The Gamma Function

An Eclectic Tour

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Role played special functions in modern mathematics

- Theory of Group Representations
- Study of completely integrable systems of PDEs
- Painlévé Analysis
- Mathematical Physics
- Millenium Prize Problem (Clay Inst)

1. Genesis of the Gamma Function
2. Contributions from the Great Masters of the 19th century
3. Early twentieth century developments
4. Some Proofs - particularly on a formula of Ramanujan
5. Some recent results.

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For a detailed account of this see J. Lützen: Joseph Liouville 1809-1882, Master of pure and applied mathematics, Springer Verlag, 1990.

The most basic results:

The gamma function is defined on the right half plane by the absolutely convergent integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

It is immediate that

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1, \quad \Gamma(n+1) = n!, \quad n = 1, 2, 3, \ldots.$$

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For values of z other than integers and half integers, very little is known about the gamma values. For example if x is positive rational and not an integer then one of $\Gamma(x)$ or $\Gamma(2x)$ is transcendental. It is not known whether $\Gamma(1/3)$ is transcendental. C. L. Siegel, Transcendental Numbers, Ann. Math. Studies, Princeton, p. 100.

The Beta Function

This is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

Euler's Beta-Gamma relation reads:

$$\Gamma(p+q)B(p,q) = \Gamma(p)\Gamma(q).$$

In particular if $0 < s < 1$,

$$B(s, 1-s) = \int_0^\infty \frac{x^{s-1}dx}{1+x}$$

In principle the integral can be computed for rational values of s but requires a very clever book-keeping. See G. Lejeune-Dirichlet. Vorlesungen über die Lehre von der einfachen und mehrfachen bestimmten Integralen, Friedrich Vieweg und Sohn, Braunschweig, 1904.

Contibutions by Great Masters of the nineteenth Century:

Almost every mathematician of great repute has contributed to the study of the Gamma function - Here is an incomplete list:

1. Legendre (1809 - 1817)
2. C. F. Gauss (1812)
3. M.-J. Binet (1839)
4. L. Dirichlet (1836)
5. C. G. J. Jacobi (1834)
6. R. Dedekind (1852)
7. K. Weierstrass (1856)
8. B. Riemann (1859)
9. H. Hankel (1863)
10. E. Kummer (1847)

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$$1 + \frac{\alpha \cdot \zeta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \zeta(\zeta+1)}{1 \cdot 2\gamma(\gamma+1)} x^2 + \ldots$$

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3. In 1839 a long and important memoir of Binet appeared (Memoire sur les integrales definies euleriennes et sur leur applicatione a la theorie de suites, Jor. de l'Ecole Poly. **16** (1839) 123-343) containing many interesting transformation formulas one of which would feature significantly in a later slide. An important integral formula for $\log(\Gamma(z))$ was given by Plana in 1819 and rediscovered by Binet.

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$$\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-(1-z)/2}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z), \quad z \neq 0, 1.$$

B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, Gesammelte mathematische Werke, B. G. Teubner, Leipzig, 1990.

5 Weierstrass and Hankel: Functiontheoretic Viewpoint.
K.Weierstrass, Ueber die theorie der analytische Facultäten, Crelle Journal 1856, Mathematische Werke, Volume 1, Mayer and Müller, Berlin 1894.

Early twentieth century - I: Mellin and Barnes, Work of N. Nielsen

1. The gamma function features in the works of Mellin and Barnes in their intergal representations of various hypergeometric functions and what are today known as the Mellin-Barnes integrals. Such integrals appear in the work of Hecke (in connection to problems in Algebraic Number theory), Cahen and Voronoi (in connection with analytic number theory and the Riemann zeta function). This is a very vast chapter in the theory of special functions and we shall say no more on this.
2. Many of these integrals also feature in the work of Srinivasa Ramanujan who independent of Mellin and others. Among other things he gave a formula for the Fourier transform of $|\Gamma(a+it)|^2$ where $a > 0$ is a constant. We shall return to this formula later.

N. Nielsen: Handbuch der theorie der gammafunktion, Leipzig, 1905.

Early twentieth century - II: Bohr-Mollerup, E. Artin and H. Wielandt

Characterizing the gamma function in terms of the functional relations:

$$f(z+1) = zf(z), \quad f(1) = 1$$

In the real domain we take the domain as $(0, \infty)$ and in the complex domain the right half plane. Observe that

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Question: What Additional conditions are needed to prevent us from indulging in such trivial modifications?

Theorems of Bohr-Mollerup and Helmut Wielandt

The Bohr-Mollerup theorem on the real domain is well-known (example, Rudin, Principles of Mathematical Analysis, 3rd Edition).

Theorem 1

Suppose $f : (0, \infty) \longrightarrow (0, \infty)$ *satisfies* $f(x+1) = xf(x), f(1) = 1$ *and* $\log(f(x)$ *is convex then* $f(x) = \Gamma(x)$.

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In the complex domain, we have the beautiful function theoretic characterization:

Theorem 2

Suppose f *is holomorphic in the open right half plane,* $f(z+1) = zf(z)$ *and* $f(1) = 1$*. Assume further that* $|f(z)|$ *is bounded in* $1 \leq Re\, z \leq 2$*. Then* $f(z) = \Gamma(z)$*.*

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Wielandt's great works in Group theory (e.g Automorphism tower) and matix theory practically eclipsed this beautiful theorem on the gamma function.

Proof of Wielandt's theorem:

As in the case of the gamma function the function $f(z)$ can be extended meromorphically on $\mathbb{C} - \{0, -1, -2, -3, \dots\}$ with simple poles at $0, -1, -2, \dots$. The residues at these points agree with those of $\Gamma(z)$ and so $q(z) = f(z) - \Gamma(z)$ is entire. Now $Q(z) = q(z)q(1-z)$ is also entire and satisfies $|Q(z+1)| = |Q(z)|$ (using the functional equation). If we show that $|Q(z)|$ is bounded in a vertical strip of width one then by Liouville's theorem it follows that $Q(z)$ and hence $q(z)$ is constant...

Proof that $|Q(z)|$ is bounded in a vertical strip of width one:

Already $|f(z)|$ and $|\Gamma(z)|$ are bounded in $1 \leq \mathrm{Re}\, z \leq 2$ and hence $|q(z)|$ is bounded there. But then $|q(3-z)|$ is bounded in the same strip and

$$q(1-z) = q(3-z)/(1-z)(2-z)$$

which is bounded in absolute in part of the same strip with $\mathrm{Im}(z) \geq 1$. Thus $|Q(z)|$ is bounded.

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A violet by a mossy stone
Half hidden from the eye !
-Fair as a star when only one
Is shining in the sky
William Wordsworth

Fuglede's Generalization of Wielandt's theorem:

Quite recently Bent Fuglede sharpened Wielandt's theorem. Wielandt requires that the function $f(z)$ be bounded in absolute value in the strip $S = \{1 \leq \operatorname{Re} z \leq 2\}$, Fuglede relaxes this considerably requiring only the following

1. $$\limsup_{z\to\infty, z\in S} \left(|f(z)||y|^{\frac{1}{2}-x} \exp(-\frac{3}{2}\pi|y|)\right) < \infty$$

2. $$\liminf_{z\to\infty, z\in S^{\pm}} \left(|f(z)||y|^{\frac{1}{2}-x} \exp(-\frac{3}{2}\pi|y|)\right) = 0.$$

   $S^{\pm}$ are the pieces of the strip in the upper and lower half planes.

Further he demonstrates the impossibility of any further sharpening of the result. See B. Fuglede, A sharpening of Wielnadt's characterization of the gamma function, American Math. Monthly **115** (2008) 845-850

Sketches of Some proofs of known theorems:

Theorem 3

Let us consider the nonlinear ODE

$$\frac{d^2}{dz^2}(\log(\Gamma(z)) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}$$

which has the property that if $f(z)$ is a solution then so is $f(z+1)/z$. If we fix the initial conditions $f(1) = 1$ and $f'(1) = -\gamma$ then the solution exists on the right half plane and agrees with the gamma function.

One constructs a solution by integrating along the line segment joining 1 and z:

$$\frac{f'(z)}{f(z)} = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right)$$

The series converges uniformly on compact subsets of the right half plane.

Denoting the sum by $\Lambda_1(z)$ we see that

$$f(z) = \exp\Big(\int_1^z \Lambda_1(\zeta)d\zeta\Big)$$

which shows in particular that $f(z)$ has no zeros. Integrating the equation for $f'(z)/f(z)$ in the last slide gives after a little manipulation,

$$f(2) = 1, \quad f'(2) = 1 - \gamma$$

Now we see that the two solutions $f(z)$ and $f(z+1)/z$ satisfy the same intial conditions and so $f(z+1) = zf(z)$. Basic estimates easily give us that $|f(z)|$ is bounded in the strip $1 \leq \text{Re } z \leq 2$ and so Wielandt's theorem applies and the proof is complete.

Euler's reflection formula

Theorem 4

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Before taking up the proof of the theorem let us interpolate a result from Partial Differential Equations namely a Liouville type theorem for entire harmonic functions.

A lemma on Harmonic funcions

The following theorems known as Liouville type theorems:

Theorem 5

(i) A bounded entire function is constant
(ii) A bounded harmonic function on $\mathbb{R}^n$ is constant.
(iii) An entire function $f(z)$ that satisfies an estimate of the form $|f(z)| \leq A + B|z|$ is a polynomial of degree atmost one.
(iv) A harmonic function on $\mathbb{R}^n$ that satisfies an estimate of the form $|f(x)| \leq A + B|x|$ is a linear polynomial.
(v) A harmonic function on $\mathbb{R}^n$ that is in L^2 or more generally L^p, is a constant.

For the case (iv) and $n = 2$ one can try and reduce it to (ii) but this would involve getting information about the harmonic conjugate in terms of bounds on its real part.

The requisite theorem is called Hadamard's real parts theorem - See E.C. Titchmarch, Theory of functions. Doing it for general n is quite a different matter, apart from the fact that results like Hadamard's real parts theorem involve some manipulations that are not too illuminating.

Proofs of the Liouville type theorems

We prove the following general theorem

Theorem 6

Suppose u is a harmonic function in $\mathbb{R}^n$ which is a tempered distribution then u is a polynomial.

To see this let us take the Fourier transform of the equation $\Delta u = 0$ and we get

$$|\xi|^2 \widehat{u}(\xi) = 0.$$

which means $\widehat{u}(\xi)$ is a distribution whose support is the origin. This immediately implies $\widehat{u}(\xi)$ is a linear combination of the Dirac delta and finitely many derivatives at the origin! The original function must therefore be a polynomial.

Let us define

$$g(z) = \Gamma(z)\Gamma(1-z)\sin \pi z$$

and the simple poles of the gamma function are canceled by the simple zeros of the sine function. We have observed that the solution $f(z)$ which we can now denote by $\Gamma(z)$ does not vanish in the right half plane and so $g(z)$ has no zeros. Also $g(z)$ is entire and periodic with period one. It is real when z is real so that

$$g(z) = \exp \phi(z)$$

where $\phi(z)$ is entire and periodic with period one. Now, $z\Gamma(z)\Gamma(1-z)$ grows linearly in y (the imaginary part of z) and $|z^{-1}(\sin \pi z)|$ grows like $|y|^{-1}\exp(\pi|y|)$ in the strip $-1/2 \leq x \leq 1/2$ so that the harmonic function $\mathrm{Re}\phi(z)$ satisfies the estimate

$$\mathrm{Re}(\phi(z)) \leq C_1 + C_2\pi|y|$$

which implies $\phi(z) = A + Bz$ for certain constants A and B.

So we infer

$$\Gamma(z)\Gamma(1-z)\sin \pi z = \exp(A + Bz)$$

Letting $z \to 0$ we infer $e^A = \pi$. Since $g(z)$ is real when z is real B must also be real. Letting $z \to 1$ we get $B = 0$.

Theorem 7

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}$$

On the some of the proofs of the reflection formula:

Most known proofs of the reflection formula are non-elementary. A remarkable elementary proof was given by R. Dedekind, in his dissertation on the gamma function written under the supervision of Gauss (1852). Through some clever manipulations of integrals, Dedekind obtains the following IVP for a non-linear second order ODE for the function $B(z) = B(z, 1-z)$:

$$B''B - B'^2 = B^4, \quad B(1/2) = \pi, \quad B'(1/2) = 0.$$

One can easily integrate this ODE and obtain that $B(z) = \pi/\sin \pi z$.
R. Dedekind, Über ein Eulersches Integral, J. reine angewandte Math. **45** (1853) 370-374.

Significance of the reflection formula

The reflection formula states that the gamma function is half the sine function. The sine function $g(z) = \sin \pi z$ satisfies

$$g(z) = g\left(\frac{z}{2}\right) g\left(\frac{z+1}{2}\right)$$

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Let us compare this with the following

$$\sqrt{\pi}\,\Gamma(z) = 2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right)$$

which is the famous Duplication Formula of Legendre (circa 1809). The analogy is striking !

Elementary Proof of the duplication formula:

The duplication formula can easily be proved by computing in two ways the Beta function $B(s,s)$:

$$B(s,s) = \int_0^1 (x(1-x))^{s-1}dx = 2\int_0^{\pi/2} \sin^{2s-1}\theta \cos^{2s-1}\theta d\theta$$
$$= 2^{1-2s}\int_0^{\pi/2} \sin^{2s-1}(2\theta)(2d\theta) = 2^{1-2s}\int_0^{\pi} \sin^{2s-1}\theta d\theta$$
$$= 2^{2-2s}\int_0^{\pi/2} \sin^{2s-1}\theta d\theta = 2^{1-2s}B(\frac{1}{2}, s)$$

An application of the beta gamma relation would prove the result.

The Legendre Duplication formula bears a close resemblence to the half angle formula for the sine function. However the sine function also has the following submultiple angle identity:

$$2^{1-k} \sin k\pi z = \sin \pi z \sin \pi \left(z + \frac{1}{k}\right) \ldots \sin \pi \left(z + \frac{k-1}{k}\right)$$

Is there a gamma analogue?

Gauss' Product formula and beyond

$$\Gamma(z) = k^{z-\frac{1}{2}}(2\pi)^{(1-k)/2}\Gamma\Big(\frac{z}{k}\Big)\Gamma\Big(\frac{z+1}{k}\Big)\dots\Gamma\Big(\frac{z+k-1}{k}\Big)$$

This formula appears in Gauss' 1812 memoir and the case $k = 2$ recovers the duplication formula. However one cannot easily arrive at this result some manipulations integrals ! Various proofs are available (Dirichlet, Sonine and many others).

Gauss' own proof is rather remarkable. For Dirichlet's proof (Crelle's Journal Volume 15, 1836) see G. K. Srinivasan 2007.

The Schobloch's Reciprocity formula:

A remarkable generalization of Gauss' product formula is due to Schobloch:

$$(2\pi)^{-q/2} q^{z+\frac{pq-p-q}{2}} \prod_{j=0}^{q-1} \Gamma\Big(\frac{z+pj}{q}\Big) = (2\pi)^{-p/2} p^{z+\frac{pq-p-q}{2}} \prod_{j=0}^{p-1} \Gamma\Big(\frac{z+qj}{p}\Big)$$

A proof employing Plana's formula for $\log(\Gamma(z))$ is available on pp 196-198 of N. Nielsen. See also G. K. srinivasan 2007 for an alternate argument and more details.

Factorial series

These are series of the form

$$a_0 + \sum_{n=1}^{\infty} \frac{a_n}{x(x+h)(x+2h)\dots(x+(n-1)h)}$$

These series play the same role in the theory of difference equations as power series do in the theory of differential equations (both in the analytic set up). The best known reference for a study of these series is the monograph of E. Nörlund, Leçons sur les équationes linéares aux différences finies.

Series of this kind are important in transcendental number theory.

Factorial series representations of the gamma function:

The gamma function solves the problem of interpolation of the factorial, this natually led James Stirling to look for series representations of this form. Stirling did not succeed and the first successful attempt was due to C. Hermite who gave the following (See P. J. Davis, American Mathematical Monthly **66** (1959) 849-869.

$$\log(\Gamma(1+z)) = \frac{z(z-1)}{1\cdot 2}\log 2 + \frac{z(z-1)(z-2)}{1\cdot 2\cdot 3}(\log 3 - \log 2) + \dots$$

There are several identities involving Factorial series and we shall look at one of them in detail. For more on these see N. Nielsen's Handbuch.

An identity due to M. A. Stern (1847)

Theorem 8

For z in the right half plane the following identity holds:

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + (z-1) - \frac{(z-1)(z-2)}{2 \cdot 2!} + \frac{(z-1)(z-2)(z-3)}{3 \cdot 3!} - + \dots$$

We shall not prove that the series on the right hand side converges uniformly on compact subsets of the right half plane. Let us denote the sum of the series as $\Lambda_2(z)$. Then we verify that

$$\Lambda_2(z+1) - \Lambda_2(z) = \frac{1}{z}\left(z - \frac{z(z-1)}{2!} + \frac{z(z-1)(z-2)}{3!} - + \dots\right)$$

Invoking Abel's limit theorem,

$$\Lambda_2(z+1) - \Lambda_2(z) = \lim_{t \to 1-} \frac{1}{z}\left(tz - \frac{z(z-1)t^2}{2!} + \frac{z(z-1)(z-2)t^3}{3!} - + \dots\right) = \frac{1}{z}$$

From this functional equation we get

$$\sum_{n=1}^{N} \frac{1}{(z+n)^2} = \Lambda'(z) - \Lambda_2'(z+N+1)$$

We let $N \to \infty$ and show that $\Lambda_2(z+1+N) \longrightarrow 0$. Assuming this, we get using the theorem we proved, that

$$\Lambda_2(z) = A + \frac{\Gamma'(z)}{\Gamma(z)}$$

for some constant A. Putting $z=1$ we see that $A=0$ and the proof would be complete.
In the next slide we show that $\Lambda_2(z+1+N) \longrightarrow 0$ as $N \to \infty$ and obtain some interesting by-products.

Notice that

$$\Lambda_2(z) = -\gamma + (z-1)\int_0^1 dt - \frac{(z-1)(z-2)}{2\cdot 2!}\int_0^1 tdt + \dots$$

Combining the integrals and using the binomial theorem we get the following integral formula due to Legendre, Exercices d calcul intégral..., Tome-II, p. 45

$$\Lambda_2(z) = -\gamma + \int_0^1 \Big(\frac{1-u^{z-1}}{1-u}\Big) du$$

In particular for $z = 1$ we get

$$\gamma = \int_0^1 \Big(\frac{1}{1-u} + \frac{1}{\log u}\Big) du = -\int_0^\infty e^{-t}\log t dt.$$

The last one can be traced back to L. Euler, 1769.

Ramanujan's Master Formula

Ramanujan gave a formula that embeds the reflection formula as a very special case. The formula was recast by G. H. Hardy as a form of a Paley-Wiener theorem for the Mellin transform.

Theorem 9

Let ϕ be holomorphic in a half plane $Re z > -\delta$ ($\delta > 0$) and satisfies an estimate of the form

$$|\phi(z)| \leq C \exp(pRe\, z + q|Im\, z|)$$

for certain constants C, p and q with $0 < q < \pi$. Then

$$\int_0^\infty x^{s-1}(\phi(0) - x\phi(1) + x^2\phi(2) + \dots)dx = \frac{\pi}{\sin(\pi s)}\phi(-s)$$

The displayed series converges only on $|z| < \exp(-p)$ but Hardy shows that it has analytic continuation to a sector containing $[0, \infty)$.

1. $\phi(x) = 1$ we get the Euler's reflection formula.
2. $\phi(x) = (\Gamma(1+x))^{-1}$ gives the definition of $\Gamma(z)$.
3. $\phi(x) = (1+x)^{-a}$ with $a > 0$ then we get the formula
4. Taking $\phi(x) = (\Gamma(x+1)\zeta(2+2x))^{-1}$, Ramanujan obtained

$$\int_0^\infty x^{s-1} \sum_0^\infty \frac{(-1)^n x^n}{n!\zeta(2+2n)} = \frac{\Gamma(s)}{\zeta(2-2s)}$$

   discovered independently by M. Riesz in connection with some curious necessary and sufficient conditions for the validity of the Riemann hypothesis.

For a multi-dimensional generalization of this see H. Ding, K. Gross, P. Richards, Ramanujan's Master theorem for symmetric cones, Pacific Journal of Mathematics, Volume 175 (1996) 447-490.
For the proof of the formula and general discussions see G. H. Hardy, Ramanujan, Twelve Lectures suggested by his life and works, Chelsea, New York, 1978.

Ramanujan's formula for the Fourier Transform of $|\Gamma(a+it)|^2$

Stirling's approximation theorem states that

$$\Gamma(z+1) \sim z^z \exp(-z)\sqrt{2\pi z}, \quad |z| \longrightarrow \infty$$

uniformly with respect to Arg z on any closed sector not containing the negative real axis.
It easily follows from this that the function

$$t \mapsto |\Gamma(a+it)|^2, \quad a > 0$$

is in the Schwatrz' space of rapidly decreasing functions.
It is of interest to know what is the Fourier transform of this function. The result is a member of a large family of integrals known as Mellin-Barnes integrals. Ramanujan obtained an explicit formula for the Fourier transform but through clever manipulations. We look at an alternate approach to obtaining the Fourier transform.

We recall a formula from Binet (1839) for the Beta function that is easy to prove:

$$B(p,q) = \int_{-\infty}^{\infty} \frac{e^{(p-q)s} + e^{(q-p)s}}{(e^s + e^{-s})^{p+q}} ds$$

Let us take $p = a + it$ and $q = a - it$ in this formula and we get

$$|\Gamma(a+it)|^2 = \int_{-\infty}^{\infty} \frac{e^{2its} + e^{-2its}}{(e^s + e^{-s})^{2a}} ds$$

The Fourier transform is then

$$\int_{-\infty}^{\infty} e^{-it\xi} dt \int_{-\infty}^{\infty} \frac{e^{2its} + e^{-2its}}{(e^s + e^{-s})^{2a}} ds$$

It is tempting to invert the order of integration:

The $\exp(-\epsilon t^2)$ trick in Fourier inversion:

We arrive at the following oscillatory integral:

$$\int_{-\infty}^{\infty} \frac{ds}{(e^s + e^{-s})^{2a}} \int_{-\infty}^{\infty} (e^{it(2s-\xi)} + e^{it(-\xi-2s)})dt$$

To cope with this as is usual in Fourier analysis, one introduces the factor $\exp(-\epsilon t^2)$ and take the limit as $\epsilon \to 0+$. The result is

$$\sqrt{\pi}\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\operatorname{sech}^{2a}\left(\frac{\xi}{2}\right)$$

The result must fail if $a < 0$ despite the fact that if $-1 < a < 0$ the function

$$t \mapsto |\Gamma(a + it)|^2$$

is in the Schwartz class. Along the imaginary axis the function

$$t \mapsto |\Gamma(it)|^2$$

is a tempered distribution.

Jump Across the Imaginary Axis:

Although the Fourier transform cannot be computed in closed form when $-1 < a < 0$, one obtains an integral formula from which one can compute the Jump in the Fourier transform across the imaginary axis:

Theorem 10

For each real ξ and purely imaginary $p \neq 0$, we have the formula for the jump in the Fourier transform of $|\Gamma(a+it)|^2$ across p namely

$$4\pi \cosh(p\xi)\Gamma(2p)$$

Ref: D. Chakrabarti and G. K. Srinivasan, Archiv Der Mathematik, 2012.

The idea of proof is as follows. Call $I(a,\xi)$ the integral

$$\int_{-\infty}^{\infty} a\Gamma(a-it)\Gamma(a+it)e^{-it\xi}dt$$

Note that

$$\left(\frac{d^2}{d\xi^2}-a^2\right)I(a,\xi)=\left(\frac{-a}{a+1}\right)I(a+1,\xi)$$

If $-1 < a < 0$ the RHS is known and we can solve this ODE using say the method of variation of parameters. To fix the values of the arbitrary constants we employ the Riemann Lebesgue lemma. The jump is then computed using Binet's formula.

On the horizontal monotonocity of the gamma function

The gamma function is highly non monotone along the real axis. It has a global minimum at a point on the open interval $(1, 2)$. This is qualitatively replicated on the intervals $(-1, 0)$, $(-2, -1)$ etc.,
However, if we move away from the real axis beyond a certain threshold value and restrict the Gamma function to horizontal lines, the function becomes monotone. This property seems to have escaped notice in the literature. We discuss an elementary proof of the following

Theorem 11

If $|t| > 1.25$, the function

$$s \mapsto |\Gamma(s + it)|$$

is a monotone function on $\mathbb{R}$.

A basic lemma in complex analysis

Theorem 12

Assume that f is holomorphic on a convex open subset of the complex plane. Write $z = s + it$ for an arbitrary point $z \in \Omega$. Then

$$Re\left(\frac{f'(z)}{f(z)}\right) = \frac{1}{|f(z)|}\frac{\partial |f(z)|}{\partial s}.$$

Hence $s \mapsto |f(s+it)|$ is monotone increasing if and only if

$$Re\left(\frac{f'(z)}{f(z)}\right) > 0.$$

along horizontal line segments of Ω.

The proof is simple and we shall not discuss it.

Another lemma on the log derivative of $\Gamma(z)$

Theorem 13

$$Re\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) = Re\Big(\frac{\Gamma'(z+1)}{\Gamma(z+1)}\Big) - \frac{s}{|z|^2}$$

Hence if for a given value of t, the real part of $\Gamma'(s+it)/\Gamma(s+it)$ is positive for $0 \leq s \leq 1$ then it is positive for all $s \leq 1$.

The proof follows at once from the functional equation for the gamma function.

Another lemma on the log derivative of $\Gamma(z)$

Theorem 13

$$Re\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) = Re\Big(\frac{\Gamma'(z+1)}{\Gamma(z+1)}\Big) - \frac{s}{|z|^2}$$

Hence if for a given value of t, the real part of $\Gamma'(s+it)/\Gamma(s+it)$ is positive for $0 \leq s \leq 1$ then it is positive for all $s \leq 1$.

The proof follows at once from the functional equation for the gamma function.
Let us now use this to discuss the horizontal monotonocity of $|\Gamma(z)|$. We shall give an elementary argument that establishes the theorem only when $|t| > 2$.

Using the identity we proved for the log derivative of $\Gamma(z)$,

$$\operatorname{Re}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) = -\gamma + \sum_{n=0}^{\infty}\Big(\frac{1}{n+1} - \frac{n+s}{(n+s)^2+t^2}\Big)$$

Separating the terms $n = 0$ we get

$$\operatorname{Re}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) = 1 - \gamma - \frac{s}{s^2+t^2} + \sum_{n=1}^{\infty}\Big(\frac{s^2+t^2+(ns-s-n)}{(n+1)((n+s)^2+t^2)}\Big)$$

Now if $s > 1$ then the terms under the summation signs are all positive. Whereas $1 - \gamma \geq 0.4$ and $s/(s^2+t^2) \leq 1/2t < 1/4$.

So $\operatorname{Re}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) > 0$ in this case.

Now let us take up the case $0 \leq s \leq 1$. We separate the terms for $n = 0$ as well as $n = 1$ and we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = 1 - \gamma - \frac{s}{s^2 + t^2} + \frac{1}{2}\Big(\frac{s^2 + t^2 + 1}{(s+1)^2 + t^2}\Big) + \sum_{n=2}^{\infty}\Big(\frac{s^2 + t^2 + (ns - n - s)}{(n+1)((n+s)^2 + t^2)}\Big)$$

The first three terms together exceeds $3/20$. The fourth term exceeds $3/16$. Assume that the the terms within the summation sign are positive ' for $n = 5, \ldots, N - 1$, the summation from $n = N$ onwards is less than

$$\sum_{n=5}^{\infty} \frac{n-4}{(n+1)(n^2+4)} < \frac{\pi^2}{6} - \Big(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\Big) < 1/4$$

So that in case $0 \leq s \leq 1$ we have

$$\mathrm{Re}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) > \frac{3}{20} + \frac{3}{16} - \frac{1}{4} > 0.$$

By virtue of the basic lemma in complex analysis, we see that

$$s \mapsto |\Gamma(s + it)|$$

is monotone if $|t| > 2$. Improving this result along the elementary lines suggested above seems quite troublesome. To cope with this situation we need to use the delicate remainder estimates in Stirling's formula for the log-derivative of the gamma function. This is carried out in our paper: G. K. Srinivasan and P. Zvengrowski, On the horizontal monotonocity of $|\Gamma(s)|$, Canadian Math. Bulletin,**54**, 2011, 538-543.

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Thank You !