chapter we introduced the space of symmetric tensors $\mathcal{S}^k(V^*)$. Take $V = T_pM$, look at the case $k = 2$ and take two bases of $T_p^*M$
$$\{dx_1(p), \ldots, dx_n(p)\}, \quad \{dy_1(p), \ldots, dy_n(p)\}.$$
Express the symmetric tensor
$$\sum_{i,j} a_{ij} dx_i \odot dx_j \tag{10}$$
in terms of the basis $\{dy_i(p) \odot dy_j(p) \ : \ i, j = 1, 2, \ldots n\}$ of $\mathcal{S}^2(T_p^*M)$.

6. Consider the disjoint union
$$\mathcal{S}^k(T^*M) = \bigcup_{p \in M} \mathcal{S}^k(T_p^*)$$
with the projection map $\pi$ that maps the entire vector space $\mathcal{S}^k(T_p^*M)$ to the point $p$. A smooth section of this bundle is called a symmetric covariant tensor of order $k$. How would you precisely define the notion of smoothness? For the case $k = 2$, explain how such a tensor can be defined in terms of a family of smooth real valued functions indexed by $(U, i, j)$ where $U$ varies over the charts of an atlas and $i, j$ are indices taking values from 1 to $n$.

   **Remark:** A symmetric tensor of order two assigns to each point $p \in M$ a symmetric bilinear form on $T_pM$. If this symmetric bilinear form is positive definite then this tensor is called a metric tensor (or a Riemannian Metric) on $M$. The coefficients of this with respect to a basis (10) are usually denoted by $g_{ij}$. We shall return to this in the differential geometry course.

7. Explain how to put a topology on the space $\mathcal{S}^k(T^*M)$ and make it into a smooth manifold. For notational convenience work with the case $k = 2$.

8. Consider the first order differential equation

$$\sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j} + bu + c = 0$$

where the coefficients are smooth functions on $\mathbb{R}^n$. Explain how the principal symbol can be thought of as a differential one form and the principal part can be regarded as a vector field.

**Proof:** subject the PDE to a coordinate transformation

$$x = \psi(y)$$

where $\psi$ is a diffeomorphism. The tranformed equation reads:

$$\sum_{j,k=1}^{n} a_j \frac{\partial y_k}{\partial x_j} \frac{\partial u}{\partial y_k} + \text{l. o. t} = \sum_{k=1}^{n} b_k \frac{\partial u}{\partial y_k} + \text{l. o. t}$$

where l.o.t refer to lower order terms. Denoting by $\mathbf{a}$ and $\mathbf{b}$ the coefficients of the principal parts,

$$\mathbf{b} = [\frac{\partial y_k}{\partial x_j}]\mathbf{a}$$

Thus the principal part is a contravariant tensor of rank one. This is not surprising since the principal part may be identified with the operator

$$\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}$$

and we know that such operators are Vector Fields. Now the principal symbol is the linear form

$$\sum_{j=1}^{n} a_j(x)\xi_j,$$

or in other words a linear map $\mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\xi \mapsto \sum_{j=1}^{n} a_j(x)\xi_j = \langle \mathbf{a}, \xi \rangle.$$

Now for the transformed equation the linear form is

$$\eta \mapsto \sum_{k=1}^{n} b_k(y)\eta_j = \langle \mathbf{b}, \eta \rangle = \langle [\frac{\partial y_k}{\partial x_j}]^{-1}\mathbf{a}, \eta \rangle = \langle \mathbf{a}, {}^T[\frac{\partial y_k}{\partial x_j}]^{-1}\eta \rangle$$

Thus the principal symbol transforms as a covariant tensor of rank one.

9. Consider a second order differential operator and show that the principal part and principal symbol behave like a symmetric tensor of order two - covariant or contravariant? What do you think would be the case for the principal part/symbol of a PDE of order $k$?

**Densities on a smooth manifold:** To define the integral of a $k$ on a $k$ dimensional manifold it is essential that the manifold be orientable (we work with an orientation atlas). What can one do with a non-orientable manifold of dimension $k$? The objects that can be integrated on ANY $k$ dimensional manifold are smooth densities. To understand what is a smooth density recall the material on integration we covered before midsem.

10. Let $\Sigma$ be a parametrized surface in $\mathbb{R}^3$ given as the image of a smooth injective map

$$G : R \longrightarrow \mathbb{R}^3$$

where $R$ is open in $\mathbb{R}^2$ and $DG$ has rank two throughout $R$ and image of $R$ under $G$ is $\Sigma$. If $f : \Sigma \longrightarrow \mathbb{R}$ is continuous and bounded we define

$$\int_\Sigma f dS = \int_R f(G(u,v))\|G_u \times G_v\| dudv$$

Suppose we reparametrize the surface via a diffeomorphism $\phi : R' \longrightarrow R$ and use $G' = G \circ R'$. What happens to the above formula. Show that that the expressions

$$\|G_u \times G_v\|, \quad dudv \text{ and } \|G_u \times G_v\| dudv \tag{11}$$

behaves almost like a differential two form. What is the difference? We call them smooth densities[1] on $\Sigma$ (of different orders)

11. What would be the analogue of (11) for a $k$ dimensional manifold in $\mathbb{R}^n$? We have looked at important examples of entities that are susceptible to generalization leading to the important notion of densities. We have no time for any further discussion on densities.

12. Consider the one form in $\mathbb{R}^2 - \{0\}$ given by

$$\omega = \frac{ydx - xdy}{x^2 + y^2}$$

Compute $d\omega$. Is there a smooth zero form $f$ such that $df = \omega$? That is to say is $\omega$ exact?

13. Consider the $n-1$ form on $S^{n-1}$ defined locally in terms of charts as:

$$\omega = x_1 dx_2 \wedge \cdots \wedge dx_n - x_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \cdots + (-1)^{n-1} x_n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}.$$

Here it is understood that we have an atlas of $2n$ charts. On the northern and southern hemispheres we need to regard $x_n = \pm\sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}$ as a zero form and $dx_n$ its exterior derivative. Carry out this computation and express it in terms of $dx_1 \wedge \ldots dx_{n-1}$. Show that this form is always positive throughout $S^{n-1}$.

14. A smooth one form $\omega$ is said to admit an integrating factor if there is a smooth function $f$ such that $f\omega$ is in the image of $d : \Omega^0(M) \longrightarrow \Omega^1(M)$. That is if there is a $f \in C^\infty(M)$ such that $f\omega$ is exact. Show that the necessary condition for $\omega$ to be exact is

$$\omega \wedge d\omega = 0.$$

[1]Referred to as relative tensors in L. P. Eisenhart, Introduction to differential Geometry, Princeton University Press, p.

# *Differential Forms and the Exterior Derivative*

In low dimensions at least, differential forms made their appearance in analysis more than three centuries ago originating in the works of Euler, Lagrange, Clairut and others. For arbitrary orders they were introduced by Poincaré and E. Cartan. The interesting historical development is available in the papers of Samelson [8], Katz [6], [5] and a fragment of a paper of J. Dieudonne [3]. A significant role was played by Pfaff (thesis advisor of C. F. Gauss), Jacobi and many other mathematicians. The final form in which this topic is currently is the culmination of efforts a few centuries.

**The Exterior Derivative** Recall that we defined the important concept of Lie brackets on the family of smooth vector fields on $M$. We shall introduce a related notion on differential forms called the exterior derivative. The exterior derivative is one of the most important ideas in the theory of differential manifolds leading directly to the de Rham cohomology of manifolds - that is to algebraic topology ! It was used by Cartan in his formulation of differential geometry via moving frames (see [1]).

The exterior derivative plays a role dual to the one played by Lie bracket in the context of vector fields. In fact the Poincare lemma

$$d^2 = 0$$

is the analogue of Jacobi's identity.

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \quad X, Y, Z \in \mathfrak{X}(M)$$

The exterior derivative of $k$-forms as defined in Hicks [//] is completely coordinate free and described as a $k+1$ linear form acting on the $C^\infty(M)$ module $\mathfrak{X}(M)$. The defining formula (stated here for simplicity only for $k = 2$) as used by Hicks is:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Though the ultimate goal has been reached, the treatment in Hicks is somewhat austere. This formula is also available in Chern et al.,[2] as well as Lee [7]. The proofs in [2] and [7] on the existence and uniqueness of the exterior differential operator (satisfying certain conditions) employs a mixture of local and global arguments (see also [9]). Although the construction of the exterior derivative carried out in [2] or [7] (through its characterization in terms of its properties) is elegant we feel it is unsuitable for the present audience and masks certain features that we would like to see. Specifically the exterior derivative is an example of a covariant derivative resulting in a tensor. Thus when changing coordiantes, certain undesirable terms ought to cancel out and we would like to see this happen explictly.

We show that the essential ingredient needed for defining the exterior derivative is a basic calculus identity involving determinants. This is not surprising and seems more natural than the other standard proofs inasmuch as when changing coordinates we actually witness the internal cancellations of terms involving the second derivatives of the transition maps.

In the following lemma $J$ would denote an ordered $k-$set $\{j_1, j_2, \ldots j_k\}$, $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. We shall consider pairs $(s, J)$ such that $1 \leq s \leq n$ and $s \neq j_1, j_2, \ldots, j_k$. Let $N$ be the total number of such pairs. For a given pair $(s, J)$ and $j_p \in J$ denote by $(j_p, J')$ the complementary pair $(j_p, J')$ obtained by removing $j_p$ from $J$ and inserting $s$ in the right place.

**Lemma:** Let $\phi_1, \phi_2, \dots, \phi_k$ be $k$ smooth functions of $z_1, z_2, \dots, z_n$. Then

$$\sum_{s,J} \frac{\partial}{\partial z_s} \frac{\partial(\phi_1, \phi_2, \dots, \phi_k)}{\partial(z_{j_1}, z_{j_2}, \dots, z_{j_k})} dz_s \wedge dz_{j_1} \wedge \dots \wedge dz_{j_k} = 0.$$

**Proof:** Assume $j_{q-1} < s < j_q$ so that $dz_s$ would need to move through $q-1$ transpositions to bring the monomial in standard form leading to a factor of $(-1)^{q-1}$. Also carrying out the indicated differentiation would produce $k$ determinats out of each summand leading to $kN$ monomials in all. We need to show that the monomials can be paired off in such a way that the sum is ultimately zero.

Let us consider the terms coming from the complementary pairs $(s, J)$ and $(j_p, J')$. We may assume at the outset that $s < j_p$ for in the opposite case we can interchange the roles of $(s, J)$ and $(j_p, J')$. The determinant will be written in such a way that the second derivatives appear in the first column which necessitates a book-keeping of the number of column exchanges.

Computing the derivative of the determinant, the term wherein the $p$th column is differentiated is:

$$(-1)^{p-1} \begin{vmatrix} \frac{\partial^2 \phi_1}{\partial z_s \partial z_{j_p}} & \frac{\partial \phi_1}{\partial z_{j_1}} & \dots & \frac{\partial \phi_1}{\partial z_{j_k}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial^2 \phi_1}{\partial z_s \partial z_{j_p}} & \frac{\partial \phi_1}{\partial z_{j_1}} & \dots & \frac{\partial \phi_1}{\partial z_{j_k}} \end{vmatrix}$$

Together with the differentials, we get the monomial:

$$(-1)^{p+q-2} \begin{vmatrix} \frac{\partial^2 \phi_1}{\partial z_s \partial z_{j_p}} & \frac{\partial \phi_1}{\partial z_{j_1}} & \dots & \frac{\partial \phi_1}{\partial z_{j_k}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial^2 \phi_1}{\partial z_s \partial z_{j_p}} & \frac{\partial \phi_1}{\partial z_{j_1}} & \dots & \frac{\partial \phi_1}{\partial z_{j_k}} \end{vmatrix} dz_{j_1} \wedge \dots \wedge dz_{j_{q-1}} \wedge dz_s \wedge dz_{j_q} \wedge \dots \wedge dz_{j_k} \qquad (*)$$

Now we consider the term arising out of the complementary pair $(j_p, J')$ and look at the relevant monomial namely the one in which the second derivatives

$$\frac{\partial^2 \phi_i}{\partial z_{j_p} \partial z_s}, \quad i = 1, 2, \dots, k$$

appear in the determinant. These second derivatives appear in the $q$th column $(j_{q-1} < s < j_q)$ and so we need $q-1$ column exchanges to bring them to the first column thereby producing a $(-1)^{q-1}$ sign. We also have in addition the factor

$$dz_{j_p} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_{q-1}} \wedge dz_s \wedge dz_{j_q} \wedge \dots \wedge dz_{p-1} \wedge dz_{p+1} \wedge \dots \wedge dz_{j_k}, \quad (s < j_p).$$

Owing to the presence of $dz_s$, the $dz_{j_p}$ has to now move through $p$ transpositions to get this in standard form. Thus we get the term (*) but with $(-1)^{p+q-1}$ instead. Thus the terms arising from complementary pairs cancel out. The proof is complete.

**Definition (The exterior derivative $d$):** We introduce the $\mathbb{R}$-linear map

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M).$$

Let $\omega$ be a differential $k$ form and on the chart $U$ let $\omega$ be given by

$$\omega = \sum_i a^U_{i_1 i_2 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

We define on each chart

$$d\omega = \sum_i da^U_{i_1 i_2 \dots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \qquad (**)$$

The notation $\sum_i$ stands for the sum over all standard $k-$tuples $(i_1, i_2, \dots, i_k)$ with $i_1 < i_2 < \cdots < i_k$. However we now have the job of verifying consistency on overlapping charts.

**Theorem:** The operator $d$ given by (**) is a well-defined element of $\Omega^{k+1}(M)$.

**Proof:** Let $U$ and $V$ be two overlapping charts. Need to check that

$$\sum_i da^U_{i_1 i_2 \dots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_j da^V_{j_1 j_2 \dots j_k} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k}, \quad \text{on } U \cap V.$$

Since

$$a^V_{j_1 j_2 \dots j_k} = \sum_i a^U_{i_1 i_2 \dots i_k} \frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}$$

our job is to check that

$$\sum_i da^U_{i_1 i_2 \dots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_i \sum_j d\Big(a^U_{i_1 i_2 \dots i_k} \frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}\Big) \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k}$$

Well, the right hand side breaks up into two sums:

$$\sum_i \sum_j d\Big(a^U_{i_1 i_2 \dots i_k} \frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}\Big) \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k} = I + II.$$

The first sum $I$ displayed below is tensorial namely,

$$\sum_i \sum_j da^U_{i_1 i_2 \dots i_k} \Big(\frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}\Big) \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k} = \sum_i da^U_{i_1 i_2 \dots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

which is the desired result and so we must show that the second (non-tensorial) term $II$ is identically zero namely,

$$II = \sum_i \sum_j a^U_{i_1 i_2 \dots i_k} d\Big(\frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}\Big) \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k} = 0.$$

We shall in fact show that each of the pieces

$$\sum_j d\Big(\frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}\Big) \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k}$$

individually vanishes. That is to say for each fixed $i_1 < i_2 < \cdots < i_k$ we have

$$\sum_j \frac{\partial}{\partial y_s}\Big(\frac{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}{\partial(y_{j_1}, y_{j_2}, \dots, y_{j_k})}\Big) dy_s \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k} = 0.$$

But this exactly the calculus lemma. The proof is complete.

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# *X - Manifolds with boundary*

We now introduce a notion we need for the formulation of Stokes' theorem. Observe that the closed ball

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

is NOT a manifold in the sense we have defined since the points on the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ do not have a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. The closed ball $B$ is a *manifold with boundary.*

To define these we introduce the closed half space

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\} \tag{10.1}$$

and its boundary

$$\partial H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n = 0\} \tag{10.2}$$

A point $p = (p_1, \dots, p_n) \in H$ with $x_n = 0$, that is $p \in \partial H$, has a neighborhood $N_p$ which is the union of the open *half ball* centered at $p$ together with the *flat piece* included (hereafter referred to as the base of $N_p$):

$$\partial N_p = \{(p_1, \dots, p_{n-1}, 0) \ : \ p_1^2 + p_2^2 + \cdots + p_{n-1}^2 < \epsilon^2\}$$

where $\epsilon$ is the radius. Thus $\partial N_p \subset N_p$. Observe that $\partial N_p$ is NOT the boundary of $N_p$ in the sense of general topology (why?). We shall refer to such a $N_p$ as a *half open ball* or more often simply a *half-ball.*

Pictures

Let $U_p$ denote the full open ball centered at $p$ and radius $\epsilon$.

**Exercise:** Show that $U_p$ is not homeomorphic to $N_p$. Recall the statement of the Brouwer's theorem on invariance of domain from chapter 3.

**Definition 10.1:** A map $f : N_p \longrightarrow \mathbb{R}^m$ on the half closed open ball is said to be smooth if there exists a smooth map $F : U_p \longrightarrow \mathbb{R}^m$ such that $F = f$ on the half closed ball $N_p$. That is to say $F$ is said to be smooth if it has a smooth extension to the full open ball $U_p$. If there is an extension $F$ which is a diffeomorphism we say that $f$ is a diffeomorphism of $N_p$ onto its image.

**Exercise:** Suppose $F, G$ are two smooth extensions of $f$ such that

$$F\Big|_{N_P} = G\Big|_{N_p}$$

then at any point $q$ on the base of $N_p$,

$$DF(q) = DG(q)$$

and we may take this to be the derivative of $f$ at $p$. Thus

$$Df(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is a linear transformation on $\mathbb{R}^n$.

**Definition 10.2 (Manifolds with boundary):** A manifold with boundary is a metric space $M$ with the following data:

(i) To each point $p \in M$ there is a neighborhood $G_p$ of $p$ in $M$ and a homeomorphism

$$\phi : G_p \longrightarrow G$$

onto a full open ball $G$ with center $\phi(p)$ or a half-ball $G$ with center $\phi(p)$. If the ball or half-ball has dimension $n$ then we say $M$ has dimension $n$.

(ii) Further, whenever we have two such homeomorphisms

$$\phi : G_p \longrightarrow G, \quad \psi : G'_p \longrightarrow G'$$

the transitions maps

$$\psi \circ \phi^{-1} : \phi(G_p \cap G'_p) \longrightarrow \psi(G_p \cap G'_p)$$

is smooth diffeomorphism.

The collection of all pairs $(G_p, \phi)$ satisfying the two stated conditions is called an atlas and its members are called charts.

**Theorem 10.1:** Suppose there are two homeomorphisms as above. Then both $G$ and $G'$ are full open balls with centers $\phi(p)$ and $\psi(p)$ respectively or they are both half-balls with these centres.

**Proof:** Suppose that $G$ is a full open ball and $G'$ is a half ball. Then $\phi(p)$ is the interior point of a full open ball in $\mathbb{R}^n$ whereas $\psi(p)$ is in the base of a half-ball. Let us write the components of the diffeomorphism $\psi \circ \phi^{-1}$:

$$\psi \circ \phi^{-1} = (\eta_1, \eta_2, \dots, \eta_n)$$

Since $\eta_n \geq 0$ and equals zero at $\phi(p)$ we see that $\eta_n$ has a local minimum at $\phi(p)$ and so all its partials must vanish at $\phi(p)$. Thus

$$D(\psi \circ \phi^{-1})$$

fails to be invertible at $\phi(p)$ which is a contradiction.

**Remarks:** (i) We are proving this result using the differentiability of the transition maps which is the case of interest here. We are avoiding the use of Brouwer's theorem on invariance of domain which is why we refrain from proving it in the purely topological setting.

(ii) The above theorem also makes the following definition meaningful.

**Definition 10.3 (Boundary of a manifold):** Suppose $M$ is a smooth manifold with boundary, boundary of $M$ is the subspace $\partial M$ consisting of all points $p \in M$ such that there is a chart $(G_p, \phi)$ such that $\phi(G_p)$ is a half-ball.

**Example:** (i) As per this definition, $H$ is a manifold with boundary and the boundary $\partial H$ is

$$\{(x_1, x_2, \ldots, x_{n-1}, 0) \ : \ x_1, \ldots, x_{n-1} \in \mathbb{R}\}.$$

(ii) Let $B$ be the closed unit ball in $\mathbb{R}^n$. Then the boundary $\partial B$ is the sphere $S^{n-1}$.
(iii) Can you find an example of a manifold $M$ with boundary such that $\partial M$ is not the topological boundary of $M$. That is to say $\partial M$ is not the boundary of $M$ in the sense of general topology.

**Theorem 10.2:** Suppose $M$ is a smooth manifold of dimension $n$ with boundary then $\partial M$ is a smooth manifold of dimension $n-1$ without boundary.

**Proof:** Exercise. Hint: Take a point $p \in \partial M$ and take a chart $(U, \phi)$ containing $p$. Look at

$$\phi\Big|_{U \cap \partial M} : U \cap \partial M \longrightarrow H.$$

**An Important Example:** Consider the half ball

$$B^+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1^2 + x_2^2 + \cdots + x_2^2 \leq 1, \ x_n \geq 0\} \tag{10.3}$$

The points on $S^{n-1} \cap \{x_n = 0\}$ form a sharp edge of $B^+$ and we shall remove these points and consider

$$B_0^+ = B^+ - S^{n-1} \cap \{x_n = 0\} \tag{10.4}$$

The base of this half ball is

$$\{(x_1, x_2, \ldots, x_{n-1}, 0) \ : \ x_1^2 + x_2^2 + \cdots + x_{n-1}^2 < 1\}$$

We must exhibit charts around points $p$ of $S^{n-1}$ with $x_n > 0$ that map the curved part to the flat piece $x_n = 0$. That such charts exist is intuitively clear but honesty demands that we work out at least one example in detail by explicitly exhibiting the charts. For this purpose consider the family of hyper-surfaces

$$(x_1, x_2, \ldots, x_{n-1}) \mapsto (x_1, x_2, \ldots, x_{n-1}, (1 - x_n)\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}) \tag{10.5}$$

This map depends on $x_n$ and so must denote it by $f_{x_n}(x_1, x_2, \ldots, x_{n-1})$. Let $D$ be the cylinder

$$D = \{(x_1, x_2, \ldots, x_n) \ : \ x_1^2 + \cdots + x_{n-1}^2 < 1, 0 \leq x_n \leq 1\}. \tag{10.6}$$

Thus we can consider the map

$$D \longrightarrow B_0^+, \quad (x_1, x_2, \ldots, x_n) \mapsto f_{x_n}(x_1, \ldots, x_{n-1}) \tag{10.7}$$

$\phi : B_0^+ \longrightarrow D$ be the inverse map.

**Exercises:**

1. Check that $\phi^{-1}$ given by (10.7) is continuous, bijective and open mapping. Thus $\phi$ is a homeomorphism providing us the desired chart. Also $\phi$ maps the curved part of $\partial B_0^+$ onto a portion of $x_n = 0$.

2. Show that the paraboloid
$$\{(x, y, z) \in \mathbb{R}^3 \ : \ z \geq x^2 + y^2\}$$
is a manifold with boundary.

3. Show that the closed unit disc in $\mathbb{R}^2$ is a manifold with boundary. Use complex analysis (Möbuis maps) to construct an atlas with two charts.

**Definition 10.4 (The tangent space at a point):** Suppose $M$ is a manifold with boundary and $p \in M - \partial M$. then $T_pM$ has the usual meaning. If $p \in \partial M$ then $T_p(\partial M)$ has the usual meaning since $\partial M$ is a manifold without boundary of dimension $n - 1$. However if $p \in \partial M$ then we shall define $T_pM$ as a $n-$ dimensional vector space containing $T_p(\partial M)$ as a vector subspace.

If $p \in \partial M$ a curve through $p$ is a smooth map
$$\gamma : [0, \epsilon) \longrightarrow M$$
such that $\gamma(0) = p$, meaning $\phi \circ \gamma : [0, \epsilon) \longrightarrow H$ is smooth. Recall that this means $\phi \circ \gamma$ has a smooth extension to $(-\epsilon, \epsilon)$. As observed in the beginning of this chapter we can take any extension, differentiate it at the origin and declare it as $(\phi \circ \gamma)^{\cdot}(0)$. A tangent vector is then an equivalence class under the equivalence relation
$$\gamma_1 \sim \gamma_2 \ \text{ if } \ \gamma_1(0) = p = \gamma_2(0) \ \text{ and } \ (\phi \circ \gamma_1)^{\cdot}(0) = (\phi \circ \gamma_2)^{\cdot}(0)$$
$T_pM$ is then the equivalence class of all curves through $p$ under the above equivalence relation. As before $T_pM$ is a vector space but this is best seen via the physicist's definition. One establishes a bijective correspondence between the physicist's and geometer's definition and simply transfers the algebraic operations via this. The physicist's definition and algebraist's definition go through easily.

Note that the dimension of $T_pM$ is $n$ and not $n - 1$ since the curve in general originates at $p$ but need not be tangential to $\partial M$.

**Exercise:** Discuss scalar multiplication with respect to negative scalars.

Observe that if the image of $\gamma$ lies completely inside $\partial M$ then the equivalence class of $\gamma$ lies in $T_p(\partial M)$ so that
$$T_p(\partial M) \subset T_pM.$$
Despite the fact that $p \in \partial M$ and the chart maps into a half ball, the derivations are as usual
$$\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p$$
and they operate on smooth functions $f \circ \phi^{-1}$ defined on half balls but then they have smooth extensions to a full ball as per our definition of differentiability. After all we compute derivatives at $p$ and they are independent of the choice of the extension.

**Definition 10.5 (The cotangent space $T_p^*M$ and differential forms):** This is the dual of $T_pM$ and its exterior powers can be defined. Differential forms are smooth sections of the exterior powers. A differential $k$-form is then locally given by

$$\omega = \sum_i a^U_{i_1 i_2 \dots i_k} dx_{i_1} \wedge \dots dx_{i_k}$$

The coefficients are smooth functions on the chart $U$. If $U$ maps to a half ball then the coefficients are smooth functions on half ball and hence with a smooth extension on the full ball.

**Exercise:** Discuss whether the exterior derivative is well defined? Will there be issues such as dependence on the extension chosen? discuss the notion of pull back via a smooth function $F : M \longrightarrow M$.

**Definition 10.6 (Orientability):** A manifold with boundary is said to be orientable if there exists a compatible atlas $\mathcal{A}$ such that for $(U, \phi), (V, \psi) \in \mathcal{A}$,

$$\operatorname{Det}(D(\psi \circ \phi^{-1})) > 0, \quad U \cap V.$$

**Exercise:** Is this well-defined? For example will the Jacobian of the transition map $\psi \circ \phi^{-1}$ depend on the choice of the smooth extension in case the charts map onto half balls?

As in the case of manifolds without boundary, we can easily show that orientability is equivalent to the existence of a smooth nowhere vanishing $n-$form.

**Lemma 10.3:** Suppose $\Phi : H \longrightarrow H$ is a diffeomorphisn with everywhere positive Jacobian then $\Phi$ induces a map

$$\tilde{\Phi} : \partial H \longrightarrow \partial H \tag{10.8}$$

which also has positive Jacobian.

**Proof:** Let

$$x_j = \Phi_j(y_1, y_2, \dots, y_n), \quad j = 1, 2, \dots, n. \tag{10.9}$$

Observe that $x_n = 0$ whenever $y_n = 0$ that is to say

$$\Phi_n(y_1, y_2, \dots, y_{n-1}, 0) = 0. \tag{10.10}$$

Thus at each point of $\partial H$,

$$\frac{\partial \Phi_n}{\partial y_j}(y_1, y_2, \dots, y_{n-1}, 0) = 0, \quad j = 1, 2, \dots, n-1. \tag{10.11}$$

The Jacobian of $\Phi$ at $(y_1, y_2, \dots, y_{n-1}, 0)$ is then

$$\begin{vmatrix} \frac{\partial \Phi_1}{\partial y_1} & \frac{\partial \Phi_1}{\partial y_2} & \dots & \frac{\partial \Phi_1}{\partial y_{n-1}} & \frac{\partial \Phi_1}{\partial y_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial \Phi_{n-1}}{\partial y_1} & \frac{\partial \Phi_{n-1}}{\partial y_2} & \dots & \frac{\partial \Phi_{n-1}}{\partial y_{n-1}} & \frac{\partial \Phi_{n-1}}{\partial y_n} \\ 0 & 0 & \dots & 0 & \frac{\partial \Phi_n}{\partial y_n} \end{vmatrix} \tag{10.12}$$

Now (10.12) is given to be positive and if we show that

$$\frac{\partial \Phi_n}{\partial y_n} > 0 \tag{10.13}$$

our job will be over because its cofactor which is the Jacobian of $\tilde{\Phi}$ will also be positive. Suppose that

$$\frac{\partial \Phi_n}{\partial y_n} < 0$$

then for any fixed positive values of $y_1, y_2, \ldots, y_{n-1}$, the function

$$y_n \mapsto \Phi_n(y_1, y_2, \ldots, y_n)$$

is strictly decreasing at $y_n = 0$ and $\Phi_n(y_1, y_2, \ldots, 0) = 0$. So for small positive values of $y_n$ and the same values of the other arguments,

$$\Phi_n(y_1, y_2, \ldots, y_n) < 0$$

which is not possible since $\Phi$ maps $H$ into $H$. The proof is complete.

**Corollary 10.4:** If $M$ is orientable then $\partial M$ is also orientable.

**Proof:** We take an orientation atlas $\mathcal{A}$ for $M$ and for each chart $(U, \phi) \in \mathcal{A}$ containing points of $\partial M$, consider the chart $(U_0, \phi_0)$ obtained by restricting $\phi$ to $U_0 = U \cap \partial M$. The theorem shows that the collection of all such $(U_0, \phi_0)$ forms an orientation atlas for $\partial M$. We shall call this orientation on $\partial M$ the *orientation obtained from $M$ through restriction to $\partial M$ or simply the restricted orientation for short.* Equivalently

**Integrating a differential $n$ form on a manifold of dimension $n$.** Let us begin with a simple example of integrating

$$\omega = dx_1 \wedge d_2 \wedge dx_n$$

say on a single bounded open set $G$ in $\mathbb{R}^n$.

We may try to define it as

$$\int_G \omega = \int_G dx_1 dx_2 \ldots dx_n.$$

Here we are tacitly working with the single coordinate chart $G$ with the identity map on it. However if we switch two of the coordinates of $G$ and get a chart $G'$, the differential form with respect to $G'$ would change sign and therewith value of the integral also reverses sign. Here one may argue that $G$ with the identity map is the preferred chart but on a general manifold there is no *preferred chart.*

Suppose that we have an orientable manifold $M$ of dimension $n$ and $\mathcal{A}$ is an orientation atlas. Let $\omega$ be a differential $n$ form with compact support inside a chart $U \in \mathcal{A}$. On $U$, we express $\omega$ as

$$\omega = a dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \tag{10.14}$$

and declare

$$\int_M \omega = \int_U a dx_1 dx_2 \ldots dx_n \tag{10.15}$$

Suppose that the form $\omega$ has support inside $U \cap V$ where $U, V$ are two overlapping charts. On $V$ we express $\omega$ as

$$\omega = b dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n \tag{10.16}$$

and we must now ensure that the value (10.15) agrees with

$$\int_M \omega = \int_V b dy_1 dy_2 \dots dy_n \tag{10.17}$$

Recall that

$$b = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} a$$

and since the Jacobian factor in the above equation is positive the change of variables formula proved earlier in the course confirms that the values of the integrals in (10.17) and (10.15) agree and the integral is well defined for forms supported inside a chart belonging to an orientation atlas.

Thus the issue of integrating a differential $n-$form on an orientable manifold of dimension $n$ seems tractable once we fix an orientation atlas on $M$. However on a connected manifold if $\mathcal{A}$ is an orientation atlas then we can switch the first two coordinates of each chart and recover another orientation atlas and so we see that certain *apriori* choice has to be made before defining the integral.

**Exercise:** Prove that fixing an orientation atlas on an orientable manifold $M$ is equivalent to choosing a nowhere vanishing $n$ form $\theta$ on $M$. Hint: Take a connected chart $U$ and look at the sign of

$$\theta(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}) \tag{10.18}$$

If this sign is positive retain $U$ as it is. Else switch the first two coordinates of $U$ and take the collection of all corrected charts.

**Definition 10.7 (Integral of an $n$ form on an $n$-dim manifold):** Let $M$ be an orientable manifold with a given orientation atlas $\mathcal{A}$ and $\omega$ be an $n$ form with compact support. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the atlas $\mathcal{A}$. Define

$$\int_M \omega = \sum_\alpha \int_M \rho_\alpha \omega. \tag{10.19}$$

Note that each $\rho_\alpha \omega$ has support inside some chart in $\mathcal{A}$ and for these the integral has been defined.

**Theorem 10.5:** The value of the integral (10.19) does not depend on the choice of the partition of unity.

**Proof:** Suppose $\{\sigma_\beta\}$ is another partition of unity subordinate to the atlas $\mathcal{A}$. Then so is the collection of products$\{\sigma_\beta \rho_\alpha\}$.

For each $\rho_\alpha$,

$$\int_M \rho_\alpha \omega = \sum_\beta \int_M \rho_\alpha \sigma_\beta \omega$$

and so summing over $\alpha$ we get

$$\sum_{\alpha} \int_M \rho_\alpha \omega = \sum_{\alpha} \sum_{\beta} \int_M \rho_\alpha \sigma_\beta \omega$$

Similarly we get reversing the roles of the two partitions of identity,

$$\sum_{\beta} \int_M \sigma_\beta \omega = \sum_{\beta} \sum_{\alpha} \int_M \rho_\alpha \sigma_\beta \omega$$

from which the result follows.

**Remark:** The same arguments go through for manifolds with boundary.

**Induced orientation on the boundary:** Let us consider a simple example. Look at the differential $(n-1)-$form

$$\omega = y_n dy_1 \wedge \cdots \wedge dy_{n-1}$$

in $H$ and we see that

$$d\omega = (-1)^{n-1} dy_1 \wedge \cdots \wedge dy_n.$$

Writing this in terms of the coordinates that we have meticulously set up in (10.7),

$$\begin{aligned} d\omega &= (-1)^{n-1} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1} \wedge d((1-x_n)\sqrt{1-x_1^2-\cdots-x_{n-1}^2}) \\ &= (-1)^n \sqrt{1-x_1^2-\cdots-x_{n-1}^2}\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \end{aligned}$$

Integrate this over the domain $D$ over which $(x_1, x_2, \ldots, x_n)$ vary, we get

$$\int_{B_0^+} d\omega = (-1)^n \int_D \sqrt{1-x_1^2-\cdots-x_{n-1}^2}\, dx_1 dx_2 \ldots dx_{n-1} dx_n$$

Using Fubini theorem we get finally

$$\int_{B_0^+} d\omega = (-1)^n \int_U \sqrt{1-x_1^2-\cdots-x_{n-1}^2}\, dx_1 dx_2 \ldots dx_{n-1}, \tag{10.19}$$

Where $U$ is the open unit ball in $\mathbb{R}^n$. On the other hand let us integrate $\omega$ over the boundary of this half ball. That is to say, we compute

$$\int_{\partial B_0^+} i^*(\omega)$$

where $i : \partial B_0^+ \longrightarrow B_0^+$ is the inclusion map. Since we need to understand the relation between the form on $B_0^+$ in relation to its boundary explicitly, we shall have to write this also in terms of the coordinates (10.7). Along the base $\omega = 0$ and so we need to worry only about the part of the sphere $S^{n-1}$. Along the curved part we use the chart map (10.7) and write the differential form in this chart and integrate leading to:

$$\int_{\partial B_0^+} i^*(\omega) = \int_U \sqrt{1-x_1^2-\cdots-x_{n-1}^2}\, dx_1 dx_2 \ldots dx_{n-1}. \tag{10.20}$$

where $U$ is the open unit disc in $\mathbb{R}^{n-1}$. We have simply used the form $dx_1 \wedge \cdots \wedge dx_{n-1}$ to prescribe an orientation on $U$.

**Exercise:** Compute the integral on the right hand side of (10.20) and check that it equals $\mathrm{Vol}(B_0^+)$.

Thus we see that the basic equation (Stokes' theorem)

$$\int_M d\omega = \int_{\partial M} i^*(\omega) \tag{10.21}$$

holds only up to sign if we arbitrarily choose the orientations on $M$ and $\partial M$. We need to set down certain conventions in order that (10.21) holds.

**Definition 10.8 (Induced Orientation on $\partial M$):** Suppose $M$ is an orientable manifold of dimension $n$ with boundary and assume that an orientation on $M$ has already been chosen either by specifying an orientation atlas or equivalently by prescribing a nowhere vanishing $n$-form. We have two choices for the orientation of $\partial M$ and this can be done locally say at any chosen point[2] of $\partial M$.

We pick a point $p \in \partial M$ and a chart $(U, \phi)$ mapping into a half ball contained in $H$. Suppose that the orientation on $M$ is given by the form

$$\lambda dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

on $U$ then on the boundary $\partial U$ the orientation is given by the form

$$(-1)^n \lambda dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}.$$

The sign $(-1)^n$ is chosen to free the statement of Stokes' theorem of a sign depending on the dimension of the manifold. In terms of orientation atlas $\mathcal{A}$ if $U \in \mathcal{A}$ and the local frame

$$\Big[\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\Big]$$

decides the orientation on $M$, the induced orientation on $\partial M$ is given by the local frame

$$\Big[\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \dots, \frac{\partial}{\partial x_{n-1}}\Big|_p\Big], \text{ if } n \text{ is even.} \tag{10.22}$$

and

$$\Big[\frac{\partial}{\partial x_2}\Big|_p, \frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_{n-1}}\Big|_p\Big], \text{ if } n \text{ is odd.} \tag{10.23}$$

**Definition (The inward and outward pointing transversals/normals at a boundary point):** Let $M$ be a manifold with boundary. We have seen that if $p \in \partial M$, we have the inclusion of vector spaces

$$T_p(\partial M) \subset T_p M.$$

Since the dimension of $T_p(\partial M)$ is one less than the dimension of $T_pM$ the complement

$$T_pM - T_p(\partial M)$$

[2]It may happen that $M$ is connected but $\partial M$ has many connected components in which case each component of $\partial M$ has to be consistently oriented and so we select a representative point on each component of $\partial M$.

breaks off into two pieces. Let $v \in T_pM - T_p(\partial M)$ and use the physicist's convention to write (with respect to a chart $(U, \phi)$)

$$v = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_{n-1} \\ c \end{bmatrix}$$

and call $v$ an inward pointing transversal if $c > 0$ and outward pointing if $c < 0$. Equations (10.12) and (10.13) confirm that these are independent of the choice of the coordinate charts.

**Exercises:**

1. Check that $c = 0$ if and only if $v \in T_p(\partial M)$.
2. Show that the set of inward/outward pointing transversals are closed under addition and multiplication by positive scalars.
3. Show that $v$ is inward pointing if and only if there exists a curve $\gamma : [0, \epsilon) \longrightarrow M$ such that $\gamma(0) = p$ and $(\phi \circ \gamma)^{\cdot}(0) = c > 0$.
4. Show that the induced orientation on $T_p\partial M$ is characterized by the property. An ordered basis $[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}]$ of $T_p(\partial M)$ is in agreement with the induced orientation of $\partial M$ if and only if for every outward pointing transversal $\mathbf{v}$ at $p$, the orientation on $T_pM$ is given by
$$[\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}] \tag{10.24}$$
Hint: Write (10.22) and (10.23) in physicist's convention.

**Theorem 10.6 (Stokes's theorem):** Let $M$ be an orientable manifold of dimension $n$ with boundary, $\omega$ be a $n-1$ form with compact support. Assume that an orientation on $M$ is prescribed and $\partial M$ is assigned the induced orientation. then

$$\int_M d\omega = \int_{\partial M} i^*(\omega) \tag{10.25}$$

Note that since the pull back $i^*(\omega)$ is the restriction of $\omega$ to $\partial M$, we often see Stokes's theorem being stated as

$$\int_M d\omega = \int_{\partial M} \omega \tag{10.26}$$

**Proof:** Case (i): $M = \mathbb{R}^n$. By linearity we may assume that

$$\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$$

Then

$$d\omega = (-1)^{n-1} \frac{\partial f}{\partial x_n} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1} \wedge dx_n$$

Now it is a matter of appealing to the Fubini theorem and the fundamental theorem of calculus:

$$\begin{aligned}\int_M d\omega &= (-1)^{n-1}\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_n}dx_1dx_2\dots dx_{n-1}dx_n\\ &= (-1)^{n-1}\int_{\mathbb{R}^{n-1}}dx_1dx_2\dots dx_{n-1}\int_{\mathbb{R}}\frac{\partial f}{\partial x_n}dx_n = 0\end{aligned}$$

since $f$ has compact support. On the other hand since for this case $\partial M = \emptyset$, the right hand side of (10.25) is also zero and the theorem is verified in this case.

Case (ii) $M = H$. Let

$$\omega = a_1dx_2\wedge\dots\wedge dx_n - a_2dx_1\wedge dx_3\wedge\dots\wedge dx_n + \dots + (-1)^{n-1}a_ndx_1\wedge dx_2\wedge\dots\wedge dx_{n-1}.$$

Then

$$d\omega = \Big(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\dots+\frac{\partial a_n}{\partial x_n}\Big)dx_1\wedge dx_2\wedge\dots\wedge dx_n \tag{10.27}$$

Let us compute the left hand side of (10.25). If $1\le j\le n-1$

$$\int_H\frac{\partial a_j}{\partial x_j}dx_1dx_2\dots dx_n = 0$$

using Fubini theorem and the fundamental theorem of calculus. The only surviving term is

$$\begin{aligned}\int_H\frac{\partial a_n}{\partial x_n}dx_1dx_2\dots dx_n &= \int_{\mathbb{R}^{n-1}}dx_1dx_2\dots dx_{n-1}\int_0^\infty\frac{\partial a_n}{\partial x_n}dx_n\\ &= \int_{\mathbb{R}^{n-1}}(a_n(x_1,\dots,x_{n-1},\infty)-a_n(x_1,\dots,x_{n-1},0))dx_1\dots dx_{n-1}\\ &= -\int_{\mathbb{R}^{n-1}}a_n(x_1,\dots,x_{n-1},0)dx_1\dots dx_{n-1}\end{aligned}$$

Let us now compute the right hand side of (10.25) and verify the theorem. Since $x_n = 0$ along $\partial H$ we see that the terms involving $dx_n$ are all zero and so the only surviving term is $(-1)^{n-1}a_ndx_1\wedge\dots\wedge dx_{n-1}$ and recalling that the orientation convention,

$$\begin{aligned}\int_{\partial H}\omega &= (-1)^{n-1}\int_{\partial H}a_n(x_1,\dots,x_{n-1},0)dx_1\wedge\dots\wedge dx_{n-1}\\ &= (-1)^{n-1}(-1)^n\int_{\mathbb{R}^{n-1}}a_n(x_1,\dots,x_{n-1},0)dx_1dx_2\dots dx_{n-1}\end{aligned}$$

and the proof of this case is complete.

Case (iii): The general case. Take an orientation atlas $\{U_\alpha\}$ on $M$ and a partition of unity $\{\rho_\alpha\}$ subordinate to this cover. Thus

$$\omega = \sum_\alpha \rho_\alpha\omega$$

We have to prove

$$\int_M d(\rho_\alpha\omega) = \int_{\partial M}\rho_\alpha\omega \tag{10.28}$$

Since support of $\rho_\alpha\omega$ is contained in $U_\alpha$, the chart $U_\alpha$ is diffeomorphic to a full open ball in $\mathbb{R}^n$ or a half-ball and $\rho_\alpha\omega$ assumes one of the forms in case (i) or (ii) the result holds for each $\alpha$. the proof is complete.

**The classical version of Stokes' theorem - Gauss's Divergence Theorem:** We now show how the general Stokes' theorem proved above reduces to the classical version of the Stokes's theorem namely the Gauss divergence theorem. The student may have guessed this by looking at equation (10.27). A smooth bounded domain in $\mathbb{R}^n$ is by definition a bounded open set such that its topological closure in the ambient space has smooth boundary (forming a smooth manifold of dimension $n-1$ without boundary). Let us consider a smooth map

$$F : \Omega \longrightarrow \mathbb{R}^n$$

defined on a smooth bounded domain $\Omega$. We create the $n-1$ form

$$\omega = F_1 dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n - F_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \cdots + (-1)^{n-1} F_n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}.$$

Let $\Omega$ be assigned the natural orientation namely $dx_1 \wedge \dots dx_n$ and we see that the left hand side of (10.23) reads

$$\int_\Omega (\text{Div } F) dx_1 dx_2 \dots dx_n. \tag{10.29}$$

To compute the right hand side, let us consider a parametrization of the manifold $\partial\Omega$:

$$G : R \longrightarrow \partial M, \quad G(t_1, t_2, \dots, t_{n-1}) = (x_1, x_2, \dots, x_n).$$

where $R$ is an open set in $\mathbb{R}^{n-1}$ and $DG$ has rank $n-1$ throughout $R$. This $G$ is the inverse of a chart map where the chart comes from an atlas compatible with the induced orientation on $\partial\Omega$ (that is a chart which is corrected according to (10.22)-(10.23)).

Note that the partial derivatives of $G$ form a basis for the tangent space $T_p(\partial\Omega)$ and we shall assume that the induced orientation on $\partial M$ is in agreement with the $(n-1)-$tuple

$$\Big(\frac{\partial G}{\partial t_1}, \dots, \frac{\partial G}{\partial t_{k-1}}\Big).$$

Exercise (4) shows that for an outward pointing normal $v$, the ordered basis

$$\Big(v, \frac{\partial G}{\partial t_1}, \dots, \frac{\partial G}{\partial t_{k-1}}\Big) \tag{10.30}$$

agrees with the natural orientation on $\Omega$.

**Lemma:** The $(n-1) \times (n-1)$ signed-minors of $DG$ form a vector $v = (\nu_1, \nu_2, \dots, \nu_n)$ that is normal to $\partial\Omega$ and the determinant (10.30) is positive which means that the $(n-1) \times (n-1)$ signed-minors of $DG$ form the components of the outward pointing normal (need not be a unit vector).

**Proof:** Exercise in linear algebra. Suppose $A$ is an $n \times (n-1)$ matrix of rank $n-1$. Write down a system of equations for a vector $(y_1, y_2, \dots, y_n)$ to be orthogonal to each column of $A$ and argue that the signed $(n-1) \times (n-1)$ minors must be a basis of the solution space.

Now we are ready to compute the right hand side of (10.25)

$$i^*(dx_k) = dG_k = \sum_{l=1}^{n-1} \frac{\partial G_k}{\partial t_l}$$

so that

$$\begin{aligned} i^*(F_1 dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n) &= (F_1 \circ G)\frac{\partial(G_2, G_3, \ldots, G_{n-1})}{\partial(t_1, t_2, \ldots, t_{n-1})} dt_1 \wedge \cdots \wedge dt_{n-1} \\ &= (F_1 \circ G)\nu_1 dt_1 \wedge \cdots \wedge dt_{n-1}, \quad \text{etc.,} \end{aligned}$$

whereby

$$\int_{\partial\Omega} i^*(\omega) = \int_R (F \circ G) \cdot v dt_1 dt_2 \ldots dt_{n-1}$$

But if we recall the formula for surface element on $\partial\Omega$,

$$dS = \|v\| dt_1 dt_2 \ldots dt_{n-1}$$

we get finally denoting $v/\|v\| = \mathbf{n}$,

$$\int_\Omega (\text{Div } F) dx_1 dx_2 \ldots dx_n = \int_{\partial\Omega} F \cdot \mathbf{n} dS \tag{10.31}$$

**Remark:** Note that we are only addressing the issue of how the classical theorem of Gauss follows from the Stokes' theorem. How does one ensure that the $(n-1)-$tuple of partial derivatives of $G$ are in agreement with the induced orientation? This is a matter that cannot be resolved universally via *formulas* but rather needs to be addressed at a notional or conceptual level. One chooses a convenient point $p$ on each connected component of the boundary $\partial M$ and selects[3] an outward pointing normal $\mathbf{v}$ at $p$. One must check the sign of the determinant (10.30) and if found negative switch two the coordinates $t_1$ and $t_2$.

**Corollary (Rule for integration by parts):** Let $\Omega$ be the closed of a bounded domain with smooth boundary in $\mathbb{R}^n$ and $u, v$ be smooth real valued functions defined on the closure of $\Omega$ then

$$\int_\Omega u \frac{\partial v}{\partial x_j} dx_1 dx_2 \ldots dx_n = -\int_\Omega v \frac{\partial u}{\partial x_j} dx_1 dx_2 \ldots dx_n + \int_{\partial\Omega} uv\nu_j dS \tag{10.32}$$

where $(\nu_1, \nu_2, \ldots, \nu_n)$ is the unit outer normal to $\Omega$ and $dS$ is the area element on the boundary of $\Omega$.

1. Prove the rule for integration by parts
2. Show that the Gauss's theorem in the plane is equivalent to Green's theorem. Hint. Let us denote by $\mathbf{t}$ the unit tangent vector in the direction of bounding curve traced counter clockwise and $\mathbf{n}$ the outward unit normal. Then $\mathbf{t} \times \mathbf{k} = \mathbf{n}$.
3. Suppose that $\Delta u = 0$ in $\Omega$. Show that

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0.$$

[3]Theoretically this means we choose an appropriate curve $\gamma : [0\epsilon) \longrightarrow M$ *going into* $M$, compute its derivative at 0 and then attach a negative sign. However in practice this would be done by inspection.

4. Apply the previous to a ball and prove the mean value property for harmonic functions on spheres and balls.

5. Show that if $\Delta u = 0$ in $\Omega$ and $u = 0$ on $\partial\Omega$ then

$$\int_\Omega |\nabla u|^2 dx_1 dx_2 \dots dx_n = 0$$

**Brouwer's Fixed Point Theorem** As an application of Stokes' theorem, we prove the Brouwer's fixed point theorem first for smooth maps of the closed unit ball and later extend it to continuous maps. $D$ denotes the closed unit ball in $\mathbb{R}^n$ and $S^{n-1}$ denotes its boundary.

**Definition:** A map $r : D \longrightarrow S^{n-1}$ is said to be a retraction if $r(x) = x$ when $|x| = 1$. More generally if $A$ is a closed subspace of $X$ a map $r : X \longrightarrow A$ is said to be a retraction of $X$ onto $A$ if $r(a) = a$ for all $a \in A$.

**No Retraction Theorem:** There is no smooth retraction of $D$ onto its boundary $S^{n-1}$.

**Proof:** Suppose $F : D \longrightarrow S^{n-1}$ is a smooth retraction of $D$ onto its boundary, let $F = (f_1, f_2, \dots, f_n)$. Now $\nabla f_1, \dots, \nabla f_n$ are everywhere linearly dependent for if at some point $p$ in the interior of $D$ they are linearly independent, by inverse function theorem $F$ is a local diffeomorphism of a neighborhood $N$ of $p$ and so its image would contain a full open ball which is not the case since the image is contained in $S^{n-1}$. Thus,

$$df_1 \wedge df_2 \wedge \cdots \wedge df_n = 0, \quad \text{in } D.$$

Consider now the $n-1$ form $\omega$ on $D$ given by

$$\omega = \sum_{j=1}^{n} (-1)^j f_j df_1 \wedge df_{j-1} \wedge df_{j+1} \cdots \wedge df_n$$

A simple calculation gives $d\omega = 0$. Applying Stokes' theorem we get

$$\int_{S^{n-1}} i^*(\omega) = 0. \tag{10.33}$$

Since $F(x) = x$ along $S^{n-1}$, we see that $i^*(\omega)$ is given by

$$i^*(\omega) = \sum_{j=1}^{n} (-1)^j x_j dx_1 \wedge dx_{j-1} \wedge dx_{j+1} \cdots \wedge dx_n, \tag{10.34}$$

**Exercise:** Compute this integral and show that

$$\int_{S^{n-1}} \sum_{j=1}^{n} (-1)^j x_j dx_1 \wedge dx_{j-1} \wedge dx_{j+1} \cdots \wedge dx_n = n \int_D dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = n\text{vol}(D)$$

which is a contradiction.

**Theorem:** The following are equivalent:

(i) There is no smooth retraction of $D$ onto its boundary

(ii) Every smooth map $f : D \longrightarrow D$ has a fixed point.

**Proof:** Assume (i). Suppose $f : D \longrightarrow D$ has no fixed point. Then $f(x) \neq x$ for any $x \in D$. Let the line segment joining $x$ and $f(x)$ meet the boundary at the point $y(x)$. We show that $y(x)$ is a smooth retraction of $D$ onto its boundary. First note that

$$y = tf(x) + (1-t)x.$$

The condition $|y| = 1$ gives

$$t^2|f(x)|^2 + (1-t)^2|x|^2 + 2t(1-t)x \cdot f(x) = 1,$$

which can be recast as

$$t^2|x - f(x)|^2 - 2tx \cdot (x - f(x)) + |x|^2 - 1 = 0.$$

Consider the negative root of this quadratic (this means of the two points of intersection of the line segment with the boundary we are taking the point $y$ for which $x$ lies between $f(x)$ and $y$):

$$t = \frac{x \cdot (x - f(x)) - \sqrt{(x \cdot (x - f(x))^2 + |x - f(x)|^2(1 - |x|^2)}}{|x - f(x)|^2}$$

With this choice of $t$ we see that $y(x)$ is a smooth map of $D$ to $S^{n-1}$. We now show that it is a retraction. Let $|x| = 1$ so that $t = 0$ and hence that $y(x) = x$. Note that this would not be so if we take the root with positive sign in front of the radical!

Assume (ii) and that (i) is false. Then there is a smooth retraction $f$ of $D$ onto the boundary. The map $-f(x)$ is a smooth map of $D$ to $D$ without fixed points.

**Corollary (Brouwer's fixed point theorem):** Every smooth map of $D$ to itself has a fixed point.

**Proof:** We have proved that there is no smooth retraction of $D$ onto its boundary. The result follows from the previous theorem.

**Exercises:**

1. Show that if $B$ is the unit ball in $\mathbb{R}^n$ and $\omega$ is any differential $n-$form, show that $\omega$ is exact. Deduce that $H^n(B) = 0$.

2. Prove that if $\omega$ is a closed two-form on the closed unit ball $B$ in $\mathbb{R}^3$ then $\omega$ is exact. See Rudin's principles. Deduce that $H^2(B) = \{0\}$.

3. Suppose $M$ is a compact *orientable* manifold of dimension $n$ *without* boundary, show that $\mathrm{Dim}(H^n(M)) \neq 0$. Use Stokes' theorem. In fact the dimension equals one whereas if $M$ is a compact non-orientable manifold of dimension $n$ without boundary, the dimension of $H^n(M)$ is zero. This is related to the so called *Poincaré Duality.*

4. Use the last two exercises to provide another proof of the no retraction theorem. Hint: Suppose $r$ is a retraction and $i : S^2 \longrightarrow B$ is the inclusion map then $r \circ i$ is the identity map on $S^2$. Now look at the maps induced on cohomology. The proof generalizes easily to any dimension.