# Dedekind's proof of Euler's reflection formula via ODEs

Gopala Krishna Srinivasan *

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400 076

Among the higher transcendental functions, Euler's gamma function enjoys the previlage of being most popularly studied. The text [4] contains a short but elegant account of it in the real domain via its characterization due to Bohr and Mollerup. Among its properties the most striking is the reflection formula of Euler:

$$\Gamma(a)\Gamma(1-a) = \pi \csc \pi a, \quad 0 < a < 1, \tag{1}$$

which is usually proved using Cauchy's residue theorem or the infinite product expansion for $\sin \pi a$. Richard Dedekind wrote his PhD dissertation ([2], [3]) on the gamma function under the supervision of C. F. Gauss in which he has given an interesting but elementary real variables proof of (1) by solving an initial value problem for a second order ordinary differential equation (7). Surprisingly, this proof seems to have escaped notice completely though it appears among the many exercises in [1]. The purpose of this brief note is to popularize Dedekind's proof. For historical details and other approaches to the gamma function see [5] and the references therein. The exchange of integrals and differentiation under the integrals carried out here fall within the scope of Fubini's theorem and Lebesgue's dominated convergence theorem.

Recalling the classical beta-gamma identity ([4], p. 193), we denote $B(a, 1-a)$ by $B(a)$ and (1) is equivalent to proving

$$B(a) = \int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi \csc \pi a. \tag{2}$$

The proof is broken up into easy lemmas, the first of which recasts (2) in a symmetric form:

**Lemma 1:** (i) $B(a) = \displaystyle\int_0^1 \frac{x^{a-1} + x^{-a}}{1+x} dx = \frac{1}{2} \int_0^\infty \frac{x^{a-1} + x^{-a}}{1+x} dx.$ (3)

(ii) The function $B(a)$ has a local minima at $a = 1/2$ and the minimum value is $\pi$. The function $B'(a)$ is strictly increasing on $(0, 1)$.

*e-mail address: gopal@math.iitb.ac.in

**Proof:** Break the integral (2) into two integrals, over $(0,1)$ and $(1,\infty)$. In the latter put $x = 1/t$ and the result follows. To prove (ii), differentiate the identity $B(a) = B(1-a)$ and substitute $a = 1/2$ to get $B'(1/2) = 0$. The second derivative is strictly positive on $(0,1)$ so that the first derivative is strictly increasing, vanishing exactly once at $1/2$. Finally the integral $B(1/2)$ is elementary and its value is seen to be $\pi$.

A rescaling of the variable of integration in (3) produces:

**Lemma 2:** (i) $$w^{a-1}B = \int_0^\infty \frac{x^{a-1}dx}{x+w} \tag{4}$$
(ii) $$w^{-a}B = \int_0^\infty \frac{x^{a-1}dx}{1+xw} \tag{5}$$

**Proof:** Setting $x = t/w$ in (2) gives (i) and the substitution $x = tw$ transforms (2) into (ii).

Equation (3) suggests adding (4) and (5), dividing through by $1/(w+1)$ and integrating over $(0,\infty)$. The easy calculation results in:

**Lemma 3:** $$B^2 = \int_0^\infty \frac{x^{a-1}\log x dx}{x-1} \tag{6}$$
We are now ready to obtain the ODE for $B(a)$.

**Lemma 4:** $B(a)$ satisfies the following initial value problem for ODEs:

$$BB'' = (B')^2 + B^4, \quad B(1/2) = \pi, \quad B'(1/2) = 0. \tag{7}$$

**Proof:** Owing to the symmetry $B(a) = B(1-a)$ there is no loss of generality in assuming that $0 < a < 1/2$. Subtract (5) from (4), divide by $(w-1)$ and integrate with respect to $w$ over $(0,\infty)$ to get

$$B\int_0^\infty \frac{w^{a-1} - w^{-a}}{w-1}dw = 2\int_0^\infty \frac{x^{a-1}\log x dx}{x+1} = 2\frac{dB}{da}. \tag{8}$$

But from (6), the left hand side of the last equation is exactly

$$B\int_{1-a}^a B^2(t)dt,$$

and (8) assumes the form

$$B\int_{1-a}^a B^2(t)dt = 2\frac{dB}{da}$$

which, in view of the symmetry $B(t) = B(1-t)$ may be rewritten as

$$B \int_{1/2}^{a} B^2(t)dt = \frac{dB}{da} \tag{9}$$

Differentiating (9) with respect to $a$ gives immediately (7).

**Theorem:** For $0 < a < 1$, $\Gamma(a)\Gamma(1-a) = \pi \csc \pi a$

**Proof:** The differential equation (7) admits two elementary integrations as we now show. We work in the in the interval $[1/2, 1)$ and denote $B'$ by $C$. Since $B' > 0$ on $(1/2, 1)$ we may regard $C$ as a function of $B$. Using the chain rule,

$$B'' = \frac{dC}{da} = \frac{dC}{dB}\frac{dB}{da} = C\frac{dC}{dB}.$$

This transforms (7) into a simple linear ODE for $C^2$ namely,

$$\frac{1}{2}\frac{dC^2}{dB} = \frac{C^2}{B} + B^3, \quad C(1/2) = 0.$$

Since $C > 0$ on $(1/2, 1)$, this gives

$$C = \frac{dB}{da} = B\sqrt{B^2 - \pi^2} \quad (a > 1/2)$$

and the result follows upon integration.

# References


[1] R. Askey, G. E. Andrews and R. Roy, *Special functions,* Cambridge University Press, New York, 1999.

[2] R. Dedekind, *Über die Elemente der Theorie der Eulerschen Integrale,* dissertation, Göttingen 1852, Gesammelte mathematische Werke, Bd I, pp 1-31.

[3] R. Dedekind, *Über die Elemente ein Eulerschen Integral,* Journal für reine und angewandte Mathematik **45**, 370-374 (1852).

[4] W. Rudin, *Principles of mathematical analysis,* Third edition, McGraw-Hill international student edition, Singapore, 1976.

[5] G. K. Srinivasan, *The gamma function - an eclectic tour,* American Math. Monthly **114**: 297-315 (2007).