Supplementary notes on Real Analysis

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Existence of n-th root of a positive real number: This concept is extremely important for instance in the statement of Cauchy's theorem on means. It is essential to provide a treatment of this from scratch and not rely on intermediate value theorem or exponentials and logarithms.

Theorem: Let $k \in \mathbb{N}$ be arbitrary. Every positive real number has a unique positive k-th root.

Proof: Let $a > 0$. Suppose b and c are both $k-$th positive roots of a and $b < c$ then $b^k < c^k$ (how?). Thus $a < a$ and we get a contradiction. Likewise $c < b$ is not possible and the uniqueness is established.

Existence Consider the set

$$S = \{x \in \mathbb{Q} \ : \ x^k < a\}.$$

Note that we could work with real x. We have chosen this approach in order to exhibit a Dedekind cut for $a^{1/k}$. The set S is bounded above. To see this by Archimedean property select $n > a$ and then n^k serves as an upper bound. Let $\sup S = l$. We show that $l^k = a$. Suppose $l^k < a$ we shall arrive at a contadiction by showing that for some $m \in \mathbb{N}$,

$$\left(l + \frac{1}{m}\right)^k < a \tag{1.1}$$

and then picking a rational number q between l and $l + \frac{1}{m}$ we get $q \in S$ and $q > l$ which is impossible.

Now we seek a $m \in \mathbb{N}$ such that (1.1) holds. Instead of (1.1) we shall secure a better inequality

$$l^k + \frac{c_1}{m} + \frac{c_2}{m} + \cdots + \frac{c_k}{m} < a,$$

where $c_1, c_2, \ldots, c_k$ are the binomial coefficients that appear when we expand $(l + 1/m)^k$. Letting $C = c_1 + c_2 + \cdots + c_k$ we need m such that

$$\frac{C}{m} < a - l^k$$

This is evidently possible by virtue of the Archimedean property. The case $l^k > a$ is left as an exercise.

Kronecker's theorem: If α is irrational then the set of fractional parts

$$n\alpha - [n\alpha], \quad n = 1, 2, 3, \ldots$$

are dense in $[0, 1]$.

proof: We construct the set

$$S = \{n\alpha + m \ : \ m, n \in \mathbb{Z} \text{ and } n > 0\}$$

We show that S is dense in $\mathbb{R}$. Let

$$\xi_n = n\alpha - [n\alpha] \tag{1.1}$$

and observe that since α is irrational the numbers ξ_n for $n = 1, 2, 3, \ldots$ are pairwise distinct. For any arbitrary $N \in \mathbb{N}$ one of the intervals

$$[0, \frac{1}{N}], [\frac{1}{N}, \frac{2}{N}], \ldots, [\frac{N-1}{N}, 1]$$

must contain at least two of the numbers in the list (1.1) say ξ_n and ξ_m with $m < n$. Thus the difference

$$\xi = \xi_n - \xi_m$$

belongs to S. We have shown

For each $N \in N$ there exists $\xi \in S$ such that

$$0 < |\xi| < \frac{1}{N}$$

Let us assume $\xi > 0$ and complete the proof. Let us consider an interval $(t - \eta, t + \eta)$ in $\mathbb{R}$ where $\eta > 0$. We have to show that this interval contains a point of S and this is already done if $t = 0$. Assume that $t > 0$ and take an $N \in N$ such that $0 < 1/N < \eta$. Then a simple use of the Archimedean property and well ordering property gives a least natural number k_0 such that

$$t < k_0\xi.$$

Note that we are assuming that $t > 0$ here. Obviously then

$$(k_0 - 1)\xi \leq t$$

which implies $k_0\xi \leq \xi + t < 1/N \ + \ t < \eta + t$. The requisite element of S is $k_0\xi$.

Exercises: Discuss the cases $\xi < 0$ and $t < 0$. For $\xi < 0$ let l be the least natural number such that $l\xi < -1$. Then

$$-1 \leq (l - 1)\xi$$

which means

$$0 < 1 + (l - 1)\xi < \frac{1}{N} \quad \text{(check)}$$

Work with $1 + (l - 1)\xi$ which is also in S. Discuss the case $t < 0$.

Now if T is the set of fractional parts $n\alpha - [n\alpha]$ where $n = 1, 2, 3, \ldots$ then $S \cap [0, 1] = T$ from which follows that T is dense in $[0, 1]$.

Remarks:

1. There are many refinements and generalizations of this result. The most well known being Weyl's equi-distribution theorem:

   Theorem (H. Weyl): Suppose α is irrational and J is any subinterval of $[0,1]$ of length $|J|$, and if k_n is the number of times

   $$j\alpha - [j\alpha]$$

   lies in interval J when j ranges from 1 to n, then

   $$\lim_{n\to\infty} \frac{k_n}{n} = |J|.$$

   Thus the numbers $n\alpha - [n\alpha]$ enter every subinterval J with the same aymptotic relative frequency namely $|J|$ and is independent of the location of J.

   Once the result has been recast in terms of asymptotic frequency of hits in an interval we see that the result is a part of a large group of theorems of a similar kind. We now state another result which is also quite classical. Let us consider a real number $x \in [0,1]$ and look at its decimal expansion:

   $$x = 0.a_1a_2a_3\ldots$$

   and k_n be the number of times a specific digit say 7 appears among the first n digits $a_1, a_2, \ldots, a_n$ and then

   $$\lim_{n\to\infty} \frac{k_n}{n} \tag{1.2}$$

   would be the asymptotic frequency of occurence of 7 in the number x. Likewise one can ask for the asymptotic relative frequency of occurence of each of the ten digits.

   Definitions: (1) A number $x \in [0,1]$ is said to be normal if all digits in the decimal expansion occur with the same asymptotic relative frequence namely $1/10$.

   (2) A set of real numbers E has zero measure if for every $\epsilon > 0$ we can cover E by a countable collection of open intervals of total length less than ϵ. We shall see later that $\mathbb{Q}$ has measure zero.

   We are now ready to state the famous theorem of E. Borel:

   Borel's theorem on normal numbers: The set of numbers in $[0,1]$ which are not normal form a set of measure zero.

   Consider the interval $J = [7/10, 8/10)$. The first digit a_1 is 7 if and only if $x \in J$. The second digit a_2 is 7 if and only if $10x \in J$ and the third digit is 7 if and only of $10^2x \in J$. Thus k_n counts the number of hits in J of the sequence $x, 10x, \ldots, 10^{n-1}x$.

Exercises: Exhibit a number for which the limit (1.2) does not exist. Explain why 1/3 and 3/4 are not normal. Are there normal rational numbers?

2. A proof of Borel's theorem along classical lines is available in Hardy and Wright and also in the first chapter of P. Billingsley, Probability theory. Borel's theorem is often deduced as a corollory of the more powerful *strong law of large numbers.* These are the starting points of a huge subfield of analysis called Ergodic theory.

3. The fact that α is irrational can be restated as the set $\{1, \alpha\}$ is linearly independent over $\mathbb{Q}$. This suggests the following generalization:

   Theorem(Multi-dimensional Kronecker's theorem): Suppose $\alpha_1, \alpha_2, \dots, \alpha_m$ are linearly independent over $\mathbb{Q}$ the the set

$$S = \{(k_1\alpha_1 - [k_1\alpha_1], k_2\alpha_2 - [k_2\alpha_2], \dots, k_m\alpha_m - [k_m\alpha_m])\}$$

   as $k_1, k_2, \dots$ vary over all the natural numbers, is dense in the unit cube $[0,1]\times[0,1]\times[0,1]$ in $\mathbb{R}^m$?

   This matter is quite non-trivial. A discussion can be found in Hardy and Wright's famous book on number theory. The book of Hlawka et al., is highly recommended.

Exercise: Let α be irrational. Is the set

$$S = \{n\alpha + 2m \ : \ n \in \mathbb{N}, \ m \in \mathbb{Z}\}$$

dense in $\mathbb{R}$? What if $2m$ is replaced by $2m+1$?

Exercise: What is the supremum of the set

$$\{\sin 1, \sin 2, \dots, \sin n, \dots\}?$$

The number 1 is clearly an upper bound. To see that it is the least upper bound, for each $\epsilon > 0$ we need an n such that

$$1 - \epsilon < \sin n.$$

Well, by uniform continuity of sin function there exists a $\delta > 0$ such that

$$\left|\frac{\pi}{2} - x\right| < \delta \quad \text{implies} \quad 1 - \sin x < \epsilon$$

Note that $\sin n = \sin(2\pi k + n)$. Use Kronecker's theorem.

Exercise: Show that the image of the curve $x = \sin t$, $y = \sin \sqrt{2}t$ is dense in $[0, 1] \times [0, 1]$. Hint: For a given point (p, q) in the square pick a x_0 such that $\sin x_0 = p$ and then look at the sequence of points on the curve corresponding to the following values of t:

$$x_0 + 2\pi k, \quad k \in \mathbb{Z}$$

What can you say about the sequence

$$\sin \sqrt{2}(x_0 + 2\pi k)?$$

First note that $\sqrt{2}k + s$ ($k \in \mathbb{N}$ and $s \in \mathbb{Z}$) is dense in $\mathbb{R}$ and hence so is

$$\sqrt{2}x_0 + 2\pi(\sqrt{2}k + s)$$

Exercise (Lissajous figures): Sketch the curves

1. $x = \sin t$ and $y = \sin 2t$.
2. $x = \sin t$ and $y = \cos 3t$.
3. $x = \sin t$ and $y = \sin 3t$.

Exercise: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be periodic if there exists a $p > 0$ such that

$$f(t + p) = f(t), \qquad \text{for all } t \in \mathbb{R}.$$

The number p is called a period. Show that the set of periods together with zero forms a subgroup of $\mathbb{R}$. Show that if a subgroup of $\mathbb{R}$ has zero as a limit point then the subgroup must be dense in $\mathbb{R}$. Deduce that a continuous function with arbitrarily small positive periods must be constant.

Exercise: Discuss whether the following functions are periodic. If so what are the periods?

1. $f(t) = \sin t + \cos 3t$
2. $f(t) = \sin t + \sin \sqrt{2}t$

Cauchy's theorem on means: Given a set of n positive real numbers $a_1, a_2, \dots a_n$ their arithmetic mean A_n is defined as

$$A_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

Their geometric mean G_n is defined as

$$G_n = \Big(a_1 a_2 \dots a_n\Big)^{1/n}$$

Their harmonic mean H_n is the reciprocal of the arithmetic mean of the reciprocals. That is,

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Theorem (Cauchy): $H_n \le G_n \le A_n$

Proof: The first inequality follows from the second (check). We now prove the first inequality using backwards induction. Check the result for $n = 2$ (exercise). Thus

$$\sqrt{b_1 b_2} \le \frac{b_1 + b_2}{2}$$

Replace b_1 and b_2 by $(a_1 + a_2)/2$ and $(a_3 + a_4)/2$ respectively and we get

$$\sqrt{\frac{(a_1 + a_2)}{2}\frac{(a_3 + a_4)}{2}} \le \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$$

Using the result proved for $n = 2$ we get

$$\sqrt{\sqrt{a_1 a_2}\sqrt{a_3 a_4}} \le A_4$$

or $G_4 \le A_4$. Now show that the result holds for $n = 8$ and generally when n is a power of two.

Thus the result is true for a sequence of values of n increasing to infinity. We now show that if the result holds for $n \ge 2$ then it holds for $n - 1$ numbers as well. Assume for any set of n numbers and $G_n \le A_n$ and let $a_1, a_2, \dots a_{n-1}$ be a given set of $n - 1$ positive reals.

We shall select now a_n such that for the n numbers $a_1, a_2, \dots a_n$ we have

$$G_n = G_{n-1} \tag{2.1}$$

Well if we write out (2.1) and raise it to the n-th power we get

$$(a_1 a_2 \dots a_{n-1}) a_n = (a_1 a_2 \dots a_{n-1})^{n/(n-1)}$$

which gives

$$a_n = G_{n-1}$$

Now using (2.1) and the result for n namely $G_n \le A_n$ we get

$$G_{n-1} = G_n \le A_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_{n-1} + G_{n-1})$$

Hence

$$(1 - \frac{1}{n})G_{n-1} \le \frac{1}{n}(a_1 + a_2 + \cdots + a_{n-1})$$

which upon simplfying gives the desired result $G_{n-1} \le A_{n-1}$.

Theorem (The sandwich theorem): Suppose $(a_n), (b_n)$ and (c_n) are three real sequences such that

$$a_n \leq b_n \leq c_n, \qquad \text{for all } n \in \mathbb{N}.$$

Assume further that (a_n) and (c_n) both converge and converge to the same limit l. Then (b_n) also converges to l.

This theorem is quite easy to prove and is available in all texts and so will not be reproduced here.

Theorem:

1. Suppose $a > 0$ then $a^{1/n} \longrightarrow \infty$ as $n \longrightarrow \infty$.
2. $n^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$
3. The sequence (a_n) given by
$$\left(1 + \frac{1}{n}\right)^n$$
is monotone increasing and the sequence (b_n) given by
$$\left(1 - \frac{1}{n}\right)^{-n}$$
is monotone decreasing.
4. The sequences (a_n) and (b_n) above both converge to a common limit. This limit is denoted by e.

Proofs:

1. consider first the case when $a \geq 1$ and apply Cauchy's theorem of means to the set of n numbers
$$a, 1, 1, \ldots, 1.$$
Next apply sandwich theorem and complete the argument.
2. Apply Cauchy's theorem of means to the set of n numbers
$$\sqrt{n}, \sqrt{n}, 1, 1, 1, \ldots, 1$$
and invoke the sandwich theorem.
3. Apply Cauchy's theorem of means to the n numbers
$$1, \left(1 + \frac{1}{n}\right), \left(1 + \frac{1}{n}\right), \ldots, \left(1 + \frac{1}{n}\right).$$
We immediately see that (a_n) is monotone increasing. The case of (b_n) is equally easy.

4. Observe that
$$a_2 \leq a_n \leq b_n \leq b_2, \qquad \text{for all } n \geq 2.$$
It is clear that both sequences converge and the limits are positive. To show that the limits are equal, look at their ratio:
$$\frac{a_n}{b_n} = \left(1 - \frac{1}{n^2}\right)^n = \left(\left(1 - \frac{1}{n^2}\right)^{n^2}\right)^{1/n} = 1/(b_{n^2})^{1/n}$$
But then
$$a_2 \leq b_{n^2} \leq b_2$$
and hence taking the n-th root and invoking sandwich theorem we conclude that
$$\lim_{n\to\infty} (b_{n^2})^{1/n} \longrightarrow 1.$$
From this we conclude that $a_n/b_n \longrightarrow 1$ as $n \to \infty$.

Theorem: The sequence (a_n) given by
$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$
is monotone increasing and bounded above. Further the sequence converges to e.

Proof: It is quite easy (exercise) to check that the sequence is monotone increasing and bounded above. Let us assume it converges to f. Expanding using the binomial theorem,
$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \quad (2.2)$$
Thus we get the inequality
$$\left(1+\frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \leq f.$$
Since the sequence (a_n) monotonically increases to f. So we conclude
$$e \leq f$$
To obtain the reverse inequality we cannot simply take the limit term by term in (2.2) (why?). Instead a careful argument is needed. Fix $m \in \mathbb{N}$ and let $n > m$. Then from (2.2) we get
$$e \geq \left(1+\frac{1}{n}\right)^n geq 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right). \quad (2.2)$$
Letting n tend to infinity which is clearly permissible since $n > m$, we conclude
$$e \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}, \qquad \text{for all } m \in \mathbb{N},$$
whence $e \geq f$ and the proof is complete.

Remark: The errenous passage to the limit as $n \to \infty$ in (2.2) is unfortuately carried out in many classical texts. The matter is carefully handled in G. Chrystal (part II). Hobson in his famous, Theory of functions of a real variable invokes Tannary's theorem (as for instance T. J. I. Bromwhich).

Theorem (Cauchy's first limit theorem:) Suppose (a_n) is a sequence of real numbers such that $a_n \longrightarrow l$ as $n \longrightarrow \infty$ then the sequence of arithmetic means

$$\frac{1}{n}\Big(a_1 + a_2 + \cdots + a_n\Big)$$

also converges to l.

Proof: Since (a_n) converges, it is bounded and hence there exists $M > 0$ such that

$$|a_n| \leq M, \qquad \text{for all } n \in \mathbb{N}.$$

Second, let $\epsilon > 0$ be arbitrary. There exists $n_1 \in \mathbb{N}$ such that

$$|a_n - l| < \frac{\epsilon}{2}, \qquad \text{for all } n \geq n_1. \tag{2.3}$$

We now set up the stage and estimate:

$$\Big|\frac{1}{n}\Big(a_1 + a_2 + \cdots + a_n\Big) - l\Big| \leq \frac{1}{n}\Big(|a_1 - l| + |a_2 - l| + \cdots + |a_n - l|\Big) \tag{2.4}$$

Assume $n > n_1$ and split the sum in the paranthesis in two pieces and note that

$$|a_{n_1+1} - l| + |a_{n_1+2} - l| + \cdots + |a_n - l| < (n - n_1)\epsilon/2 < n\epsilon/2. \tag{2.5}$$

On the other hand

$$|a_1 - l| + |a_2 - l| + \cdots + |a_{n_1} - l| \leq n_1(M + |l|) \tag{2.6}$$

Now since $n_1(M + |l|)/n$ tends to zero as $n \longrightarrow \infty$, there exists $n_2 \in \mathbb{N}$ such that

$$n_1(M + |l|)/n < \frac{\epsilon}{2}, \qquad \text{for all } n > n_2. \tag{2.7}$$

Now let $n_0 = n_1 + n_2$ and we use (2.5), (2.6) and (2.7) in (2.4) and conclude that

$$\Big|\frac{1}{n}\Big(a_1 + a_2 + \cdots + a_n\Big) - l\Big| \leq \epsilon, \qquad \text{for all } n > n_0.$$

Corollary: Suppose (a_n) is a sequence of positive real numbers converging to l then the sequence of geometric means

$$(a_1 a_2 \ldots a_n)^{1/n}$$

also converges to l.

Proof: If $l = 0$ then let A_n and G_n be the arithemtic and geometric means of the numbers $a_1, a_2, \ldots, a_n$ and we have

$$0 < G_n < A_n.$$

But $A_n \longrightarrow 0$ as $n \longrightarrow \infty$ by Cauchy's first limit theorem. Invoking the sandwich theorem we conclude that $G_n \longrightarrow 0$ as $n \longrightarrow \infty$.

We now turn to the case $l > 0$. Then we also have $1/a_n \longrightarrow 1/l$. Apply Cauchy's first limit theorem to both the sequences (a_n) as well as $(1/a_n)$ and for the latter take reciprocals. We obtain immediately

$$A_n \longrightarrow l, \quad \text{and} \quad H_n \longrightarrow l.$$

where H_n is the harmonic mean of $a_1, a_2, \ldots a_n$. Now invoking Cauchy's theorem of means and the sandwich theorem we obtain the desired result.

Corollary (Cauchy's second limit theorem): Suppose (b_n) is a sequence of positive real numbers such that

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = l$$

then

$$\lim_{n \to \infty} (b_n)^{1/n} = l$$

Proof: Define $a_1 = b_1$, $a_2 = b_2/b_1 \ldots$, $b_n = a_n/a_{n-1}$. Then we see that $a_n \longrightarrow l$ as $n \longrightarrow \infty$. Now use the previous corollary that the sequence of geometric means $(a_1 a_2 \ldots a_n)^{1/n}$ converges to l.

Example: Prove that

$$\lim_{n \to \infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}. \tag{2.8}$$

Well let us write

$$\frac{(n!)^{1/n}}{n} = \Big(\frac{n!}{n^n}\Big)^{1/n}.$$

and define

$$b_n = \frac{n!}{n^n}$$

A little algebra gives

$$\frac{b_{n+1}}{b_n} = \Big(1 + \frac{1}{n}\Big)^{-n}$$

which converges to $1/e$ hence $b_n^{1/n}$ also converges to $1/e$.

Remarks: We may rewrite (2.8) as

$$(n!)^{1/n} \sim n/e \tag{2.9}$$

It is tempting to raise both sides to the power n and write

$$n! \sim n^n e^{-n} \tag{2.10}$$

However (2.10) is WRONG. The reason is that since $(p(n))^{1/n} \longrightarrow 1$ as $n \to \infty$ for any polynomial $p(x)$ in place of (2.9) we may also write

$$(n!)^{1/n}(p(n))^{1/n} \sim n/e \tag{2.11}$$

The CORRECT replacement for (2.10) is given by the following remarkable result going back to James Stirling in his *Methodus Differentialis* published in 1730. The result is definitely the marvelous theorems in classical analysis. Its importance was immediately recognized and numerous proofs as well as generalizations appeared. It was used by Laplace in his monumental treatise on Probability theory in establishing the De Moivre Laplace theorem. A good starting point to learn about these topics is the book of H. L. Rietz, Mathematical Statistics.

A proof of Stirling's formula due to Robbin's is given in Feller's book. We shall discuss Robbin's proof later in the course.

Theorem (Stirling's Approximation Formula, James Stirling, Methodus Differentialis, 1730) For large values of n we have

$$n! \sim n^n e^{-n}\sqrt{2\pi n}$$

More precisely

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1 \tag{2.12}$$

Exercises:

1. Show that the sequence (a_n) given by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

   is monotone increasing and bounded above. Further show that the limit lies between $1/2$ and 1.

2. Is the sequence (a_n) given by

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

   monotonic for any real x? Denoting by $f(x)$ the limit of this sequence, show that $f(x+y) = f(x)f(y)$.

3. Let a and α be a positive real numbers. How would you define

$$a^{\alpha}$$

and establish the basic rules of indices? Remember we have not yet introduced the notion of logarithms.

4. A dyadic rational number is a rational number a/b where $a, b \in \mathbb{Z}$ and $b > 0$, such that the denominator b is a power of 2. Show that the dyadic rationals are dense in $\mathbb{R}$. Explicitly find a sequence of dyadic rationals converging to $1/3$.

5. Show that if (a_n) is a monotone increasing/decreasing sequence of real numbers then the sequence of arithemtic means (A_n) given by

$$A_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

is also monotone increasing/decreasing. What can you say about the sequence of geometric means?

6. Let k be a fixed positive natural number and (a_n) be a sequence of positive reals converging to l then show that

$$a_n^{1/k} \longrightarrow l^{1/k}.$$

7. Let a and b be two positive real numbers. Define two sequences (a_n) and (b_n) inductively as follows. Set $a_1 = a, b_1 = b$. For $n \geq 2$

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), \quad b_n = \sqrt{a_{n-1}b_{n-1}}$$

Discuss the convergence of the sequences (a_n) and (b_n).

Limit Superior and limit inferior: Throughout this section we shall work with a given a real sequence (a_n).

Definition: We say that the sequence (a_n) converges[1] to $+\infty$ if for any real number $T > 0$, there exists an n_0 such that

$$a_n > T, \qquad \text{for all } n \geq n_0.$$

Likewise we say (a_n) converges to $-\infty$ if for any real number $T < 0$, there exists an n_0 such that

$$a_n < T, \qquad \text{for all } n \geq n_0.$$

Exercise: Suppose that a real sequence (a_n) does not converge to $-\infty$ and is bounded above, then show that there is a subsequence which converges to a finite limit.

Definition (Limit Superior): Now suppose that the given sequence (a_n) is unbounded above, we declare the limit superior of (a_n) to be $+\infty$. If on the othe hand the sequence (a_n) converges to $-\infty$ we simply declare the limit superior of (a_n) to be $-\infty$. Turning now to the remaining case we know that there are subsequences that converge to finite limits. So let

$$E = \{p \in \mathbb{R} \ : \ \text{some subsequence of } (a_n) \text{ converges to } p\}$$

Then the supremum of E is called the limit superior of (a_n).

We now provide an alternate formulation of the notion of limit superior. For this we introduce for each $n \in \mathbb{N}$,

$$M_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

Then notice that the sequence (M_n) is monotone decreasing. Each M_n is $+\infty$ if and only if the sequence (a_n) in unbounded above and we see that (M_n) converges to $+\infty$ which is the limit superior of the given sequence.

Exercise: Show that if the sequence (a_n) converges to $-\infty$ then so does (M_n).

The case that remains is when (a_n) is bounded above and does not converge to $-\infty$ in which case $\sup E$ is finite and by the exercise above l is also finite. Denoting $\sup E$ by λ and by l the limit of the sequence (M_j) we shall show that $\lambda = l$. Let $\epsilon > 0$ be arbitrary. There exists a $j_0 \in \mathbb{N}$ such that

$$M_j < l + \epsilon, \qquad \text{for all } j \geq j_0.$$

Thus

$$\sup\{a_{j_0}, a_{j_0+1}, \dots\} < l + \epsilon.$$

[1]Note that the use of the phrase "*converges to plus infinity*"is consistent with the notion of convergence in point-set toplogy since the stated notion is precisely convergence in the extended real line - with its usual topology making it homeomorphic to $[0, 1]$.

This means there can only be finitely many terms of the sequence (a_n) that exceed $l+\epsilon$ and so every convergent subsequence must have a limit less than or equal to $l+\epsilon$. Thus $l+\epsilon$ is an upper bound for E and so $\lambda \leq l+\epsilon$. Since $\epsilon > 0$ was arbitrary, we get

$$\lambda \leq l.$$

Suppose the inequality is strict. Then choose $\epsilon > 0$ such that

$$\lambda < l - \epsilon$$

Select n_1' such that

$$l \leq M_{n_1'} < l + \epsilon/2.$$

Since $l - \epsilon/2 < M_{n_1'}$, there exists $n_1 \geq n_1'$ such that

$$l - \epsilon/2 < a_{n_1} \leq M_{n_1'} < l + \epsilon/2$$

We now select $n_2' > n_1$ such that

$$l \leq M_{n_2'} < l + \epsilon/4$$

and as above an $n_2 \geq n_2'$ such that

$$l - \epsilon/4 < a_{n_2} \leq M_{n_2'} < l + \epsilon/4$$

and continuing thus construct a subesequence (a_{n_k}) converging $l > \lambda$ which gives a contradiction. Thus in all cases (that is including the ones where $\lambda = \pm\infty$) we have $\lambda = l$.

Exercise: Formulate the notion of limit inferior and carry out a detailed analysis as above. Show that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

Further show that equality holds if and only if (a_n) converges (cases of convergence to $\pm\infty$ are included).

Limit superior and limit inferior of sets: This is a very useful notion in measure theory and probability theory and this seems to be a good place for discussion. Assume given a sequence of subsets (A_n) of some universal set X. The supremum of a family of sets ought to be the smallest set that contains all the given sets namely the union of the family. Likewise the infimum of a family of sets ought to be the largest set contained in each member of the family. Thus

$$M_j = \sup\{A_j, A_{j+1}, A_{j+2}\ldots\} = \bigcup_{n=j}^{\infty} A_n$$

The sets M_j decrease as j increases and we define

$$\limsup_{n\to\infty} A_n = \bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty} A_n.$$

Similarly we define the limit inferior of the sequence of sets (A_n) to be

$$\liminf_{n\to\infty} A_n = \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} A_n.$$

Exercise: Show that $\limsup A_n$ is the set of points in X that lie in infinitely many of the sets A_n and $\liminf A_n$ is the set of points of X that lie in all but finitely many of the A_n. Thus

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$$

The Ratio test and root test: We state and prove two of the most important results on series. Although these are weak tests, they are convenient for computing the radius of convergence of power series.

Theorem (D'Alembert's Ratio Test): (i) Suppose that (a_n) is a sequence of non-zero complex numbers and

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

then the series $\sum a_n$ converges absolutely.

(ii) If on the other hand

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1$$

for ALL BUT FINITELY MANY n then the series $\sum a_n$ diverges.

Remark: We see here the elegance and precision of Rudin's writing. Reformulation of the second part in the language of liminf is not useful since that would be subsumed in the above formulation. Nevertheless try it out as an excuse to review liminf. Suppose that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

then the series diverges. In particular if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists and has a value greater than one or plus infinity[2] then the series diverges.

If

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$$

the test is in-conclusive.

[2]Remember the convention that when dealing with real sequences we considered "a sequence converging to plus infinity".

Proof: (i) Call the limit superior α and take any number β such that $\alpha < \beta < 1$. Then there can only be finitely many values of n for which

$$\left|\frac{a_{n+1}}{a_n}\right| \geq \beta$$

So there is an $n_0 \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta, \quad n \geq n_0$$

From this it follows that

$$|a_n| \leq |a_{n_0}|\beta^{n-n_0} = c\beta^n, \quad n \geq n_0$$

for some positive constant c. Since $0 < \beta < 1$, we infer that $\sum |a_n|$ is dominated by a convergent geometric series from which the conclusion follows.

(ii) If on the other hand there is an n_0 such that

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1, \quad n \geq n_0$$

then multiplying out $(n - n_0)$ of the above inequalities we get

$$|a_n| \geq |a_{n_0}|, \quad n \geq n_0$$

which means the $n-$th term of the series $\sum a_n$ does not go to zero and the series cannot converge.

Exercises on sequences and series:

1. Construct a number in $[0, 1]$ whose decimal expansion has the property that only the digits 0 and 1 appear and further if k_n is the number of occurrences of 1 in the first n digits,
$$\limsup_{n\to\infty} \frac{k_n}{n} = 1, \quad \liminf_{n\to\infty} \frac{k_n}{n} = 0.$$
Find a number again only with zeros and ones as digits but with limsup being $2/3$ and lininf being $1/3$.

2. Prove that $\left(\log\left(1 + \frac{1}{n}\right)\right)^{1/n} \longrightarrow 1$ as $n \longrightarrow 1$.

3.

4. Use the result $1 + z + z^2 \cdots + z^n = (z^{n+1} - 1)/(z - 1)$ to sum the series (to n-terms)
$$1 + \cos\theta + \cos 2\theta + \ldots, \quad \text{and} \quad \sin\theta + \sin 2\theta + \ldots.$$
Show that the partial sums of the series $\sum \cos n\theta$ and $\sum \sin n\theta$ are bounded when $0 < \theta < \pi/2$.

5. Show that if (a_n) is monotone decreasing positive and converges to zero as $n \to \infty$ then the series

$$\sum \frac{(-1)^n}{n}\Big(a_1 + a_2 + \cdots + a_n)$$

converges.

6. Suppose $\sum a_n$ and $\sum b_n$ are two series of positive real numbers such that

$$\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}, \qquad \text{for all} \;\; n \in \mathbb{N}$$

Show that if $\sum b_n$ converges then $\sum a_n$ also converges.

7. Show that if $\sum a_n$ is a series of positive reals such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{p}{n} + \frac{k_n}{n^2}$$

where (k_n) is a bounded sequence then $\sum a_n$ converges if $p < -1$ and diverges if $p \geq -1$. Hint: Take $\sum b_n$ appropriately.

8. Let $p_1, p_2, \dots p_k$ be the first k primes. Using the expression

$$\frac{1}{1-x} = 1 + x + x^2 + \dots, \quad 0 < x < 1$$

, show that

$$\frac{1}{1-\frac{1}{p_1}}\frac{1}{1-\frac{1}{p_2}}\cdots\frac{1}{1-\frac{1}{p_k}}$$

is precisely the sum of $1/n$ where n ranges over 1 and all those natural numbers whose prime factors are powers of $p_1, p_2, \dots, p_k$. Deduce that if we assume that there are only finitely many primes then the above expression would equal

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

thereby arriving at a contradiction. This proof of the infinitude of primes is due to Euler.

9. Show that if $k \in \mathbb{N}$ and $k \geq 4$.

$$\frac{(kn)!}{(n!)^k} \sim \frac{C}{n^{(k-1)/2}}$$

for some constant C. Hence prove that the power series

$$\sum_{n=1}^{\infty} \frac{(kn)! z^n}{(n!)^k}$$

converges everywhere on the boundary of the disc of convergence.

Special functions of mathematical analysis:

Exercises: These have to be done from first principles starting from the definition of the exponential function as an infinite series and the exponential addition theorem. The functions $\exp x$, $\sin x$ and $\cos x$ are all defined via infinite series.

1. Prove that for all $x \in \mathbb{R}$,
$$\exp x = \left(1 + \frac{x}{n}\right)^n.$$
Show further, that if $k \in \mathbb{N}$ then $\exp(1/k)$ is the unique positive k-th root of e. For rational values of x we thus have $\exp x = e^x$. For x irrational define
$$S = \{e^q \ : \ q \in \mathbb{Q}, \ q < x\}$$
and define $e^x = \sup S$. Show that $e^x = \exp x$.

2. Prove that if $x < y$ then
$$0 < \exp y - \exp x \leq (y - x) \exp y$$
Hint: Begin by estimating the difference
$$\left(1 + \frac{y}{n}\right)^n - \left(1 + \frac{x}{n}\right)^n$$

3. Prove that if $a > 0$ then there exists a unique $b \in \mathbb{R}$ such that $\exp b = a$. Define
$$S = \{x \in \mathbb{R} \ : \ \exp x < a\}$$
Then S is non-empty and bounded above since $\exp x$ tends to 0 as $x \longrightarrow -\infty$ and $\exp x \longrightarrow \infty$ as $x \longrightarrow \infty$. Let $b = \sup S$ and use the preceeding to prove that $\exp b > a$ and $\exp b < a$ both lead to contradictions. Deduce that
$$\exp \ : \ \mathbb{R} \longrightarrow (0, \infty)$$
is surjective as well as injective

   Definition: The b obtained above is called the logarithm of a and denoted by $\ln a$ or $\log a$.

4. Use the exponential addition theorem to prove that
$$\log(a_1 a_2) = \log a_1 + \log a_2, \quad a_1 > 0, \ a_2 > 0.$$
Thus we have a function $\log \ : \ (0, \infty) \longrightarrow \mathbb{R}$ which is the inverse of the exponential function.

5. Prove that $\exp x > 1 + x$ for $x > 0$ and hence

   (i) $\log(1+x) < x$ for $x > 0$.

   (ii) $x - \log(1+x) < x^2$ for $0 < x \leq 1/2$. Hint: For the second, we need to show

   $$\exp x < (1+x)\exp x^2, \qquad \text{for all } \ 0 < x \leq 1.$$

   and clearly, it is enough to show $\exp x < 1 + x + x^2$ if $0 < x < 1$.

   Well, $(1+x)\exp x^2 = (1 + x + x^2 + \dots)$ where the dots refer to positive quantities.

   Next,

   $$\exp x = 1 + x + x^2 + \frac{x^2}{2}\Big(\frac{x}{3} + \frac{x^2}{3\cdot 4} + \dots\Big) - \frac{x^2}{2}$$

   The series within paranthesis is dominated by the geometric series

   $$\frac{x}{3} + \frac{x^2}{3^2} + \cdots = \frac{x}{3-x}.$$

   It is enough to show

   $$x/(3-x) < 1, \quad 0 < x < 1.$$

6. Use the above result to discuss the convergence of

   $$\sum_{n=1}^{\infty}\Big(\frac{1}{n} - \log\Big(1 + \frac{1}{n}\Big)\Big)$$

7. Prove that the sequence (a_n) given by

   $$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1)$$

   is monotone increasing.

8. Show that the sequence (a_n) defined above is bounded. Try employing the grouping idea used to prove that the harmonic series diverges. This will just about miss the target ! Try using the inequality $x - \log(1+x) < x^2$ for $0 < x \leq 1/2$

   Definition: The limit

   $$\lim_{n\to\infty}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1)\Big)$$

   is called Euler's constant. It is denoted by γ and as of today it is not known whether γ is rational or irrational.

9. Show that the sequence

   $$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

   converges to $\log 2$. Hint: Use the previous exercise.

10. Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$$

11. Discuss whether the series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

converge and if the series is convergent, find its sum. Hint: Take first the sum of $3n$ terms and group them in brackets of threes and find the general term in the nth bracket.

12. Prove that

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \end{aligned}$$

13. Prove that

$$\begin{aligned} \cos x - \cos y &= 2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{y-x}{2}\right) \\ \sin x - \sin y &= 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right) \end{aligned}$$

Theorem: (i) If $0 < x < \sqrt{6}$, $\sin x > 0$.
(ii) If $0 \le x \le \sqrt{2}$ then $\cos x > 0$.

Proof: The second is an exercise. The first one follows from the fact that each paranthesis in the following is positive

$$\sin x = x\Big(1 - \frac{x^2}{6}\Big) + \frac{x^5}{5!}\Big(1 - \frac{x^2}{6 \cdot 7}\Big) + \dots$$

14. Show that if $0 < x < \sqrt{20}$ then $\sin x < x$. Write out the series and do a re-grouping:

$$\sin x = x - \frac{x^3}{3!}\Big(1 - \frac{x^2}{4 \cdot 5}\Big) - \dots$$

Prove that $\sin x$ is strictly increasing and positive on the interval $(0, \sqrt{2})$ and $\cos x$ is strictly decreasing on $(0, \sqrt{2})$.

15. Prove that $\cos 2 < 0$. Again, group the series for $\cos 2$ appropriately:

$$\cos 2 = -1 + \frac{2^4}{4!}\Big(1 - \frac{4}{5 \cdot 6}\Big) + \frac{2^8}{8!}\Big(1 - \frac{4}{9 \cdot 10}\Big) + \dots$$

16. Show that there exists a real number $\omega \in (1,2)$ such that $\cos\omega = 0$. Look at

$$S = \{t > 0 \;:\; \cos x > 0 \text{ for all } \; x \in [0,t)\}$$

Note that $1 \in S$ and S is bounded above by 2. Let $\omega = \sup S$ and show that this does the job. You will need the fact established above that $\sin\eta < \eta$ for small positive values of η.

The number ω whose existence has been established is denoted by $\pi/2$.

17. Show that $\sin\pi/2 = 1$ and $\sin\pi = 0$ and hence $\cos\pi = \pm 1$. Prove the following:

$$\sin(x + \frac{\pi}{2}) = \cos x, \quad \cos(x + \pi/2) = -\sin x$$
$$\sin(\pi - x) = -\cos\pi\sin x, \quad \cos(\pi - x) = \cos x\cos\pi$$

Deduce that $\cos\pi = -1$.

18. Show that $\sin x$ is positive on $(0,\pi)$. Show that $\cos x$ is positive on $[0,\pi/2)$ and negative on $(\pi/2,\pi]$. Show that $\sin x$ and $\cos x$ are periodic with period 2π. Do they have any smaller positive periods?

19. **Definition:** The functions $\sinh z$ and $\cosh z$ are defined by

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$
$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$$

Show that these series converge absolutely for all complex values of z. Further $\sinh Z$ and $\cosh z$ are real when z is real. Prove that

$$\cos z = \cos x\cosh y - i\sin x\sinh y$$

where x and y are the real and imaginary parts of z. Find the real and imaginary parts of $\sin z$. Show that $\sin z$ and $\cos z$ have only real zeros.

20. Discuss the convergence and find the sum of the following series ($\theta \in \mathbb{R}$ in the following)

(a) $1 + \cos\theta + \frac{1}{2!}\cos 2\theta + \dots$ (b) $\sin\theta + \frac{1}{2!}\sin 2\theta + \dots$

(c) $1 + \frac{1}{4!}\cos 4\theta + \frac{1}{8!}\cos 8\theta + \dots$ (d) $\frac{1}{4!}\sin 4\theta + \frac{1}{8!}\sin 8\theta + \dots$ (E. W. Hobson)

21. Show that the coefficient of z^n in the expansion of $e^z\cos z$ is $\frac{\sqrt{2^n}}{n!}\cos\left(\frac{n\pi}{4}\right)$ Expand $\cos z\cosh z$ in powers of z. Hint: First prove that

$$\cos z\cosh z \mp i\sin z\sinh z = \frac{1}{2}\sum_{n=0}^{\infty} 2^{n/2}\{1 + (-1)^n\}\exp\Big(\pm\frac{n\pi i}{4}\Big)\frac{z^n}{n} \qquad \text{(G. H. Hardy)}$$

22. Show that for $0 < x \leq 1$, $\exp(-x) < 1 - \frac{x}{2}$. Replace x by $2x$ and we get

$$(1-x)^{-1} < e^{2x}, \quad 0 < x \leq 1/2$$

For a given $N \in \mathbb{N}$, let $p_1, p_2, p_3, \ldots p_k$ be the list of prime numbers that divide atleast one of the numbers $1, 2, \ldots, N$. Taking $x = p_1, p_2, \ldots, p_k$ in succession in the above inequality and multiplying out show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots$$

diverges.

Remark: This result goes back to Euler in a memoir published in 1737. Euler was aware that the n-th partial sum grows like $\log \log n$. Mertens sharpened this result in 1874. In this connection see the paper by M. B. Villarino, Merten's Proof of Merten's theorem.

Basic Topology

Definition (Compact Sets): Let (X, d) be a metric space. A subset A of X is said to be compact if every open cover of A has a finite subcover.

Theorem: A compact subset of a metric space is closed.

Proof: Suppose A is a compact subset of a metric space (X, d) and p is a limit point of A. Select a sequence (a_n) of distinct points of A converging to p. We shall assume $p \notin A$ and arrive at a contradiction. The sets

$$S_n = \{p\} \cup \{a_n, a_{n+1}, a_{n+2}, \dots\}, \quad n = 1, 2, 3, \dots.$$

is closed and hence its complement $X - S_n$ is open for each $n = 1, 2, 3, \dots$. The family $\{X - S_n \ : \ n = 1, 2, 3, \dots\}$ is an open cover of A (why?). This collection has no finite subcover because the open sets $X - S_n$ increase with n and none contains A. We thus have a contradiction.

A Basic lemma: Suppose (X, d) is a metric space, Y is a closed subset of X with induced metric d_Y and A is a subset of Y. Then A is closed in (X, d) if and only if A is closed in (Y, d_Y).

Example: Let us consider the metric space $\mathbb{Q}$ with the metric induced from $\mathbb{R}$. Consider the set

$$S = \{x \in \mathbb{Q} \ : \ x > 0, \ 2 < x^2 < 3\}$$

Then S is closed in $\mathbb{Q}$ but not closed in $\mathbb{R}$. Thus the property of being closed depends heavily on the ambient space. Here $X = \mathbb{R}$, $Y = \mathbb{Q}$ is not closed in X. We shall show that in stark contrast compactness is a property that is independent of the ambient space which is one of the pleasant features of compactness.

Finite intersection property of sets: A family of sets $\{A_\alpha \ : \ \alpha \in \Lambda\}$ indexed by Λ is said to have the finite intersection property if every finite subcollection has non-empty intersection.

Theorem: Let (X, d) be a metric space. Then X is compact if and only if for every family $\mathcal{F}$ of closed subsets of X having the finite intersection property the intersection we have,

$$\bigcap_{C \in \mathcal{F}} C \neq \emptyset$$

Proof: This is simply a reformulation of the definition of compactness.

One would like to have a formulation of this result for the case when A is a subset of X. We shall do this after proving the following result.

Theorem: Suppose (X, d) is a metric space and Y is a subset of X endowed with the induced metric d_Y and $A \subset Y$. Then A is compact in (X, d) if and only if A is compact in (Y, d_Y). Thus the notion of compactness is independent of the ambient space.

Proof: This is immediate from the fact to be proved that sequential compactness and compactness are identical notions for metric spaces.

The theorem can be directly proved by elementary arguments which we merely sketch leaving the details for the student. All you need is the fact that if a subset G of X is open in (X, d) then $G \cap Y$ is open in (Y, d_Y) and conversely any open set in Y is of the form $Y \cap G$ for some open set G in X (Exercise). Now suppose A is compact in Y. To show that A is compact in X take an arbitrary open cover $\{G_\alpha\}$ of A and look at the family $\{G_\alpha \cap Y\}$ which is a covering of A by open subsets of Y. Now extract a finite subcover and complete one half of the theorem. Similarly prove the converse.

Theorem: Let X be a metric space and A be a subset of X. Then A is compact if and only if for every family $\mathcal{F}$ of closed subsets of A having the finite intersection propery, we have

$$\bigcap_{C \in \mathcal{F}} C \neq \emptyset$$

Proof: The proof is left as an exercise. However we must insert an important note. We are vague here as to whether the sets in the family $\mathcal{F}$ are closed in X or closed in A with respect to the induced metric d_A. In one direction namely, if A is compact then A is closed in X and there is really no ambiguity by the basic lemma. In the converse direction, if we assume that all the sets are closed in X then the argument is completely trivial whereas if the sets are all closed in A then we infer first that the metric space (A, d_A) is compact and so A is compact in X as well and again there is no issue. We shall say no more on this matter.

Theorem: Suppose K is a compact subset of a metric space X and A is a closed subset of K then A is also compact in X.

Proof: Thanks to the lemma it is immaterial whether A is closed in X or closed in K with respect to the induced metric. Let $\mathcal{G}$ be an open cover of A. Include in this family the open set $X - A$ and we get an open cover $\mathcal{G} \cup \{X - A\}$ for X and hence of K. Since K is compact we can extract a finte subcover $\mathcal{G}_0 \subset \mathcal{G} \cup \{X - A\}$ of K. From this finite subcover delete $X - A$ (in case $X - A$ belongs to it) and we get a finite subcover for A.

Theorem:

Definition (Sequential compactness): A subset A of a metric space is compact if and only if every sequence of points in A has a convergent subsequence converging to a point in A. This property is called the Bolzano Weierstrass property.

Remark: It is a fact that IN A METRIC SPACE compactness and sequential compactness are equivalent notions but we have to wait a while to see this completely.

Theorem (Compactness implies sequential compactness): Suppose the A is a compact subset of a metric space then A is sequentially compact.

Proof: First, A is closed in X. Let (a_n) be a sequence of points of A if this sequence has a convergent subsequence then its limit will automatically be in A since A is closed and the second part of the theorem is proved. We show that there is a convergent subsequence of (a_n). Suppose the given sequence has NO convergent subsequence then the sets

$$S_n = \{a_n, a_{n+1}, a_{n+2}, \dots\}, \quad n = 1, 2, 3, \dots.$$

are closed and their complements form an open cover for A without a finite subcover whereby we arrive at a contradiction.

Exercise: Suppose that (X, d) is sequentially compact and A is sequentially compact in X. Assume $\{G_n \ : \ n = 1, 2, 3, \dots\}$ is a sequence of open sets covering A then there is a finite subcover.

Hint: First it is enough to show that one of the sets

$$G_1, G_1 \cup G_2, G_1 \cup G_2 \cup G_3, \dots$$

contains A. Well suppose not. Pick a point $a_n \in A$ and not belonging to $G_1 \cup G_2 \cup \cdots \cup G_n$. We get a sequence (a_n) in A which must have a convergent subsequence (a_{n_k}) converging to say $p \in A$. Then since the family $\{G_n \ : \ n \in \mathbb{N}\}$ covers A, one of them say G_N contains p. Then there exists k_0 such that

$$a_{n_k} \in G_N \quad \text{for all } k \geq k_0.$$

Choose $k \geq k_0$ such that $n_k > N$. Then

$$a_{n_k} \in G_N \subset G_1 \cup G_2 \cup \cdots \cup G_{n_k}$$

which contradicts the choice of a_{n_k}.

Main Issue: We have shown that if A is sequentially compact then every countable cover of A has a finite subcover. We must now show that every open cover has a countable subcover. This theorem, known as the Lindelöf covering lemma, requires some preparations.

Definition (A finite ϵ net): Let X be a metric space and A be a subset of X. A finite subset E of A is called a finite ϵ net if every point of A is within ϵ distance from some point of E.

Theorem: Suppose A is a non-empty sequentially compact subset of a metric space X then for every $\epsilon > 0$ there is a finite ϵ net for A.

Proof: Suppose not. Then pick a point $a_1 \in A$. Then the singleton $\{a_1\}$ is not an ϵ net so there must be a point $a_2 \in A$ such that $d(a_1, a_2) \geq \epsilon$. Now the two element set $\{a_1, a_2\}$ is not a finite ϵ net and so there exists $a_3 \in A$ such that $d(a_1, a_3) \geq \epsilon$ and $d(a_2, a_3) \geq \epsilon$. Again the three element set $\{a_1, a_2, a_3\}$ is not an ϵ net and so we can continue the process. The sequence (a_n) so obtained is such that

$$d(a_m, a_n) \geq \epsilon, \quad \text{for all } n > m.$$

It is clear that this sequence cannot have a convergent subsequence contradicting the hypothesis that A is sequentially compact.

Definition (Separable metric space): A metric space (X, d) is separable if it has a countable dense subset. That is to say if it has a countable subset D such that $\overline{D} = X$.

Example: (i) In $\mathbb{R}$, the countable set $\mathbb{Q}$ is dense and so $\mathbb{R}$ with the usual metric is separable.

(ii) Endow $\mathbb{R}$ with the discrete metric. Show that every set is closed and so there cannot be a countable dense subset. That is to say $\mathbb{R}$ endowed with the discrete metric is not separable.

(iii) Explain why the circle $\{z \in \mathbb{C} \ : \ |z| = 1\}$ is separable. Exhibit a countable dense subset. Here the circle inherits the metric from the ambient space $\mathbb{C}$.

Theorem: A sequentially compact metric space is separable.

Proof: This is very easy. Take $\epsilon = 1/n$. Then there is a finite $1/n$ net say E_n. Now show that the countable union of finite sets

$$D = \bigcup_{n=1}^{\infty} E_n$$

is provides a countable dense subset.

Lindelöf covering lemma: In a metric space (X, d) let A be a sequentially compact subset. Then every open cover of A has a countable subcover.

Proof: Let D be a countable dense subset of A. Consider the set $\mathcal{B}$ of all open balls with centers at points of D and positive rational radii. The collection $\mathcal{B}$ is obviously countable. Now let $\mathcal{G}$ be a covering of A by open subsets of X. As an exercise, prove that for each point $p \in A$, there exists a $G \in \mathcal{G}$ and a $B_p \in calB$ such that

$$p \in B_p \subset G.$$

Thus there are balls in the collection $\mathcal{B}$ that are contained in atleast one member of $\mathcal{G}$. Take the subcollection $\mathcal{C} \subset \mathcal{B}$ of ALL such balls. Then $\mathcal{C}$ is also countable. Enumerate the members of $\mathcal{C}$ as a sequence:

$$B_1, B_2, \ldots$$

Now for each n pick one open set G_n from $\mathcal{G}$ such that $B_n \subset G_n$. Thus we get a countable subcollection $\mathcal{H}$ out of $\mathcal{G}$. This subcollection $\mathcal{H}$ then covers A (why?).

Corollary (Sequential compactness implies compactness): In a metric space, a sequentially compact subset is compact.

Theorem (Existence of Lebesgue Number): Let X be a metric space and A be a compact subset of X. Then every open cover of A has a Lebesgue number.

Proof: Suppose there is an open cover $\mathcal{U}$ for which there is no Lebesgue number. In particular $1/n$ is not a Lebesgue number. So there is a ball $B_{1/n}(x_n)$ centered at a point of A that is not contained in any open set in $\mathcal{U}$. We thus obtain a sequence (x_n) (consisting of the centers of these open balls). Extract a convergent subsequence (x_{n_k}) converging to say $p \in A$. Now p must lie in some open set U and then there exists $r > 0$ such that

$$p \in B_r(p) \subset U$$

and there exists $k_0 \in \mathbb{N}$ such that $1/k_0 < r/3$ and

$$x_{n_k} \in B_{r/3}(p) \subset U, \quad k \geq k_0.$$

Now we shall arrive at the contradiction that for $k \geq k_0$, $B_{1/n_k}(x_{n_k}) \subset U$. Well, if $z \in B_{1/n_k}(x_{n_k})$,

$$d(z, x_{n_k}) < 1/n_k \leq 1/k \leq 1/k_0 < r/3, \quad k \geq k_0.$$

and $d(x_{n_k}, p) < r/3$ so that

$$d(z, p) < \frac{r}{3} + \frac{r}{3} < r$$

so that $z \in B_r(p) \subset U$.

1. Show that in the metric space $\mathbb{Q}$ with the usual metric (given by absolute value), the set

$$\{x \in \mathbb{Q} \ : \ x > 0, \ 2 < x^2 < 3\}$$

is both closed and open.

2. Determine the set of all limit points of

$$\left\{\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \ : \ k, m, n \in \mathbb{N}\right\}$$

3. How would you prove that the set

$$\left\{(x, y) \in \mathbb{R}^2 \ : \ \frac{x^2}{4} + \frac{y^2}{9} < 1\right\}$$

is open?

4. **Theorem:** In $\mathbb{R}$ with the usual metric, the only subsets of $\mathbb{R}$ that are simultaneously closed and open are $\emptyset$ and $\mathbb{R}$.

   Proof: The details are left as exercises. Here are the main points. Assume that A is a non-empty subset of $\mathbb{R}$ that is both open and closed. Then A cannot be a singleton (why?). We now show

   (i) A is an interval. That is if $p, q \in A$ then the closed interval $[p, q]$ is contained in A.

   (ii) $A = \mathbb{R}$.

   To prove (i) construct the set

$$S = \{t \in \mathbb{R} \ : \ p < t < q, \ (p, t) \subset A\}$$

   Then show that S is non-empty and bounded above by q. Let $l = \sup S$. Then $l \leq q$. Show that $l = q$ and then $(p, l) \subset A$ so that $[p, q] \subset A$.

   Remark: What we have really established is that if X is a set with a total order which is dense and Dedekind complete then X endowed with the order topology is connected. This fact is often useful. For instance if X is a compact connected metric space with exactly two non cut points then X is $[0, 1]$ (that is homeomorphic to the closed unit interval in $\mathbb{R}$).

5. Show that the only subsets of $\mathbb{R}^2$ that are simultaneously open and closed are $\emptyset$ and $\mathbb{R}^2$. Hint: Suppose A is a non-empty subset of $\mathbb{R}^2$. Pick $p \in A$ and assume that $A \neq \mathbb{R}^2$. Then there exists $q \in A^c$. Now look at the line segment joining p and q and adapt the idea used in the previous exercise.

6. Here is another example employing the same ideas. Show that every odd degree monic polynomial has a real root. Let $f(x)$ be a monic odd degree polynomial. First show that for any polynomial (odd or even degree), given any x_0 there exists $r > 0$ and $M_{x_0} > 0$ such that

$$|f(x) - f(x_0)| \leq M_{x_0}|x - x_0|, \quad |x - x_0| < r$$

Deduce that if $f(x_0) > 0$ there is an interval $(x_0 - \delta, x_0 + \delta)$ throughout which $f(x)$ stays positive. Now let

$$S = \{t \in \mathbb{R} \,:\, f(x) > 0 \text{ for all } x > t\}$$

Show that S is bounded below and $x_0 = \inf S$. Then $f(x_0) = 0$.

7. Prove that if A is open and B is an arbitrary subset of $\mathbb{R}^n$ then

$$A + B = \{x + y \,:\, x \in A,\ y \in B\}$$

is open. Show that if A and B are closed subsets of $\mathbb{R}$ then $A + B$ need not be closed.

8. Show that if A and B are closed subsets of $\mathbb{R}^n$ and one of them is compact then $A + B$ is closed.

9. Let C be the Cantor set. Show that the set

$$C + C = \{x + y \,:\, x, y \in C\}$$

is the interval $[0, 2]$.

10. Find an open cover for $\mathbb{R}$ that has no Lebesgue number.

11. Think of an $n \times n$ matrix with real entries as a point in $\mathbb{R}^{n^2}$. Thus the set of all $n \times n$ matrices (this set is denoted by $M_n(\mathbb{R})$) is a metric space. Show that the set of all orthogonal matrices is compact.

12. Show that the set of all invertible $n \times n$ matrices with real entries (this set is denoted by $GL_n(\mathbb{R})$ is an open set in $M_n(\mathbb{R})$.

13. Show that a connected subset of a metric space with atleast two points must be uncountable.

14. A subset K of $\mathbb{R}^n$ is convex if given $x, y \in K$,

$$tx + (1 - t)y \in K, \qquad \text{for all } 0 \leq t \leq 1.$$

Show that the interior of the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{9} < 1$$

is convex. Show that a convex subset of $\mathbb{R}^n$ is connected.

15. Suppose a, b, c are positve real numbers such that $a + b \geq c$ then show that

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c}$$

Prove that if (X, d) is a metric space and $D : X \times X \longrightarrow \mathbb{R}$ is given by

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric. Prove that any set that is open with respect to D is also open with respect to d and vice-versa.

16. Let (X, d) be a metric space and $D : X \times X \longrightarrow \mathbb{R}$ is given by

$$D(x, y) = \min\{1, d(x, y)\}$$

Show that D is also a metric. Any set that is open with respect to D is also open with respect to d and vice-versa.

17. **Distance between two sets:** Let A and B be subsets of a metric space. The distance between A and B denoted by $d(A, B)$ is defined as:

$$\inf\{d(a, b) \ : \ a \in A, \ b \in B\}$$

In particular if B is a singleton set we talk of the distance between a point and a set.

Show that if $x, y \in X$ then

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Find two disjoint closed subsets A and B of $\mathbb{R}^2$ such that $d(A, B) = 0$. Find such a pair in $\mathbb{R}$.

18. Show that if A is closed and p is a point of X then p lies in A if and only if $d(p, A) = 0$. Show that the result fails if A is not closed. If A and B are disjoint closed sets then

$$d(x, A) + b(x, B)$$

is never zero. Thus we have a function $f : X \longrightarrow \mathbb{R}$

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

with the following properties:

(i) $0 \leq f(x) \leq 1$

(ii) $f(x) = 0$ if and only if $x \in A$

(iii) $f(x) = 1$ if and only if $x \in B$

(iv) $f(x)$ is continuous (we shall discuss this aspect in the next chapter).

Continuity

Theorem (Tietze's extension theorem): Suppose A is a closed subset of a metric space X and $f : A \longrightarrow \mathbb{R}$ is a continuous function then there exists a continuous function $F : X \longrightarrow \mathbb{R}$ such that

$$F\big|_A = f$$

that is to say f has a continuous extension to whole of X. Further, in case f is bounded above/below, then so is F with the same bounds.

Note that it is essential that A be closed. For example the function $f : (0, \infty) \longrightarrow \mathbb{R}$ given by

$$f(x) = \sin(1/x)$$

has no continuous extension to $\mathbb{R}$. We shall not prove the Tietze's extension theorem here. It will be proved in the general topology course.

We now prove a useful lemma

Pasting lemma: Suppose $\{G_\alpha\}_{\alpha \in \Lambda}$ is a collection of open subsets of a metric space X and for each α, assume we have a continuous function $f_\alpha : G_\alpha \longrightarrow Y$ such that

$$f_\alpha(x) = f_\beta(x), \quad x \in G_\alpha \cap G_\beta$$

whenever $G_\alpha \cap G_\beta$ is non-empty. Then there exists a continuous function

$$f : \bigcup_{\alpha \in \Lambda} \longrightarrow Y$$

such that for each $\alpha \in \Lambda$,

$$f\big|_{G_\alpha} = f_\alpha$$

Locally finite collection: A family of (distinct) subsets $\{S_\alpha\}_{\alpha \in \lambda}$ of a metric space is said to be locally finite if for each point p there is a open ball $B_r(p)$ such that

$$B_r(p) \cap S_\alpha$$

is non-empty for only finitely many α.

Is the set of all singletons locally finite? Is the set of all intervals of length 2 centered at points of $\mathbb{Z}$ locally finite? Show that if each set S_α in a locally finite collection is closed then the union is also closed.

Exercises on continuity:

1. Show that $1 - e^{-x} < x$ for all $x > 0$. Note that the inequality is trivial if $x \geq 3$.

2. Let $\zeta : (1, \infty) \longrightarrow \mathbb{R}$ be given by
$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$
Show that the function ζ is continuous. Hint: $n^{-x} = \exp(-x \log n)$

3. Show that the map $\exp : \mathbb{R} \longrightarrow (0, \infty)$ is surjective.

4. Discuss injectivity and surjectivity of the functions sin and cos on the intervals $[-\pi, \pi]$, $(0, \pi/2)$, $[0, \pi/2]$, $[0, \pi]$ and $(0, \pi)$ with the codomain $[-1, 1]$ for all cases.

5. Show that the function $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ is not surjective. Show that it misses the value zero and no other value. The little Picard theorem states that an entire function that misses two or more values is a constant.

6. Use the little Picard's theorem to show that the functions sin and cos assume all complex values.

7. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$f(x) = x \sin(1/x), \quad x \neq 0,$$
and $f(0) = 0$ is continuous but not Lipschitz.

8. Examine whether the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$f(x) = |x|^{1/3}, \quad x \neq 0,$$
is Lipschitz.

9. Show that if a continuous function has a set of periods having zero as a limit point then the function must be identically zero.

10. The set of all real valued continuous functions on $\mathbb{R}$ forms a vector space. Are the functions $\cos x$, $\sin x$, $\cos \sqrt{2}x$ and $\sin \sqrt{2}x$ linearly independent? Is the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$f(x) = \sin x + \sin \sqrt{2}x$$
periodic?

11. Use the existence of Lebesgue number for open covers of a compact metric space to prove that a continuous function on a compact domain is uniformly continuous.

12. Show that every open set in a metric space is an F_σ set and every closed set is a G_δ set. For any closed set F in $\mathbb{R}$ is there a continuous function which is continuous at points of F and discontinuous at points in $\mathbb{R} - F$?

13. Find a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that the image of $\mathbb{Z}$ is $\mathbb{Q}$.

14. show that a connected metric space with at least two points must be uncountable.

15. Let A be a subset of a metric space. Show that the function $f : X \longrightarrow \mathbb{R}$ given by

$$f(x) = \operatorname{dist}(x, A)$$

is continuous. Is this Lipschitz (in the case when $X = \mathbb{R}^n$?)

16. Suppose X is a metric space and A and B are disjoint closed sets in X then there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Hint: You must consult one of the exercises in the last chapter. Deduce that there are disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

17. Suppose X is a metric space and A and B are non-empty subsets of X such that $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. Then show that there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Hint: Look at the subset $Y = X - \overline{A} \cap \overline{B}$ which contain the sets A and B. The sets A and B have disjoint closures in Y and so we get open sets U and V in Y as in the preceeding exercise.

    Conversely show that if A and B are non-empty subsets of X such that there exists disjoint open sets U and V containing A and B then $\{A, B\}$ is a disconnection of $A \cup B$.

18. Suppose $\{A_n\}$ is a sequence of connected subsets such that $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$ then $\displaystyle\bigcup_{n=1}^{\infty} A_n$ is connected.

19. Show that a decreasing sequence of non-empty compact and connected sets is compact and connected. I have indicated a proof in class and said that we shall return to it. Now prove this using the Tietze's extension theorem.

20. Show that there is a function from $\mathbb{R}$ to $\mathbb{R}$ which is discontinuous at each rational and continuous at each irrational.

21. Show that it is impossible to construct a function from $\mathbb{R}$ to $\mathbb{R}$ which is continuous at rationals and discontinuous at irrationals. Use the Baire Category theorem.

22. Show that the pasting lemma also holds if instead of functions defined on open sets we have continuous functions defined on closed sets S_α such that on the overlaps $S_\alpha \cap S_\beta$ the functions agree and the collection of sets $\{S_\alpha\}$ is locally closed.

Let F be a closed subset of $\mathbb{R}$. Then there exists a function whose set of points of discontinuities is exactly F.

The easiest way to see this is to take a countable dense subset S of F and declare $f(x) = 1$ on S and zero outside S. Then since the function is zero on $\mathbb{R} - F$ and $\mathbb{R} - F$ is open, the function is continuous on $\mathbb{R} - F$. If now $p \in F - S$, pick a sequence of points in S converging to p and we see at once that f is discontinous at p. If $p \in S$ then since S has empty interior, we can find a sequence of points outside S converging to p and using this we againg see that f is discontinuous at p.

Cauchy's Mean Value theorem : Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b), then there exists a $c \in (a,b)$ such that

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)).$$

Proof: We select $h(x)=1$ in the corollary and consider the function $\phi : [a,b] \longrightarrow \mathbb{R}$ given by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & 1 \\ f(b) & g(b) & 1 \\ f(a) & g(a) & 1 \end{vmatrix}.$$

This is differentiable on (a,b) and $\phi(a)=\phi(b)$. Applying Rolle's theorem we conclude

$$\phi'(c) = \begin{vmatrix} f'(c) & g'(c) & 0 \\ f(b) & g(b) & 1 \\ f(a) & g(a) & 1 \end{vmatrix} = 0$$

for some $c \in (a,b)$. Expand the determinant and we get

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)).$$

The proof is complete.

Corollary (The L'Hospital's Rule): Suppose that f and g are differentiable at each point of (a,b) and for $p \in (a,b)$ assume that

(i) $f(p)=g(p)=0$, and $g(x)$ does not vanish at any other point $x \in (a,b)$.

(ii)
$$\lim_{x\to p} \frac{f'(x)}{g'(x)}$$
exists and equals l then
$$\lim_{x\to p} \frac{f(x)}{g(x)}$$
exists and equals l.

Proof: Let (x_n) be a sequence of points in (a,b) converging to p and $x_n \neq p$ for any n. Then we apply Cauchy's Mean Value theorem to the interval $[x_n, p]$ (or $[p, x_n]$) and we get

$$\frac{f(x_n)-f(p)}{g(x_n)-g(p)} = \frac{f'(c_n)}{g'(c_n)} \tag{1}$$

for some c_n lying strictly between x_n and p and so (c_n) also comverges to p. By hypothesis (ii) we see that the right hand side of (1) converges to l and so by virtue of (i)

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = l.$$

The proof is complete.

Convex functions: Convexity plays a central role in many parts of analysis. Many important and frequently used inequalities such as the inequality of arithmetic and geometric means are special cases of convexity of the exponential function. Yet another case in point is the characterization of the Gamma function due to Bohr and Mollerup (see Rudin). We gather here a few basic facts about convex functions of one variable that we shall occasionally use. The reference for this is Rudin, Principles of Mathematical Analysis. We begin with the definition

Definition: A real valued function defined on an interval I is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \quad x, y \in I,\ 0 \leq \lambda \leq 1.$$

The parabola $y = x^2$ serves as an excellent paradigm for a convex function. The reader is encouraged to draw the figure of this parabola and take three points P, Q and R in that order along the curve and conjecture the following:

Basic slope lemma: Suppose that P, Q and R are three points on the graph of a convex function with Q between P and R, and P to the left of Q. Then the following hold:

(i) Slope PQ $\leq$ Slope QR.

(ii) Slope PQ $\leq$ Slope PR.

(iii) Slope PR $\leq$ Slope QR.

Proof: Let p, q and r be in the interval I with $p < q < r$. The proof proceeds along predictable lines. Then (it may be useful to recall the section formula from elementary coordinate geometry)

$$q = \Big(\frac{r-q}{r-p}\Big)p + \Big(\frac{q-p}{r-p}\Big)r,$$

so that

$$f(q) \leq \Big(\frac{r-q}{r-p}\Big)f(p) + \Big(\frac{q-p}{r-p}\Big)f(r), \tag{1}$$

Now we subtract off $f(p)$ on both sides after noting that

$$f(p) = \Big(\frac{r-q}{r-p}\Big)f(p) + \Big(\frac{q-p}{r-p}\Big)f(p)$$

and we get

$$f(q) - f(p) \leq \Big(\frac{q-p}{r-p}\Big)(f(r) - f(p))$$

from which (ii) follows. Now to prove (i) we subtract off $f(q)$ from either side of (1) after writing

$$f(q) = \Big(\frac{r-q}{r-p}\Big)f(q) + \Big(\frac{q-p}{r-p}\Big)f(q)$$

and we get

$$0 \le \left(\frac{r-q}{r-p}\right)(f(p)-f(q)) + \left(\frac{q-p}{r-p}\right)(f(r)-f(q)),$$

which after rearrangement gives (i). Finally we subtract off $f(r)$ from either side of (1) after writing

$$f(r) = \left(\frac{r-q}{r-p}\right)f(r) + \left(\frac{q-p}{r-p}\right)f(r)$$

to get

$$f(q) - f(r) \le \left(\frac{r-q}{r-p}\right)(f(p)-f(r)),$$

which after rearrangement proves (iii).

We now proceed to derive all the basic properties of convex functions out of the basic slope lemma.

Theorem: Suppose $f : I \longrightarrow \mathbb{R}$ is convex then,

(i) f is continuous on I

(ii) The left and right hand derivatives of f exist at each point of I.

(iii) At each point $p \in I$, $f'(p-) \le f'(p+)$.

(iv) If $p < q$ then $f'(p+) \le f'(q-)$.

(iv) If f has one sided derivatives at each point of I and satisfies (iv) then f is convex.

(v) If f'' exists and is non-negative then f is convex. Conversely if f is convex and twice differentiable then f'' is non-negative.

Proof: Let $p \in I$ and we pick q, r ans s in the interval I such that $s < p < q < r$. The basic slope lemma gives

$$\frac{f(s)-f(p)}{s-p} \le \frac{f(q)-f(p)}{q-p} \le \frac{f(q)-f(r)}{q-r}$$

Fix s, r and let $q \to p+$ to get continuity from the right. Repeat the argument with $s < q < p < r$ and we get continuity from the left. To prove (ii),

Theorem (Support theorem):

Exercises:

1. Show that a convex function on an open interval is locally Lipschitz.

2. Show that if $F : I \longrightarrow \mathbb{R}$ is convex then the epi-graph

$$\{(x,y) \in \mathbb{R}^2 \; : \; y > f(x)\}$$

is convex.

3. Show that if ψ is a strictly increasing convex function and ϕ is convex then $\psi \circ \phi$ is convex.

4. Suppose $\{\phi_\alpha\}_\alpha$ is a family of convex functions on an interval I and $\phi = \sup_\alpha$ is finite on I then ϕ is also convex. Deduce that $-\log^+$ is convex on the real line.

5. Show that if f is convex on an interval I and $x_1, \ldots, x_k$ are k points in I and $\lambda_1, \ldots, \lambda_k$ are k non-negative reals such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$ then

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_k f(x_k)$$

6. Show that $\exp x$ is a convex function on the real line

7. Prove the inequality of means, namely if $a_1, a_2, \ldots, a_n$ are n positive numbers then

$$(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{1/n} \leq \frac{1}{n}\Big(a_1 + a_2 + \cdots + a_n\Big)$$

8. Show that if $u_1, u_2, \ldots, u_n$ are n non-negative numbers and $p_1, p_2, \ldots, p_n$ positive reals such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$ then

$$u_1 \cdot u_2 \cdot \cdots \cdot u_n \leq \frac{{u_1}^{p_1}}{p_1} + \cdots + \frac{{u_n}^{p_n}}{p_n}$$

9. Show that $-\sin$ is convex on $(0, \pi/2)$ and so

$$\sin x > \frac{2x}{\pi}, \quad 0 < x < \pi/2$$

This is known as Jordan's inequality. This will be needed in MA 412 in order to prove that

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x} = \pi$$

There the issue would be to prove that

$$\lim_{R\to\infty} \int_0^\pi \exp(-R\sin\theta)d\theta = 0$$

and Jordan's ienquality can be used.

10. Given a positive function f in an interval I, prove that if $e^{cx}f(x)$ is convex on I for every $c \in \mathbb{R}$ then $\log f(x)$ is convex. Prove also that $f(x)$ must be monotone increasing. (P. Montel) Note: Convexity of $\log f(x)$ is already a very powerful condition since log is a highly concave function. In particular if $\log f(x)$ is convex then $f(x)$ must be convex. The above problem is from the Berkeley Problems in Math.

11. Show that $\exp(\exp x)$, $\sec x$ and $\cosh x$ are examples of functions whose logarithms are convex. The following exercise provides yet another.

Exercises on derivatives:

1. Use the LMVT to prove that if $f : I \longrightarrow \mathbb{R}$ is differentiable throughout I and $f'(x) = 0$ for every $x \in I$ then f is constant. Deduce that if f and g are a pair of differentiable functions on I such that $f'(x) = g'(x)$ for every $x \in I$ then $f - g$ is a constant.

2. Prove that on $(-1, \infty)$, the derivative of $(1 + x)^\alpha$ is $\alpha(1 + x)^{\alpha-1}$.

3. Determine the intervals over which the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$f(x) = \frac{1 - x + x^2}{1 + x + x^2}$$
is increasing/decreasing. Over what intervals is it convex? What is the range of the function?

4. Consider the function $f(x) = x^3 - 3x$. Determine the number of elements in $f^{-1}(a)$ for various values of a. Determine the maxima and minima of f. Discuss similarly the function $f(x) = x^5 - 5x$.

5. We have seen that the Taylor polynomial $P_k(x)$ of $\log(1 + x)$ is
$$P_k(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k-1}x^k}{k}$$
and the remainder $R_k(x)$ is
$$\frac{(-1)^{k-1}x^{k+1}}{(k+1)(1+c)^{k+1}}$$
If $-1 < x < 0$ then we have no information on c to decide whether or not $R_k(x) \longrightarrow 0$ as $k \to \infty$. However if $0 < x < 1$ then prove that $R_k(x) \longrightarrow 0$ as $k \to \infty$.

6. Prove that on the interval $(-1, 1)$,
$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$
Hint: Differentiate and use exercise (1).

7. Use the same idea to prove that on the interval $(-1, 1)$,
$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

8. Prove that if $|x| < 1$,
$$(1 + x)^\alpha = 1 + \alpha x + \alpha(\alpha - 1)x^2/2! + \alpha(\alpha - 1)(\alpha - 2)x^3/3! + \ldots.$$

9. Prove that on the interval $(-1, 1)$,
$$(\tan^{-1}(x))^2 = x^2 - (1 + \frac{1}{3})\frac{x^4}{2} + (1 + \frac{1}{3} + \frac{1}{5})\frac{x^6}{3} - \ldots$$

10. Prove Leibnitz' theorem for the nth derivative of a product:

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

11. The function sin is strictly increasing on $(-\pi/2, \pi/2)$. Denote the inverse function by $\sin^{-1}$ with domain $(-1, 1)$. What is the derivative of $\sin^{-1}$? Compute the nth derivative of $\sin^{-1} x$ at the origin and write down the Taylor expansion. Show that the function $\sin(p \sin^{-1} x)$ satisfies the Tchebychev's ODE

$$(1 - x^2)y'' - xy' + p^2 y = 0$$

Hence deduce a series expansion for $\sin px$ as a series in powers of $\sin x$. Can you write down a formula for the nth Tchebychev's polynomial and its three term recursion formula? What can you say about the zeros of $T_n(x)$? Do the zeros of $T_{n+1}(x)$ interlace the zeros of $T_n(x)$?

12. Prove the following result. If I is an open interval and $f : I \longrightarrow \mathbb{R}$ is twice differentiable at some point $p \in I$,

$$f'(p) = 0, \quad f''(p) > 0$$

then show that f has a local minumum at p. Formulate and prove a corresponding sufficient condition for local maxima. Hint: The first derivative is strictly increasing at p and hence positive on $(p, p + \delta)$. Use LMVT on $[p, p + \delta]$.

13. Note that in the last exercise we only assumed that the function has a second derivative at p. Assume more, namely that f is twice differentiable throughout I and the second derivative is continuous. Then offer a simpler proof of the fact that if $f'(p) = 0$ and $f''(p) > 0$ then f has a local minimum at p.

14. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = \exp(-1/x^2), \quad x \neq 0, \quad f(0) = 0.$$

Show that the function f is differentiable infinitely often on $\mathbb{R}$. Does the Taylor series for f converge to f in a neighborhood of the origin?

15. Show that for any $k, m \in \mathbb{N}$,

$$\lim_{x \to \infty} x^m \frac{d^k}{dx^k} \exp(-x^2) = 0$$

16. Let $\mathcal{S}$ be the set of all C^∞ functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\lim_{x \to \infty} x^m \frac{d^k}{dx^k}(f(x)) = 0$$

for all non-negative integers m, k. Show that $\mathcal{S}$ is a vector space over the reals. Show that it is infinite dimensional by exhibiting an infinite linearly independent subset. Is the function $1/(\cosh x)$ in $\mathcal{S}$?

17. Show that if $p(x)$ is continuous throughout an open interval I and $y(x)$ is a non-trivial solution of

$$y'' + p(x)y = 0$$

then the zeros of $y(x)$ do not have a limit point in I.

18. Show that if u and v are two solutions of the ODE in the previous exercise then the function $W(x) = u(x)v'(x) - v(x)u'(x)$ is constant throughout I. Prove that $W(x)$ is zero if and only if u and v are linearly dependent.

19. Prove that if u and v is a pair of linearly independent solutions of the ODE

$$y'' + p(x)y = 0$$

where $p(x)$ is continuous on I, then between two successive zeros of u there is precisely one zero of v.

20. Imitate the above idea to show that if u and v are non-trivial solutions of a pair of ODEs

$$y'' + p(x)y = 0, \quad y'' + q(x)y = 0$$

where $q(x) > p(x)$ throughout I then between two successive zeros of u there is atleast one zero of v.

21. Use Rodrigues formula to prove that the nth Legendre polynomial has n distinct real roots in $[-1, 1]$. Hint: Use Rolle's theorem. Further deduce that between two zeros of $P_n(x)$ there is precisely one zero of $P_{n+1}(x)$.

22. Manipulating power series, obtain estimates of the form

$$\frac{1 - \cos x}{x} \leq ax, \quad \frac{x - \sin x}{x} \leq bx^2$$

valid for all $x > 0$ with positive constants a and b. Would it be easier to use Taylor series with remainder terms? Can the L'Hospital's rule be used for this purpose?

23. Compute using the L'Hospital's rule, $\lim_{x \to e} \frac{\log(\log x)}{\cos(\pi e/(2x))}$

24. Use the previous exercise to show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$$

is differentiable and its derivative is the same as the series of derivatives:

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Exercises on Integration

1. Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. Show that

$$\int_a^b f(t)dt$$

exists if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every partition P with mesh less than δ,

$$U(f, P) - L(f, P) < \epsilon.$$

Discuss whether the corresponding result hold for Stieltjes integrals with a monotone integrator $\alpha : [a, b] \longrightarrow \mathbb{R}$. Hint: Suppose the function is integrable. Then, begin with a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ for which $U(f, P) - L(f, P) < \epsilon/2$. Now pass to a refinement by enclosing the points t_k by tiny open intervals J_k. We then have two sets of open intervals (t_{j-1}, t_j) and J_k together forming an open cover for $[a, b]$. Let δ be a Lebesgue number for this cover.

2. Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable over $[a, b]$ then, in order to compute the integral it is enough to take a partition P_n of $[a, b]$ into n equal parts and calculate

$$\lim_{n\to\infty} U(f, P_n).$$

Use the last exercise. Could this result be proved without using the previous exercise? Is the previous exercise a convenient reformulation of the basic necessary and sufficient condition for integrability?

3. Prove that $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if there is a sequence of partitions P_n such that

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$$

4. Prove that

(i) $\displaystyle\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$

(ii) $\displaystyle\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right) = \frac{\pi}{4}$

5. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is non-negative, $f(c) > 0$ for some $c \in [a, b]$ and is continuous there. Show that

$$\int_a^b f(t)dt > 0.$$

6. If f is integrable over $[a, c]$ as well as $[c, b]$ then is it integrable over $[a, b]$. This has been done in BSc courses for Riemann integrals. Discuss whether this also holds for Stieltjes integrals.

7. Is the sum of the infinite series
$$\sum_{n=1}^{\infty} \frac{nx - [nx]}{n^2}$$
Riemann integrable over $[0, 1]$?

8. Show that the function $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ is NOT of bounded variation over $[0, 1]$.

9. Prove that if f and g are integrable functions over I and $f \leq g$ throughout I then
$$\int_I f \leq \int_I g$$
Discuss whether the result would hold for improper integrals and Stieltjes integrals of positive functions with respect to a monotone increasing function.

10. Suppose that $f : [0, \infty) \longrightarrow \mathbb{R}$ is Riemann integrable over $[0, T]$ for every $T > 0$ then the improper integral
$$\int_0^\infty f(t)dt$$
converges if and only given any $\epsilon > 0$ there exists an $n_0 > 0$ such that
$$\Big| \int_S^T f(t)dt \Big| < \epsilon, \qquad \text{for} \quad S > T > n_0.$$
Formulate and prove a comparison test for improper integrals.

11. Suppose that the improper intergal
$$\int_0^\infty f(t)dt$$
converges, show that it is not necessary that $f(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Recall that if an infinite series $\sum a_n$ converges then the nth term goes to zero. The corresponding result for improper integrals fails. Even if we assume that f is non-negative and the improper integral converges, it is still not true that $f(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

12. Show that the improper intergals
$$\int_\pi^\infty \frac{\sin t}{t} dt, \int_\pi^\infty \frac{\cos t}{t} dt, \int_\pi^\infty \frac{\sin t}{\sqrt{t}} dt, \int_\pi^\infty \frac{\cos t}{\sqrt{t}} dt$$
all converge but none absolutely. The issue here is that in the integral
$$\int_1^T \frac{\sin t}{t} dt$$
there are internal cancellations happening exactly as in the case of
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

keeping the partical sums in control. Demonstrate this by splitting the integral as a sum of integrals over $[\pi, 2\pi], [2\pi, 3\pi], \ldots$ and using the alternating series test. Show also that over of these intervals I_n the absolute value of the integral satisfies

$$\frac{a}{n} \leq \Big| \int_{I_n} \frac{\sin t}{t} \Big| \leq \frac{b}{n}$$

for certain fixed constants a and b.

13. Prove that the improper integrals in the last exercise converge using integration by parts over $[\pi, T]$. The reason why this works is that the integrand $(\sin t)/t$ can be written as

$$\int_\pi^T \frac{d}{dt}\Big(\int_\pi^t \sin u du\Big)\frac{dt}{t}.$$

and the integral

$$\int_\pi^t \sin u du$$

remains bounded while the other factor goes to zero. This idea can be carried over to the study of the infinite series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$$

Here (like the integral $\int_\pi^T \sin u du$ of the previous exercise) the partial sums

$$\sin x + \sin 2x + \cdots + \sin nx$$

remains bounded for $x \in (0, \pi/2)$ while the other factor $1/n$ or $1/\sqrt{n}$ goes to zero. To complete the job we need an analogue of the rule for integration by parts for infinite series. We do have such an analogue namely the formula for summation by parts (see below).

14. Let $\phi(s)$ be given by

$$\phi(s) = \int_0^\infty \exp(-st)\frac{\sin t}{t}$$

Show that ϕ is differentiable on $(0, \infty)$ and continuous on $[0, \infty)$. Compute the derivative of ϕ and hence determine the value of

$$\int_0^\infty \frac{\sin t}{t} dt.$$

For proving the continuity of ϕ apply the mean value theorem to the integral representing $\phi(s) - \phi(0)$. However be careful. The "c" in Lagranges Mean-Value theorem applied to the function $g(s) = \exp(-st)$ can depend on t. The integral from 0 to 1 poses no problems. On $[1, \infty)$ the way around the difficulty is to integrate by parts first a few times to obtain a t^3 in the denominator. The problem is typical of how conditionally convergent (and some oscillatory integrals ought to be dealt with).

15. Discuss the convergence of the improper integrals

$$\int_0^\infty \sin(t^2)dt, \quad \int_0^\infty \cos(t^2)dt$$

16. Suppose $f : [0, \infty) \longrightarrow \mathbb{R}$ is monotone decreasing and positive, prove that

$$\int_0^\infty f(t)dt, \quad \text{and} \quad \sum_{n=1}^{\infty} f(n)$$

behave alike. That is to say, either both converge or both diverge.

17. Suppose that f and g are integrable on an interval I (either as a proper Riemann integral, or an improper integral or as a Stieltjes integral with respect to a monotone function) and p, q are conjugate exponents then

$$\int_I |fg| \leq \Big(\int_I |f|^p\Big)^{1/p} \Big(\int_I |g|^q\Big)^{1/q}.$$

This is Hölder's inequality for integrals. You need to review the proofs of the the result for l^p and l^q spaces we proved earlier.

18. Formulate and prove a version of Minkowski inequality for integrals.

19. On the set $C[a, b]$ define for $1 \leq p < \infty$,

$$\|f\|_p = \Big(\int_a^b |f(t)|^p dt\Big)^{1/p}$$

Show that $d(f, g) = \|f - g\|_p$ is a metric on $C[a, b]$. Show that this metric space is not complete. Remark: It is unpleasant to work with incomplete metric spaces and one needs to complete them. The process is analogous to the construction of $\mathbb{R}$ from $\mathbb{Q}$ and the resulting space is unique (in a suitable sense). The resulting complete metric space is $L^p[a, b]$. But these would be studied in greater detail in MA 408 course.

20. Suppose that $f : [0, 1] \longrightarrow \mathbb{R}$ is Riemann integrable and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is convex then show that

$$\phi\Big(\int_0^1 f(t)dt\Big) \leq \int_0^1 (\phi \circ f)(t)dt$$

Hint: The result is completely trivial for the case of a "linear function" $\phi(t) = at + b$. Can you think of some theorem on convex functions that we can use? Also observe that the result holds if ϕ is defined on any open interval containing the range of f.

21. Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous (this can be weakened but we at the moment we are not interested in generalities) and $1 \leq p < q$ then

$$\|f\|_p \leq \|f\|_q \leq \sup\{|f(t)| \ : \ t \in [0, 1]\}$$

Further show that

$$\lim_{p \to \infty} \|f\|_p = \sup_{0 \leq t \leq 1} |f(t)|$$

This is the reason why the right hand side is denoted by $\|f\|_\infty$

22. Prove that
$$\int_0^{\pi/2} \sin^n \theta d\theta = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} \theta d\theta$$
Now use the pair of inequalities
$$\sin^{2n} \theta \le \sin^{2n-1} \theta, \quad \sin^{2n-1} \theta \le \sin^{2n-2} \theta$$
to obtain Wallis product formula for π:
$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\cdots$$
The infinite product is the limit of finite products obtained by truncations.

23. Explain how to compute
$$\int_0^{\infty} \exp(-x^2)dx$$
without involving double integrals. Hint: What happens to $(1 \pm \frac{x^2}{n})^n$ when n tends to infinity? Try out the intergal of one of these over $[0, \sqrt{n}]$. The other must be considered over $[0, \infty]$.

24. Compute the integral
$$I(s) = \int_0^{\infty} \exp(-ax^2) \cos(xs)\, dx,$$
where a and s are real positive. Obtain an ODE for $I(s)$.

25. Discuss for convergence the integral
$$\int_0^{\infty} \exp\Big(-t - \frac{1}{t}\Big)\frac{dt}{\sqrt{t}}$$
and show that the value of the integral is $\sqrt{\pi}e^2$.

26. Summation by parts. Prove the following result which is the discrete analogue of the rule for integration by parts.

Given two sequences (U_n) and (A_n),
$$\sum_{j=1}^{n} U_j(A_j - A_{j-1}) + \sum_{j=1}^{n} A_{j-1}(U_j - U_{j-1}) = U_nA_n - U_1A_1.$$
The result hardly warrents a proof. One can see this by inspection. Here are two applications

(i) Suppose that $\sum u_n$ is a series with bounded partial sums and (a_n) is a monotone decreasing sequence of positive reals converging to zero then the series $\sum a_nu_n$ converges.

(ii) Suppose that $\sum u_n$ is a convergent series and (a_n) is a monotone decreasing sequence of positive reals then the series $\sum a_nu_n$ converges.

Hints: Let U_n be the nth partial sum of $\sum u_n$. Apply the rule for summation by parts.

Use these results to prove that $\sum(\sin nx)/n$ converges for all $x \in (0, \pi/2)$ but at $\pi/4$ for instance the convergence is conditional.

27. Suppose that $\sum a_n x^n$ is a power series with unit radius of convergence and sum $f(x)$. Further assume that $\sum a_n$ converges to say s. Denoting by S_n the nth partial sum of $\sum a_n$, use the summation by parts formula to prove that

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n.$$

Next, since $(1-x)\sum x^n = 1$ multiplying by s and subtracting we get

$$f(x) - s = (1-x)\sum_{n=0}^{\infty}(s_n - s)x^n$$

Let $\epsilon > 0$ be arbitrary. Choose n_0 such that

$$|s_n - s| < \epsilon/2, \quad n \geq n_0.$$

Then,

$$f(x) - s = (1-x)\sum_{n=0}^{n_0}(s_n - s)x^n + (1-x)\sum_{n=n_0+1}^{\infty}(s_n - s)x^n$$

Let us estimate the second piece

$$|(1-x)\sum_{n=n_0+1}^{\infty}(s_n - s)x^n| \leq \frac{\epsilon}{2}(1-x)\sum_{n=n_0}^{\infty} x^n < \epsilon/2$$

Now we select $\delta > 0$ such that

$$|(1-x)\sum_{n=0}^{n_0}(s_n - s)x^n| < \epsilon/2, \quad |1-x| < \delta$$

so conclude that

$$\lim_{x \to 1-} f(x) = s = \sum a_n.$$

This is Abel's limit theorem.

28. Show that
$$1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log 2, \quad \text{and} \quad 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \pi/4$$

29. Improper Integral Test: Suppose $f : [1, \infty) \longrightarrow \mathbb{R}$ is a positive monotone decreasing function, then show that the infinite series $\sum f(n)$ converges if and only if the improper integral $\displaystyle\int_1^{\infty} f(t)dt$ converges.

Exercises: Uniform Convergence and Beta-Gamma functions.

1. Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be given by

$$f_n(x) = x^n(1-x)$$

   Does the sequence f_n converge uniformly?

2. Discuss for convergence the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\Big(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\Big).$$

   Prove that the sum of the series equals $\pi^2/16$.

3. Assume that $f_n : [a,b] \longrightarrow \mathbb{R}$ is a sequence of functions converging uniformly to $[a,b]$ and $\alpha : [a,b] \longrightarrow \mathbb{R}$ is monotone increasing. Show that $f_n \in \mathcal{R}(\alpha)$ for each n then $f \in \mathcal{R}(\alpha)$.

4. Does the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

   converge uniformly on compact subsets of $(0, \pi/2)$? Use the rule for summation by parts to show that the sequence of partial sums $S_n(x)$ is uniformly Cauchy. Thus $S_n(x)$ converges uniformly. Does $S_n'(x)$ even converge pointwise? By imitating this, can you formulate and prove a "uniform" version of the Dirichlet's test for convergence?

5. Discuss for convergence (absolute and uniform) the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x^2+n)}{n^2}$$

   Hinh: For any sequence of constant functions pointwise convergence implies uniform convergence !

6. Show that if a power series $\sum a_n(z-p)^n$ has radius of convergence R and converges absolutely on one point of its boundary then the series converges uniformly on the closed disc $\{z \in \mathbb{C} : |z-p| \le R\}$.

7. Consider the binomial series

$$(1-x)^{\alpha} = 1 - \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 - \ldots$$

   Show using Stirling's approximation formula, if $\alpha = 1/2$ then the series converges absolutely at $x = \pm 1$ and hence uniformly on $|x| \le 1$. Prove that on $[-1,1]$ there is a sequence of polynomials converging uniformly to $|x|$. Further prove that on any interval $[a,b]$ there is a sequence of polynomials converging to $|x-c|$ uniformly.

8. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a fixed partition of $[a, b]$ and V be the vector space of all piecewise linear continuous functions w.r.t P namely, continuous functions $f : [a, b] \longrightarrow \mathbb{R}$ such that the restriction of f to each $[t_{j-1}, t_j]$ is given by

$$f(x) = a_j x + b_j, \quad t_{j-1} \leq x \leq t_j.$$

Continuity demands that $a_j t_j + b_j = a_{j+1} t_j + b_{j+1}$ for $j = 1, 2, \ldots, n-1$. What is the dimension of the vector space V?

9. Show that

$$1, t, |t - t_j|, \quad j = 1, 2, 3, \ldots, n-1$$

belong to V. Do they form a basis for V?

10. Prove that any continuous, piecewise linear function on $[a, b]$ is the uniform limit of a sequence of polynomials. Deduce that any continuous function on $[a, b]$ is the uniform limit of a sequence of polynomials ! This is the famous Weierstrass's approximation theorem. Hint: If f is continuous on $[a, b]$ then uniform continuity implies for every $\epsilon > 0$ there exists a continuous piecewise linear function g such that

$$\sup_{a \leq x \leq b} |f(x) - g(x)| < \frac{\epsilon}{2}$$

Now approximate g by a polynomial using the preceeding exercises.

11. Show that the set of polynomials is dense in the metric space $C[a, b]$ with the sup norm. Further show that the metric space $C[a, b]$ is separable and complete. It is a fact that if X is a compact metric space then $C(X)$ is separable but we shall not discuss this here in this course.

12. Suppose that f is a continuous function on $[a, b]$ such that

$$\int_a^b f(t) t^n dt = 0, \quad n = 0, 1, 2, \ldots$$

then $f(t) = 0$ for all $t \in [a, b]$.

13. Let X be a compact metric space and M be the set of real valued continuous functions on X vanishing at a given point $p \in X$. Show that M is an ideal in the ring $C(X)$. Conversely show that if M is any maximal ideal in $C(X)$ then there exists $p \in X$ such that all the members of M vanish at p.

14. Suppose X is a compact metric space and $f : X \longrightarrow (0, 1)$ is continuous then the sequence of functions

$$1 - (1 - f)^n$$

converges uniformly to the constant function 1. Can you explain what is happening geometrically? How does the graph of $1 - (1 - t)^n$ look for large n?

Deduce that if $\mathcal{A}$ is an algebra of continuous functions separating points of X the the following are equivalent:

(i) $\mathcal{A}$ contains all the constant functions.

(ii) There is NO point $p \in X$ such that all the functions of $\mathcal{A}$ simultaneously vanish at p.

Thus the version of Stone's theorem given in Rudin is equivalent to the version we have proved in class.

15. Show that if $f_n : [0, \infty) \longrightarrow \mathbb{R}$ $(n = 1, 2, 3, \dots)$ is a sequence of functions converging uniformly to $f : [0, \infty) \longrightarrow \mathbb{R}$ then it is NOT necessary that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx$$

16. Show that if $a > 0$, the improper integral

$$\int_0^\infty e^{-t} t^{a-1} dt$$

converges and its value is denoted by $\Gamma(a)$. You need to worry about what happens near the lower end of the integral when $0 < a < 1$. Prove that if $a > 0$ then

$$\Gamma(a + 1) = a\Gamma(a), \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}.$$

17. Note that the function $\Gamma(x)\cos(4\pi x)$ also satisfies the properties of the last exercise. Hence it is not true that if a function $f : (0, \infty) \longrightarrow (0, \infty)$ satisfies these properties then $f(x) = \Gamma(x)$. The situation can be compared with the following exercise:

Construct a function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\phi(x + y) = \phi(x) + \phi(y)$ but there is NO constant a such that $\phi(x) = ax$. If in addition ϕ is continuous then it is true that $\phi(x) = ax$ for some constant a. What is the situation with $\Gamma(x)$? What **additional conditions** must we impose on a function $f : (0, \infty) \longrightarrow (0, \infty)$ such that the conditions $f(x + 1) = xf(x)$ and $f(1) = 1$ imply that $f(x) = \Gamma(x)$? We shall answer this presently.

18. Let $\epsilon > 0$ show that there exists a n_0 such that

$$\int_{n_0}^n \left(1 - \frac{t}{n}\right)^n t^{a-1} dt < \epsilon/3, \qquad \text{for all } n \geq n_0.$$

19. Use Dini's theorem to prove that

$$\Gamma(a) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{a-1} dt.$$

Compute the integral appearing on the right hand side. The resulting formula is called the Gauss's product formula for the gamma function:

$$\Gamma(a) = \lim_{n\to\infty} \frac{n! n^a}{a(a + 1)(a + 2) \dots (a + n)}$$

Remark: This may appear as a nice application of Dini's theorem. But Dini's theorem can easily be avoided by using a more important result namely the Monotone convergence theorem from Lebesgue theory. So what is the use of Dini's theorem? Why is it important in MA 403?? There is one place where it is used namely in the proof of the Cartan-Thullen theorem in the theory of functions of several complex variables. Besides this the Dini theorem is important because it is the simplest representative of a group of theorems all of which are of the following type:

For sequences of functions (f_n) coming from certain spaces and with monotonocity hypothesis, a very weak form of convergence of (f_n) implies a strong form of convergence. A notable example is the Harnack type convergence theorem, theorems on Subharmonic functions (complex analysis/PDEs) and sequences of convex functions.

20. Prove that the function $f : (0, 1) \longrightarrow \mathbb{R}$ given by

$$f(x) = \Gamma(x)\Gamma(1 - x) \sin \pi x$$

has finite limits as x approaches 0 and 1.

Remark: It turns out that $f(x)$ is constant and has value π. The formula

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}$$

is called Euler's reflection formula.

21. Use the Gauss's product formula to prove the Duplication formula of Legendre:

$$2^{a-1}\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{a+1}{2}\Big) = \sqrt{\pi}\Gamma(a)$$

22. Use the Gauss' product formula to prove that $\log(\Gamma(a))$ is convex on $(0, \infty)$. Provide also a second and direct proof of log-convexity using Hölder's inequality.

23. The Bohr-Mollerup theorem (You will be a guided through the proof in class): Suppose that $f : (0, \infty) \longrightarrow (0, \infty)$ is a function such that

    (i) $f(x + 1) = xf(x)$

    (ii) $f(1) = 1$ and

    (iii) $\log(f(x))$ is convex.

    Then $f(x) = \Gamma(x)$.

Proof: It is enough to prove this for the range $0 < x < 1$. Then

$$n < n + x < n + 1 \qquad (i)$$

So find a $\lambda \in [0,1]$ such that $n+x = \lambda n + (1-\lambda)(n+1)$. Since $\log f$ is convex,

$$f(x+n) \le (f(n))^{\lambda}(f(n+1))^{1-\lambda}$$

From this obtain an inequality of the type

$$(x+n)f(x)/n \le \frac{n!n^x}{x(x+1)(x+2)\ldots(x+n)}$$

Now we need another inequality in order to apply the sandwich theorem. Note that we started out with (i) where $x+n$ was flanked on either side by n and $n+1$. Now we need an inequality with n flanked between $n+x$ and $n-1+x$ namely

$$n+x < n+1 < n+x+1$$

Again find a $\lambda \in [0,1]$ such that $n+1 = \lambda(n+x) + (1-\lambda)(n+x+1)$ and repeat the same steps and you get another inequality. The sandwich theorem now can be used to show:

$$\lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\ldots(x+n)} \tag{ii}$$

exists and equals $f(x)$. So if an f satisfying the conditions of the theorem exists then it must be the limit in (ii). Since $\Gamma(x)$ satisfies these conditions we get the result. We have incidentally given yet another proof of the Gauss's product formula.

24. The Beta Function: Suppose that x, y are both positive real numbers, the integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

converges and the value is denoted by $B(x,y)$. The notation is due to Binet (1839). It has since then been called the beta function. Prove the following (here x and y are positive):

(i) $B(x,y) = B(y,x)$

(ii) $B(x+1,y) + B(x,y+1) = B(x,y)$

(iii) $xB(x,y+1) - yB(y,x+1) = 0$

(iv) $B(x,y) = \Big(\dfrac{x+y}{y}\Big)B(x,y+1)$

25. Prove the following relations (assume $x > 0$ and $y \ge 1$).

(i) $\lim_{n\to\infty} n^x B(x,y+n) = \Gamma(x)$. Explain how to relax the assumption $y \ge 1$ to $y > 0$.

(ii) $B(x,y)B(x+y,n) = B(x,y+n)B(y,n)$

(iii) Use (i) and (ii) to establish the Famous Beta-Gamma relation namely

$$\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y).$$

The remarkable feature about this proof is that it avoids the use of multiple integrals and uses only single integrals. The argument is due to Burkhardt [3] though vestiges of such an argument can be traced back to Dirichlet's Vorlesungen über die Lehre von einfachen und mehrfachen bestimmten Integralen, Druck und Verlag von Friederich Vieweg und Sohn, Braunschweig, 1904.

26. Prove that
$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{x-1}dt}{1+t}, \quad 0 < x < 1.$$
Discuss how to evaluate the integral on the right hand side for rational values of $x \in (0,1)$ by reducing it to the integral of a rational function. The integral of any rational function can in principle be computed via partial fractions but the computation can get messy unless you are clever ! For instance try evaluating the integral for $x = 1/8$. Putting $t^{1/8} = u$ we are are led to computing
$$\int_0^\infty \frac{du}{1+u^8}$$
In principle, and with some cleverness, the Euler's reflection formula can be verified for rational values of x. For irrational values one appeals to continuity. This approach is due to Dirichlet (See his *Vorlesungen* cited above).

27. Prove that
$$B(p,q) = \int_{-\infty}^\infty \frac{e^{(p-q)x} + e^{(q-p)x}}{(e^x + e^{-x})^{p+q}} dx$$

28. Stirling's formula for the gamma function:
$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}} = 1.$$
As an application prove that if $\alpha > 0$ then the binomial series
$$(1-x)^\alpha = 1 - \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 - \dots$$
Converges absolutely when $x = \pm 1$. You may also need Euler's reflection formula. Deduce that the series converges uniformly on $[-1,1]$.

29. Integral representation for the Bessel functions. We begin with the series
$$J_0(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!^2}\Big(\frac{x}{2}\Big)^{2n}.$$
Compute the integral
$$\int_{-\pi}^{\pi} \sin^{2n}\theta d\theta$$

[3]Zur Theorie der Gammafunktionen, Jahrbericht der Deutschen Mathematiker Vereinigung, **22** (1913) 223-224.

to show that

$$J_0(x) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int_{-\pi}^{\pi}(x\sin\theta)^{2n}d\theta$$

Explain why the summation and integral can be exchanged and deduce that

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta)d\theta$$

30. Prove using the differentiation theorem for power series that

$$xJ_{n+1}(x) = nJ_n(x) - xJ_n'(x).$$

Use induction to prove that

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - n\theta)d\theta. \qquad (i)$$

31. Here is an alternate argument: We start with the two identities

$$\Big(x^p J_p(x)\Big)' = x^p J_{p-1}(x), \quad \Big(x^{-p} J_p(x)\Big)' = -x^p J_{p+1}(x).$$

These follow easily from the differentiation theorem for power series. The series for Bessel functions converge for all complex values of x. Now for convenience we declare $J_n(x) = (-1)^n J_n(x)$ and let

$$G(x,t) = \sum_{n=-\infty}^{\infty} t^n J_n(x) \qquad (ii)$$

This function is called the generating function for the sequence $\{J_n(x) : n \in \mathbb{Z}\}$. We now let t vary over a fixed compact subset L of $\mathbb{C} - \{0\}$ and x vary over a compact subset K of $\mathbb{R}$. Discuss whether for a fixed $t \neq 0$, the series can be differentiated term by term. First obtain an estimate $|J_n(x)| \leq \exp|x|$. Obtain a first order ODE for G and then show that

$$G(x,t) = \exp\Big(\frac{xt}{2} - \frac{x}{2t}\Big) \qquad (iii)$$

Put $t = \exp(i\theta)$ in (ii), multiply by $\exp(-im\theta)$ and integrate term by term and prove (i)

32. Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty}\int_a^b f(x)\cos nx dx = 0, \quad \lim_{n\to\infty}\int_a^b f(x)\sin nx dx = 0$$

Hint: The result is true if $f(x)$ is a polynomial. Use the Weierstrass' approximation theorem. The result is called the Riemann Lebesgue lemma.

33. Show that if $f : [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable then given any $\epsilon > 0$, there is a continuous function $g : [a,b] \longrightarrow \mathbb{R}$ such that

$$\Big|\int_a^b f(x)dx - \int_a^b g(x)dx\Big| < \epsilon.$$

Extend the Riemann Lebesgue lemma to Riemann-integrable functions.

34. Prove that if

$$\int_0^\infty |f(t)|dt$$

converges, show that

(i)

$$\lim_{\xi\to\infty}\int_0^\infty f(t)e^{-it\xi}dt = 0.$$

This is the Riemann Lebesgue Lemma for Fourier transforms.

(ii)

$$\lim_{\xi\to\infty}\int_0^\infty f(t)e^{-st}dt = 0.$$

This is the Riemann Lebesgue Lemma for Laplace transforms.

First you may begin by assuming that f is continuous.

Exercises: Ascoli-Arzela theorem:

1. We have proved that if X is compact metric and S is a subset of $C(X)$ which is equicontinuous and pointwise bounded then S is pre-compact. Show conversely that if S is precompact then S is equicontinuous and uniformly bounded. Hint: Let $\epsilon > 0$ be arbitrary. Pick an $\epsilon/3$ net for the closure of S.

2. Suppose X is compact metric and S is a subset of $C(X)$ that is pointwise bounded and equi-continuous then it is uniformly bounded. This follows at once from Ascoli-Arzela theorem but give a direct proof by taking $\epsilon = 1$ and a suitable δ net of X.

3. Let $F : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be a continuous function and for each $f \in C[0,1]$, Tf denotes the function
$$Tf(x) = \int_0^1 F(x,y)f(y)dy.$$
Show that the family $\{Tf : \sup_{0 \leq t \leq 1} |f(t)| \leq 1\}$ is equi-continuous and uniformly bounded.

4. Let us consider the set of all solutions on the fixed interval $[0,1]$ of
$$y'' + k^2 y = f, \quad y(0) = 1, y'(0) = 0,$$
as f varies over $C[0,1]$ with absolute value bounded by 1. Show that the family is equicontinuous.