THE RADIUS OF CONVERGENCE AND THE WELL-POSEDNESS OF THE PAINLEVÉ EXPANSIONS OF THE KORTEWEG-DE-VRIES EQUATION

Short Title: Well Posedness of Painlevé expansions

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Abstract. In this paper we obtain explicit lower bounds for the radius of convergence of the Painlevé expansions of the Korteweg-de-Vries equation around a movable singularity manifold $\mathcal{S}$ in terms of the sup norms of the arbitrary functions involved. We use this estimate to prove the well-posedness of the singular Cauchy problem on $\mathcal{S}$ in the form of continuous dependence of the meromorphic solution on the arbitrary data.

§1 **Introduction.** Given a holomorphic manifold $\mathcal{S}$ described locally by

$$\mathcal{S} : x - \psi(t) = 0, \tag{1.1}$$

the Painlevé expansion of the Korteweg de Vries (KdV) equation

$$w_t = w_{xxx} + w w_x \tag{1.2}$$

is a formal series solution of the of the form

$$w(x,t) = \sum_{n=0}^{\infty} u_n(t)(x - \psi(t))^{n+\nu}, \tag{1.3}$$

where $\nu = -2$, and $u_4(t)$, $u_6(t)$ are arbitrary functions.

Such series expansions were suggested by J.Weiss, M.Tabor and G.Carnevale [7] as a practical test of the Painlevé property for partial differential equations (PDEs), i.e. the property that all solutions are single-valued around all noncharacteristic holomorphic given singularity manifolds. The Painlevé property has become a widely used indicator for integrability (see [1, 3]) meaning exact solvability through

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an associated linear problem. The KdV equation is well known to be integrable. Hence it is widely believed that it should possess the Painlevé property.

The convergence of the Painlevé expansions has been discussed in [4, 6]. However, the issue of estimating their radii of convergence has not been addressed. Here, we prove a lower bound on the radius of convergence of the Painlevé expansion for the KdV equation and use it to prove the well-posedness of the singular Cauchy problem with respect to the arbitrary data given on $\mathcal{S}$.

To describe the well-posedness result in terms of such data we define a collective name for them:

Definition. *The* WTC data *for the KdV equation is the set* $\{\psi(t), u_4(t), u_6(t)\}$ *of arbitrary functions describing the Painlevé expansion (1.3).*

We restrict our attention to the space of holomorphic WTC data topologized by uniform convergence on compact sets.

Our results give a lower bound for the radius of convergence in terms of the sup-norm of the WTC data and, moreover, show that the meromorphic function given by the (convergent) Painlevé expansion varies continuously, in the sup-norm, as the WTC data are varied. Before we state our main results, we need to recall the construction and the convergence result for the Painlevé expansion of the KdV equation (see Refs [4] or [7]).

Theorem 1.1. *Given an analytic manifold* $\mathcal{S} : x - \psi(t) = 0$*, with* $\psi(0) = 0$*, and two arbitrary analytic functions*

$$\lim_{x\to\psi(t)} \left(\frac{\partial}{\partial x}\right)^4 [w(x,t)(x-\psi(t))^2], \quad \lim_{x\to\psi(t)} \left(\frac{\partial}{\partial x}\right)^6 [w(x,t)(x-\psi(t))^2] \tag{1.4}$$

there exists in a neighbourhood of $(0,0)$ *a meromorphic solution of the KdV equation (1.2) of the form*

$$w(x,t) = \frac{-12}{\left(x-\psi(t)\right)^2} + h(x,t)$$

where h *is holomorphic.*

The expansion (1.3) can be written in terms of the new variable $X := x - \psi(t)$ as

$$w(x,t) = \sum_{n=0}^{\infty} u_n(t) X^{n-2}. \tag{1.5}$$

Substitution into the KdV equation shows that the coefficients u_n must satisfy the recursion relation

$$Q(n)u_n = u'_{n-3} - (n-4)\psi'(t)u_{n-2} - \sum_{j=1}^{n-1}(j-2)u_j u_{n-j} \quad (n>0), \tag{1.6}$$

where $Q(n) := (n+1)(n-4)(n-6)$. It is easily checked that u_4 and u_6 are arbitrary and

$$u_0 = -12, \quad u_1 = u_3 = 0, \quad u_2 = -\psi'(t), \quad u_5 = \frac{\psi''(t)}{6}. \tag{1.7}$$

The convergence of the series may then be established by the majorant method of Ref. [7] or the iteration method of Ref.[4].

The main results of this paper are stated below as Theorems A and B.

Theorem A (Radius of Convergence of the Series for w). *Given WTC data $\psi(t)$, $u_4(t)$, $u_6(t)$ analytic in the ball $B_{2\rho+\epsilon}(0) = \{t \in \mathbb{C} : |t| < 2\rho + \epsilon\}$, let*

$$M = \sup_{|t|=2\rho} \{1, |\psi(t)|, |u_4(t)|, |u_6(t)|\}. \tag{1.8}$$

The radius of convergence $R_\rho = R$ of the power series (1.5) satisfies

$$R \geq \frac{\min\{1, \rho\}}{10M}. \tag{1.9}$$

Notice that, because of the Maximum Principle, the supremum over $|t| = 2\rho$ is the same as the supremum over $|t| < 2\rho$. To describe the well-posedness result we need to be slightly more precise in defining our notation for domains. If the WTC data are given on a connected bounded open set Ω in $\mathbb{C}$ and furthermore $\delta > 0$ we define Ω_δ as

$$\Omega_\delta = \{t \in \Omega : \operatorname{dist}(t, \mathbb{C} - \Omega) \geq 3\delta\}.$$

We will always assume that δ is small enough that the set Ω_δ is a connected set. Note that if $t \in \Omega_\delta$ then the *closed disk* of radius 2δ centred at t is contained in $\Omega_{\delta/3}$. We denote by M_δ, B_δ the numbers

$$M_\delta = \sup_{t \in \Omega_{\delta/3}} \{1, |\psi(t)|, |u_4(t)|, |u_6(t)|\},$$
$$B_\delta = \frac{10M_\delta}{\min\{1, \delta\}},$$

given by Theorem A and take

$$\mathcal{O}_\delta = \{(x,t) \in \mathbb{C}^2 : t \in \Omega_\delta \text{ and } |x - \psi(t)| < B_\delta^{-1}\},$$
$$\mathcal{O} = \bigcup_{\delta > 0} \mathcal{O}_\delta.$$

Theorem B (Continuity as a Function of WTC Data).

(1) *Let $\big(\psi_j(t), u_{4,j}(t), u_{6,j}(t)\big)$ be a sequence of holomorphic WTC data defined on a bounded connected domain $\Omega \in \mathbb{C}$ and converging uniformly on compact subsets to $\big(\psi(t), u_4(t), u_6(t)\big)$. Assume $\psi_j(0) = 0$ for every j. Let w_j be the solution of the equation (1.2) given by (1.5) with WTC data $\big(\psi_j(t), u_{4,j}(t), u_{6,j}(t)\big)$. Then there exists a polydisk*

$$P(r_1, r_2) := \{(x,t) : |x| < r_1, |t| < r_2\} \subset \mathcal{N},$$

where $\mathcal{N} := \bigcap_j \mathcal{O}(\psi_j, u_{4,j}, u_{6,j}) \cap \mathcal{O}(\psi, u_4, u_6)$.

(2) *The sequence of functions $(x - \psi_j(t))^2 w_j$ are all defined on a common open subdomain $G \subset \mathcal{N}$, and converge uniformly on compact subsets of G to $(x - \psi(t))^2 w$, where w is the solution of (1.2) with data $\big(\psi(t), u_4(t), u_6(t)$.*

We prove these two theorems in sections 2 and 3 respectively.

§2 Proof of Theorem A – Estimate of Radius of Convergence.

In this section, we prove Theorem A. The method we use is a combination of a majorant method and iteration on a shrinking sequence of disks. Let $v := X^2 w$.

Proof. We estimate the sequence of numbers $\sup_{|t|=\rho} |u_n(t)|$. Define the sequence R_n as follows:

$$R_n = \frac{6\rho}{\pi^2}\Big(\frac{\pi^2}{3} - \sum_{j=1}^{n} \frac{1}{j^2}\Big). \tag{2.1}$$

Clearly $\rho < R_n < R_{n-1} < 2\rho$ for all n and $R_n \to \rho$. Define

$$d_n = R_{n-1} - R_n = \frac{6\rho}{\pi^2 n^2}, \quad A := \pi^2/6\rho \quad \Longrightarrow \frac{1}{d_n} = An^2. \tag{2.2}$$

We consider the sequence of recurrence formulae

(2.3)

$$Q(n)M_n = An^2 M_{n-3} + AnMM_{n-2} + \sum_{j=1}^{n-1}(j-2)M_{n-j}M_j \quad (n \geq 7),$$

(2.4)

$$Q(n)U_n = An^2 U_{n-3} + AnMU_{n-2} + \sum_{j=1}^{n-1}(j-2)U_{n-j}U_j.$$

For $1 \leq i \leq 6$, choose $M_i = U_i = \sup_{|t|=R_i} |u_i(t)|$. We determine M_n, U_n using the equations (2.3, 2.4).

Claim. $M_i \leq U_i$ for all i.

Proof of the claim is by induction on n. It is true for $1 \leq i \leq 6$ by the choice of U_i.

$$\begin{aligned} Q(n)U_n &= An^2 U_{n-3} + AnMU_{n-2} + \sum_{j=1}^{n-1}(j-2)U_{n-j}U_j, \quad \text{using (2.4)}, \\ &\geq An^2 M_{n-3} + AnMM_{n-2} + \sum_{j=1}^{n-1}(j-2)M_{n-j}M_j = Q(n)M_n \end{aligned}$$

completing the induction.

Claim. $M_n \geq \sup_{|t|=R_n} |u_n(t)|$.

We prove this by induction on n. It is true for $0 \leq i \leq 6$ by definition. By Cauchy's estimate we get

$$\sup_{|t|=R_n} |u'_{n-3}(t)| \leq \big(\sup_{|t|=R_{n-3}} |u_{n-3}(t)|\big)/(R_{n-3} - R_n) \leq \frac{M_{n-3}}{(R_{n-1} - R_n)}.$$

$$\therefore \sup_{|t|=R_n} |u'_{n-3}(t)| \leq \frac{M_{n-3}}{d_n} = An^2 M_{n-3}, \text{ and}$$

$$\sup_{|t|=R_n} |\psi'(t)| \leq \sup_{|t|=2\rho} |\psi(t)|/(2\rho - R_n) \leq \frac{M}{2\rho - R_1} = \frac{M}{A}.$$

Now the recursion relation defining M_n gives

$$\begin{aligned}
Q(n)M_n &= \frac{M_{n-3}}{d_n} + \frac{nMM_{n-2}}{2\rho - R_1} + \sum_{j=1}^{n-1}(j-2)M_{n-j}M_j \\
&\geq \sup_{|t|=R_n} |u'_{n-3}(t)| + n \sup_{|t|=R_n} |\psi^{'}(t)| \sup_{|t|=R_{n-2}} |u_{n-2}| \\
&\qquad + \sum_{j=1}^{n-1}(j-2) \sup_{|t|=R_{n-j}} |u_{n-j}| \sup_{|t|=R_j} |u_j| \\
&\geq \sup_{|t|=R_n} \Big[|u^{'}_{n-3}(t)| + (n-4)|\psi^{'}(t)||u_{n-2}| + \sum_{j=1}^{n-1}(j-2)|u_{n-j}u_j|\Big] \\
&\geq \sup_{|t|=R_n} \Big|u'_{n-3} - (n-4)\psi^{'}(t)u_{n-2} - \sum_{j=1}^{n-1}(j-2)u_{n-j}u_j\Big| \\
\therefore Q(n)M_n &\geq \sup_{|t|=R_n} Q(n)|u_n(t)|
\end{aligned}$$

completing the induction.

Claim. If we choose K, B such that

$$B \geq \max\{M, 10\sqrt{M/\rho}, 3\rho^{-1/3}, \pi(M/\rho^2)^{1/5}, \frac{2}{\rho}\}, \quad K = 1/4 \tag{2.5}$$

then $U_n \leq nKB^n$ for all n.

In the following, we will use the self-evident facts

$$n^2/Q(n) < 3, \quad \frac{n(n-1)}{(n-4)(n-6)} \leq 14 \quad \text{for } n \geq 7.$$

The choice of K, B and the values of u_n for $1 \leq n \leq 6$ implies that $U_n \leq nKB^n$ for $1 \leq n \leq 6$. We will prove this later. Assume inductively that the estimate is true for all $i < n$. Now, for $n \geq 7$, we get

$$\begin{aligned}
Q(n)U_n &= An^2U_{n-3} + AnMU_{n-2} + \sum_{j=1}^{n-1}(j-2)U_{n-j}U_j \\
&\leq nKB^n\Big(An(n-3)/B^3 + \frac{(n-2)AM}{B^2} + K\sum_{j=1}^{n} j^2(n-j)/n\Big)
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
U_n &\leq nKB^n\Big(\frac{An^2}{Q(n)B^3} + \frac{AM}{B^2(n-4)(n-6)} + \frac{Kn(n-1)}{12(n-4)(n-6)}\Big) \\
&\leq nKB^n\Big(\frac{3A}{B^3} + \frac{A}{B(n-4)(n-6)} + \frac{7K}{6}\Big) \\
&\leq nKB^n\Big(\frac{3A}{B^3} + \frac{\pi^2}{6\rho B(n-4)(n-6)} + \frac{1}{3}\Big) \\
&\leq nKB^n\Big(\frac{3A}{B^3} + \frac{2}{3\rho B} + \frac{1}{3}\Big)
\end{aligned}$$

To complete the induction we need to show that

(1) $3A/B^3 + 2/(3\rho B) + 1/3 \leq 1$

Proof: From (2.13) follows $2\rho - R_1 = 6\rho/\pi^2$.

$$\frac{3A}{B^3} = \frac{3\pi^2}{6B^3\rho} < \frac{5}{\rho B^3} < \frac{1}{3},$$

because $B > 3\rho^{-1/3}$. Also,

$$\frac{2}{3\rho B} \leq \frac{1}{3}$$

because $B > 2/\rho$.

(2) $U_n \leq nKB^n$ for $n = 1, \ldots, 6$. Since $u_1 = u_3 = 0$ and $|u_4(t)| \leq M$, $|u_6(t)| \leq M$, and $B \geq M \geq 1$ we need to consider only $n = 2, 5$. From the definitions of u_2, u_5 we get

$$\begin{aligned}
\sup_{|t|=R_2} |u_2(t)| &= \sup_{|t|=R_2} |\psi'(t)| \\
&\leq \frac{2M}{R_1 - R_2} = \frac{4M\pi^2}{3\rho} \\
&< \frac{16M}{\rho} < 2KB^2, \\
\sup_{|t|=R_5} |u_5(t)| &= \sup_{|t|=R_5} \frac{|\psi''(t)|}{6} \\
&\leq \frac{M}{3(R_4 - R_5)(2\rho - R_1)} \\
&= \frac{25M\pi^4}{108\rho^2} < 5KB^5.
\end{aligned}$$

(Note, in the latter, we have used Cauchy's estimate twice on two concentric disks.) So we get for all n, $U_n \leq nKB^n$ completing the induction. The arithmetic-geometric series $\sum_{n=1}^{\infty} nK(XB)^n$ dominates the series $\sum_{j=1}^{\infty} |u_j(t)|X^j$ uniformly in the set

$$\{(X, t); |X| < B^{-1}, |t| \leq \rho\} \subset \mathbb{C}^2.$$

Since $M \geq 1$ and $\rho_0 := \min\{1, \rho\} \leq 1$, the required estimate (2.7) on B holds if we take $B = 10M/\rho_0$, completing the proof of the theorem. □

Remark. Note that the majorant $\sum_{n=1}^{\infty} nK(XB)^n$ has a double pole on its circle of convergence. This is consistent with the suspected double pole of the solution as the next singularity away from the initial manifold (1.1).

§3 Proof of Theorem B — Continuity With Respect To The Arbitrary Functions.

In this section, we prove Theorem B. Recall the notation defined for domains $\mathcal{O}_\delta$, $\mathcal{O}$ in the Introduction. For emphasis we may sometimes use the notations $\mathcal{O}_\delta(\psi, u_4, u_6)$, $B_\delta(\psi, u_4, u_6)$, $M_\delta(\psi, u_4, u_6)$, or simply $\mathcal{O}_\delta(\psi)$, $B_\delta(\psi)$, $M_\delta(\psi)$.

Lemma 3.1. *Given $\Omega \subset \mathbb{C}$ containing the origin and ψ, u_4, u_6 holomorphic in Ω the meromorphic solution obtained in Theorem [1.1] exist in the domain $\mathcal{O}$.*

Proof. Pick a δ arbitrary and small. Let $t \in \Omega_\delta$. Then the closed disk $\overline{D}_{2\delta}(t)$ is contained in $\Omega_{\delta/3}$. By maximum modulus theorem,

$$\max_{\overline{D}_{2\delta}(t)} \{1, |\psi(\tau)|, |u_4(\tau)|, |u_6(\tau)|\} \leq M_\delta.$$

The proof of Theorem A can be generalised by replacing all suprema over $|t| = R_i$ by suprema over $|t-a| = R_i$. This shows that the solution is defined on $\{(x,\tau) : \tau \in D_{2\delta}(t), |x - \psi(\tau)| < B_\delta^{-1}\}$ and since the B_δ is the same for all the disks $\overline{D}_{2\delta}(t)$ as t varies over Ω_δ, the solution is defined on

$$\bigcup_{t\in\Omega_\delta} \{(x,\tau) : \tau \in \overline{D}_{2\delta}(t), |x - \psi(\tau)| < B_\delta^{-1}\} \supset \mathcal{O}_\delta.$$

Since $\delta > 0$ is arbitrary, the solution is defined on $\bigcup_{\delta>0} \mathcal{O}_\delta = \mathcal{O}$.

This lemma is used below to enable us to work in any polydisk in $\mathcal{O}_\delta$.

Proof of Theorem.

(1) Proof of the first part of the theorem:

Since we have $\psi(0) = 0$ it follows that $\mathcal{O}$ contains the origin. Choose $\delta > 0$ such that $0 \in \Omega_\delta$ and choose a polydisk $P(a_1, a_2)$ whose closure is contained in $\mathcal{O}_\delta$. Since $\psi_j(t)$, $u_{4,j}(t)$, $u_{6,j}(t)$ converge uniformly on compact subsets of Ω, there exists a $J = J_\delta$ such that

$$1 \leq \sup_j \sup_{\Omega_{\delta/3}} \{1, |\psi_j(t)|, |u_{4,j}(t)|, |u_{6,j}(t)|\} \leq J_\delta$$

Let

$$H = \frac{10J_\delta}{\min\{1,\delta\}}.$$

There exists j_0 such that

$$\sup_{\Omega_\delta} \{|\psi_j(t) - \psi(t)|, |u_{4,j}(t) - u_4(t)|, |u_{6,j}(t) - u_6(t)|\} < \frac{1}{3H} \quad \forall j \geq j_0$$

Now choose a $b_2 > 0$ with $b_2 < a_2$ such that $|\psi(t)| < \frac{1}{3H}$ for all $|t| < b_2$ and now if $|x| < \frac{1}{3H}$, $|t| < b_2$,

$$|x - \psi_j(t)| \leq |x| + |\psi(t)| + |\psi_j(t) - \psi(t)| < 3\frac{1}{3H} = \frac{1}{H} \quad \forall j \geq j_0$$

But note that

$$H = \frac{10J_\delta}{\min\{1,\delta\}} \geq \frac{10M_\delta(\psi_j)}{\min\{1,\delta\}} = B_\delta(\psi_j).$$

This shows that if we take $b_1 = \frac{1}{3H}$ and b_2 as above then $P(b_1, b_2) \subset \mathcal{O}_\delta(\psi_j)$ for all $j \geq j_0$. Finally, since $\cap_{j=1}^{j_0} \mathcal{O}_\delta(\psi_j)$ is an open set containing the origin it follows that there exists c_1, c_2 such that $P(c_1, c_2) \subset \bigcap_{j=1}^{j_0} \mathcal{O}_\delta(\psi_j)$ and then $r_i = \min\{a_i, b_i, c_i\}$ $(i = 1, 2)$ suffices. The lower bound on the radius of convergence given in Theorem A and the uniform convergence of the WTC data imply that $r_i \neq 0$. This finishes the proof of the first part of the theorem.

(2) Proof of the second part of the theorem:

To prove that the convergence is uniform on compact subsets we proceed as follows. Let C be a compact subset of $G := P(r_1, r_2)$. Choose a compact subset $\widetilde{C}$ of G such that $C \Subset \widetilde{C}$.[1] By the maximum modulus theorem we have

$$\sup_{(x,t)\in C} |x - \psi(t)| < \beta < \gamma < \sup_{(x,t)\in \widetilde{C}} |x - \psi(t)|$$

for some pair of numbers β and γ. Uniform convergence of ψ_j gives a j_0 such that for all $j \geq j_0$,

$$\sup_{(x,t)\in C} |x - \psi_j(t)| < \beta < \gamma < \sup_{(x,t)\in \widetilde{C}} |x - \psi_j(t)|$$

Since $\widetilde{C}$ is compact, there is a $\delta > 0$ such that $\widetilde{C} \subset \mathcal{O}_\delta$. (Note $\mathcal{O}_\delta$ are nested domains.) In $\mathcal{O}_\delta(\psi_j)$ we have $|x - \psi_j(t)| < B_\delta(\psi_j)^{-1}$ and consequently $B_\delta(\psi_j)^{-1} > \gamma$ for all $j \geq j_0$. Let $u_{n,j}$ be given by the recursion formula (1.6) with data $\psi_j, u_{4,j}, u_{6,j}$. Then we have the estimates:

$$\sup_{t\in\Omega_\delta(\psi_j)} |u_{n,j}(t)| \leq nK(B_\delta(\psi_j))^n < nK\gamma^{-n} \quad \forall j \geq j_0,$$

$$\therefore \sup_{(x,t)\in C} |u_{n,j}(t)||x - \psi_j(t)|^n < nK\beta^n\gamma^{-n} \quad \forall j \geq j_0.$$

Meanwhile by induction on n and (1.6), it follows easily that

$$\lim_{j\to\infty} u_{n,j} \to u_n, \quad \lim_{j\to\infty} u'_{n,j} \to u'_n \quad \text{unif. comp. sets.}$$

The remainder of the proof is now a consequence of the following result which is a special case of Lebesgue dominated convergence theorem.

A Dominated Convergence Theorem. *Suppose that $f : \mathbb{N} \times \mathbb{N} \times C \to \mathbb{C}$ such that*

1) $\lim_{j\to\infty} f(n, j, v)$ converges uniformly in v to a limit denoted by $f(n, v)$.
2) $\sup_{j,v} |f(n, j, v)| \leq g(n)$ and
3) $\sum_{n=1}^{\infty} g(n)$ converges
Then,

$$\lim_{j\to\infty} \sum_{n=1}^{\infty} \sup_v |f(n, j, v) - f(n, v)| = 0$$

In particular $\sum_{n=1}^{\infty} f(n, j, v)$ converges to $\sum_{n=1}^{\infty} f(n, v)$ uniformly on C.

Apply the above to the case (with $v = (x, t)$)

$$f(n, j, v) = u_{n,j}(x - \psi_j(t))^n, \quad f(n, v) = u_n(x - \psi(t))^n, \quad g(n) = nK(\frac{\beta}{\gamma})^n$$

This finishes the proof of the second part of the theorem. $\square$

[1] The notation $C \Subset \widetilde{C}$ means closure of C is compact and is contained in the interior of $\widetilde{C}$.

§4 **Conclusion.** In this paper we have obtained an explicit lower bound for the radius of convergence of the Painlevé expansion of solutions of the KdV equation and used this estimate to prove the continuous dependence of solutions on the WTC data. Note that these cannot be given in the usual way *i.e.* as one would give Cauchy data on a regular manifold. The question of how to relate WTC data to regular Cauchy data elsewhere still remains open. The methods used in this paper readily extend to other integrable PDEs that are analytic in the dependent variable u and its derivatives, including those in $2+1$-dimensions such as the Kadomtsev-Petviashvili equation.

A proof that the KdV equation possesses the Painlevé property is still lacking in the literature. The major problem is the absence so far of a method of global analysis in $\mathbb{C}^n$ that applies to the whole space of solutions.

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