# The Exterior Derivative - A direct approach 

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#### Abstract

In this note we provide a direct approach to the most basic operator in this theory namely the exterior derivative. The crucial ingredient is a calculus lemma based on determinants. We maintain the view that in a first course at least this direct approach is preferable to the more abstract one based on characterization of the exterior derivative in terms of its properties.


## 1. Introduction:

In low dimensions at least, differential forms made their appearance in analysis more than three centuries ago originating in the works of Euler, Lagrange, Clairut and others. For arbitrary orders they were introduced by Poincaré and E. Cartan. The interesting historical development is available in the papers of Samelson [4. A significant role was played by Pfaff (thesis advisor of C. F. Gauss), Jacobi and many other mathematicians. The final form in which this topic is currently is the culmination of efforts a few centuries.

The exterior derivative is one of the most important ideas in the theory of differential manifolds leading directly to the de Rham cohomology of manifolds - that is to algebraic topology! It was used by Cartan in his formulation of differential geometry via moving frames [7] (chapter 7).

The treatment of exterior derivative of $k$-forms in Hicks [2] is completely coordinate free and described as a $k+1$ linear form acting on the $C^{\infty}(M)$ module $\mathfrak{X}(M)$. The defining formula (stated here for simplicity only for $k=2$ ) is:

$$
d \omega(X, Y)=X(\omega(Y))-Y(\omega(X))-\omega([X, Y])
$$

Though the ultimate goal of introducing this operator in a coordinate free manner has been reached, the treatment in Hicks is somewhat austere. This formula is also available in Chern et al., [1 or [6] (p. 213) as well. The proofs in [1] and [6] on the existence and uniqueness of the exterior differential operator (satisfying certain conditions) employs a mixture of local and global arguments (see also [8]). Although the construction of the exterior derivative carried out in [1] (through its characterization in terms of its properties) is quite elegant we feel an alternate treatment which is direct would be useful for audience in a first course (following the books of M. Spivak [5] or J. R. Munkres [3]) keeping in focus certain special tensorial features. Specifically, the case with the exterior derivative is reminiscent

[^0]of that of covariant derivative namely, when changing coordinates, certain undesirable terms ought to cancel out. We see this happen explicitly here in our discussion of the exterior derivative.

We show that the key ingredient needed for defining the exterior derivative is a basic calculus identity involving determinants. We find this interesting inasmuch as when changing coordinates we actually witness the internal cancellations of terms involving the second derivatives of the transition maps.

## 2. Basic Calculus Lemma and the existence of $d$ :

In the following lemma $J$ would denote an ordered $k$-set $\left\{j_{1}, j_{2}, \ldots j_{k}\right\}, 1 \leq j_{1}<j_{2}<\cdots<j_{k} \leq n$. We shall consider pairs $(s, J)$ such that $1 \leq s \leq n$ and $s \neq j_{1}, j_{2}, \ldots, j_{k}$. Let $N$ be the total number of such pairs. For a given pair $(s, J)$ and $j_{p} \in J$ denote by $\left(j_{p}, J^{\prime}\right)$ the complementary pair $\left(j_{p}, J^{\prime}\right)$ obtained by removing $j_{p}$ from $J$ and inserting $s$ in the right place.

Lemma: Let $\phi_{1}, \phi_{2}, \ldots, \phi_{k}$ be $k$ smooth functions of $z_{1}, z_{2}, \ldots, z_{n}$. Then

$$
\sum_{s, J} \frac{\partial}{\partial z_{s}} \frac{\partial\left(\phi_{1}, \phi_{2}, \ldots, \phi_{k}\right)}{\partial\left(z_{j_{1}}, z_{j_{2}}, \ldots, z_{j_{k}}\right)} d z_{s} \wedge d z_{j_{1}} \wedge \cdots \wedge d z_{j_{k}}=0
$$

Proof: Assume $j_{q-1}<s<j_{q}$ so that $d z_{s}$ would need to move through $q-1$ transpositions to bring the monomial in standard form leading to a factor of $(-1)^{q-1}$. Also carrying out the indicated differentiation would produce $k$ determinants out of each summand leading to $k N$ monomials in all. We need to show that the monomials can be paired off in such a way that the sum is ultimately zero.

Let us consider the terms coming from the complementary pairs $(s, J)$ and $\left(j_{p}, J^{\prime}\right)$. We may assume at the outset that $s<j_{p}$ for in the opposite case we can interchange the roles of $(s, J)$ and $\left(j_{p}, J^{\prime}\right)$. The determinant will be written in such a way that the second derivatives appear in the first column which necessitates a book-keeping of the number of column exchanges.

Computing the derivative of the determinant, the term wherein the $p$ th column is differentiated is:

$$
(-1)^{p-1}\left|\begin{array}{cccc}
\frac{\partial^{2} \phi_{1}}{\partial z_{s} \partial z_{j_{p}}} & \frac{\partial \phi_{1}}{\partial z_{j_{1}}} & \ldots & \frac{\partial \phi_{1}}{\partial z_{j_{k}}} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\partial^{2} \phi_{1}}{\partial z_{s} \partial z_{j_{p}}} & \frac{\partial \phi_{1}}{\partial z_{j_{1}}} & \ldots & \frac{\partial \phi_{1}}{\partial z_{j_{k}}}
\end{array}\right|
$$

Together with the differentials, we get the monomial:

$$
(-1)^{p+q-2}\left|\begin{array}{cccc}
\frac{\partial^{2} \phi_{1}}{\partial z_{s} \partial z_{j_{p}}} & \frac{\partial \phi_{1}}{\partial z_{j_{1}}} & \ldots & \frac{\partial \phi_{1}}{\partial z_{j_{k}}}  \tag{1}\\
\cdots & \ldots & \ldots & \ldots \\
\frac{\partial^{2} \phi_{1}}{\partial z_{s} \partial z_{j_{p}}} & \frac{\partial \phi_{1}}{\partial z_{j_{1}}} & \ldots & \frac{\partial \phi_{1}}{\partial z_{j_{k}}}
\end{array}\right| d z_{j_{1}} \wedge \cdots \wedge d z_{j_{q-1}} \wedge d z_{s} \wedge d z_{j_{q}} \wedge \cdots \wedge d z_{j_{k}}
$$

Now we consider the term arising out of the complementary pair ( $j_{p}, J^{\prime}$ ) and look at the relevant monomial namely the one in which the second derivatives

$$
\frac{\partial^{2} \phi_{i}}{\partial z_{j_{p}} \partial z_{s}}, \quad i=1,2, \ldots, k
$$

appear in the determinant. These second derivatives appear in the $q$ th column ( $j_{q-1}<s<j_{q}$ ) and so we need $q-1$ column exchanges to bring them to the first column thereby producing a $(-1)^{q-1}$ sign. We also have in addition the factor

$$
d z_{j_{p}} \wedge d z_{j_{1}} \wedge \cdots \wedge d z_{j_{q-1}} \wedge d z_{s} \wedge d z_{j_{q}} \wedge \cdots \wedge d z_{p-1} \wedge d z_{p+1} \wedge \cdots \wedge d z_{j_{k}}, \quad\left(s<j_{p}\right)
$$

Owing to the presence of $d z_{s}$, the $d z_{j_{p}}$ has to now move through $p$ transpositions to get this in standard form. Thus we get the term (1) but with $(-1)^{p+q-1}$ instead. Thus the terms arising from complementary pairs cancel out. The proof is complete.

Definition (The exterior derivative $d$ ): The standard notation for the set of all smooth $k$-forms on $M$ is $\Omega^{k}(M)$. We introduce the $\mathbb{R}$-linear map

$$
d: \Omega^{k}(M) \longrightarrow \Omega^{k+1}(M)
$$

Let $\omega$ be a differential $k$ form and on the chart $U$ let $\omega$ be given by

$$
\omega=\sum_{i} a_{i_{1} i_{2} \ldots i_{k}}^{U} d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}}
$$

We define on each chart

$$
\begin{equation*}
d \omega=\sum_{i} d a_{i_{1} i_{2} \ldots i_{k}}^{U} \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}} \tag{2}
\end{equation*}
$$

The notation $\sum_{i}$ stands for the sum over all standard $k$-tuples $\left(i_{1}, i_{2}, \ldots, i_{k}\right)$ with $i_{1}<i_{2}<\cdots<i_{k}$.
The basic properties of this operator can almost be read off from this definition except for one hurdle. We do have the job of showing that $d$ is well defined namely, of verifying consistency on overlapping charts but we shall demonstrate that this is nothing but the basic calculus lemma!

Theorem: The operator $d$ given by (2) is a well-defined element of $\Omega^{k+1}(M)$.

Proof: Let $U$ and $V$ be two overlapping charts. Need to check that

$$
\sum_{i} d a_{i_{1} i_{2} \ldots i_{k}}^{U} \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}}=\sum_{j} d a_{j_{1} j_{2} \ldots j_{k}}^{V} \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}, \quad \text { on } U \cap V
$$

Since

$$
\begin{equation*}
a_{j_{1} j_{2} \ldots j_{k}}^{V}=\sum_{i} a_{i_{1} i_{2} \ldots i_{k}}^{U} \frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)} \tag{3}
\end{equation*}
$$

our job is to check that

$$
\sum_{i} d a_{i_{1} i_{2} \ldots i_{k}}^{U} \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}}=\sum_{i} \sum_{j} d\left(a_{i_{1} i_{2} \ldots i_{k}}^{U} \frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)}\right) \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}
$$

Well, the right hand side breaks up into two sums:

$$
\sum_{i} \sum_{j} d\left(a_{i_{1} i_{2} \ldots i_{k}}^{U} \frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)}\right) \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}=I+I I
$$

The first sum $I$ displayed below is tensorial namely,

$$
\sum_{i} \sum_{j} d a_{i_{1} i_{2} \ldots i_{k}}^{U}\left(\frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)}\right) \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}=\sum_{i} d a_{i_{1} i_{2} \ldots i_{k}}^{U} \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}}
$$

which is the desired result and so we must show that the second (non-tensorial) term $I I$ is identically zero namely,

$$
I I=\sum_{i} \sum_{j} a_{i_{1} i_{2} \ldots i_{k}}^{U} d\left(\frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)}\right) \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}=0
$$

Since the coefficients, apart (3), are arbitrary smooth functions, we must show that each of the pieces

$$
\sum_{j} d\left(\frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)}\right) \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}
$$

individually vanishes. That is to say for each fixed $i_{1}<i_{2}<\cdots<i_{k}$ we have

$$
\sum_{j} \frac{\partial}{\partial y_{s}}\left(\frac{\partial\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{k}}\right)}{\partial\left(y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k}}\right)}\right) d y_{s} \wedge d y_{j_{1}} \wedge \cdots \wedge d y_{j_{k}}=0
$$

But this exactly the calculus lemma. The proof is complete.

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    ${ }^{0} 2010$ Mathematics Subject Classification 58A10
    ${ }^{0}$ Key words: Manifolds, Exterior derivative, Tensors

