MACAULAY 2 TUTORIALS

Clare D'Cruz
Chennai Mathematical Institute, Chennai

A. V. Jayanthan
Indian Institute of Technology, Bombay, Mumbai

Bharat Adsul
Indian Institute of Technology, Bombay, Mumbai

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A1. Basic Commands in Macaulay2

Getting Started with Macaulay2 :

Start Macaulay2 with the command M2, and you will be provided with an input prompt
Algebraic Operations :

```
i1 : 2+2

o1 : 4

i2 : 3/5 + 7/11

     68
o2 : --
     55

o2 : QQ
```

The symbol QQ is the notation for the class of all rational numbers. Multiplication is with $\star$ and powers can be obtained with $\wedge$

```
i3 : 2*3*4

o3 : 24

i4 : 2^8

o4 : 256
```

and the factorial is given as usual :

```
i5 : 6!

o5 : 720
```

* Rings and Fields in Macaulay2 :
ZZ - The ring of integers
QQ - The field of rationals
RR - The field of reals
CC - The field of complex numbers
ZZ/p - Finite field of characteristic p=prime.
Warning : Macaulay2 does not accept ring with characteristic a composite integer. Polynomial rings are defined over ZZ, ZZ/p and QQ.

```
i6 : k=ZZ/32003

o6 : k

o6 : QuotientRing

i7 : R=k[x,y,z]
```

```
o7 : R

o7 : PolynomialRing
```

If the number of variables is large we can define:

```
i8 : R=k[a..f]

o8 : R

o8 : PolynomialRing
```

This will define the ring $k[a, b, c, d, e, f]$

```
i9 : R=k[x_0 .. x_6]

o9 : R

o9 : PolynomialRing
```

This gives the ring $k[x_0, x_1, .., x_6]$

Different orders in Macaulay2 :
The default order in Macaulay2 is the graded reverse lexicographic order. Other orders one can define are :

GLex	graded lexicographic order
Lex	lexicographic order
RevLex	reverse lexicographic order
Eliminate n	elim order, eliminating first n variables
ProductOrder {n1, n2, ..., nv}	product order

```
i10 : R=k[x,y,z, MonomialOrder => GLex]

o10 : R

o10 : PolynomialRing
```

This defines polynomial ring R with graded lexicographic ordering.

To print a variable already defined, use describe :

```
i11 : describe R
ZZ/32003[x,y,z]
```

Ideals of Polynomial ring:

```
i12 : I = ideal (x^2+y^2, x^3*y+x*y^3)

              2 2   3   3
o12 : ideal (x+y , xy+x*y )
```

```
o12 : ideal of R

i13 : J = ideal(x^3*y+x*y^2,x^4,x^3*y^2)

              3     2   4   3 2
o13 = ideal(x*y+x*y , x , x*y )

o13 : Ideal of R

i14 : I + J

              2    2   3       3   3       2   4   3 2
o14 : ideal (x  + y , x y + x*y , x y + x*y , x , x y )

o14 : Ideal of R

i15 : mingens (I * J)

o15 : | x3y3+xy5 x4y2+x2y4 x5y+x3y3+x3y2+xy4 x6+x4y2 |

              1       4
o15 : Matrix R  <--- R
```

This calculates the minimal generators of IJ.

The quotient rings are defined as follows :

```
i16 : S = R/I

           R
o15 : -------------
       2 2   3    3
     (x+y , x*y+x*y )

o15 : QuotientRing
```

Definition : Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a Groebner Basis if

$$(\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t)) = (\mathrm{LT}(I)).$$

In Macaualay2, the command `gb` is used to compute the Groebner Basis with respect to a specified monomial ordering.

```
i1 : R = QQ[x,y]

o1 = R

o1 : PolynomialRing

i2 : I = ideal(x^6-x*y^3+x, x^6+y^5-x^4*y, x^6)

             6      3       6    4     5   6
```

```
o2 = ideal (x  - x*y  + x, x  - x y + y , x )

o2 : Ideal of R

i3 : gb I

o3 = {0} | x y5 |

o3 : GroebnerBasis
```

Exercises :

1. Define polynomial ring with 10 variables in two different ways.
2. Define the 3rd power and 4th power of the homogeneous maximal ideal in the above ring and compute I + J, I * J and R/(I + J), R/(I*J).
3. Find the GB for the ideal $I = (b^5 + a^2 + c^2 - 4, b^2 + 2a^2 - 5, bc - 1)$ with different monomial orderings.
4. Determine the following sets G are GB for the ideal they generate
   (i) $G = \{x^2 - y, x^3 - z\}$ - GLex order
   (ii) $G = \{y - x^2, z - x^3\}$ - Lex order with $x > y > z$.
   (iii) $G = \{xy^2 - xz + y, xy - z^2, x - yz^4\}$ - Lex order
   (iv) $G = \{x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z\}$ - Default order.

A2. Gauss Elimination Method

We use the Gauss elimination method to find the solution for a system of linear equations. Linear Algebra theory says that we can write the coefficient matrix, make it into row-reduced echelon form and then solve it to get the solution for the original equations. Here the row-reduced echelon form is obtained from the GB. Suppose we are given a set of linear equations

```
3x - 6y - 2z      = 0
2x - 4y      + 4w = 0
 x - 2y - z  -  w = 0.
```

Then the coefficient matrix is given by

```
| 3 -6 -2  0 |
| 2 -4  0  4 |
| 1 -4 -1 -1 |
```

We proceed as follows :

```
i1 : k=ZZ/32003

o1 = k

o1 : QuotientRing

i2 : R=k[x,y,z,w]
```

```
o2 = R

o2 : PolynomialRing

i3 : I=ideal(3*x-6*y-2*z, 2*x-4*y+4*w, x-2*y-z-w)

o3 = ideal (3*x - 6*y - 2*z, 2*x - 4*y + 4*w, x - 2*y - z - w)

o3 : Ideal of R

i5 : gb I

o5 = {0} | x-2y+2w z+3w |

o5 : GroebnerBasis
```

One can easily check that the row-reduced echelon form is

```
| 1 -2 0 2 |
| 0  0 1 3 |
| 0  0 0 0 |
```

which is also the coefficient matrix of the GB.

Exercise : Find the solution of the following set of equations

1. 2x + 4y - z = 0, 3x - 4y = 0, x - y + z = 0
2. x - 3y + z - w = 0, 4x - 2w = 0, 3y - 4z + w = 0

A3. Solutions to Polynomial equations

In this experiment we show how Groebner basis can be helpful to solve polynomial equations.

```
i1 : R=QQ[x,y,z,MonomialOrder => Lex]

o1 = R

o1 : PolynomialRing

i2 : I=ideal(x^2+y^2+z^2-1, x^2+z^2-y, x-z)

             2    2    2        2         2
o2 = ideal (x  + y  + z  - 1, x  - y + z , x - z)

o2 : Ideal of R

i3 : K = gb I
```

```
o3 = {0} | z4-1/-2z2+1/-4 y-2z2 x-z |

o3 : GroebnerBasis
```

Now the solution set of K is same as the solution set of the given polynomials. The first element of the GB is polynomial in z. Solving this equation, we get the values of z and then substituting in the second and third equations, we get values of x and y.

Exercise 1 : Find the solution to the following system of equations:
$x^2 + y^2 + z^2 = 4$
$x^2 + 2y^2 = 5$
$xz = 1$

Exercise 2 : Change the power of x to 5 in the first equation of Exercise 1 and carry out the experiment.

A4. Consistency Algorithm

This experiment describes the consistency algorithm : If we have polynomials f1, ..., fs in $\mathbb{C}[x_1, ..., x_n]$, we compute a reduced GB of the ideal they generate with respect to any ordering. If this basis is 1, then the polynomials have no common zero in $\mathbb{C}^n$.

```
i1 : k=ZZ/32003

o1 = k

o1 : QuotientRing

i2 : R=k[x,y,z]

o2 = R

o2 : PolynomialRing

i3 : f1=x^2+y^2-1

      2    2
o3 = x  + y  - 1

o3 : R

i4 : f2=y-1

o4 = y - 1

o4 : R

i5 : J=ideal(f1, f2)
```

```
               2    2
o5 = ideal (x  + y  - 1, y - 1)

o5 : Ideal of R

i6 : gb J

o6 = {0} | y-1 x2 |

o6 : GroebnerBasis
```

This shows that f1 and f2 have common solution.

Exercise : Check whether the following set of polynomials have common solution:

1. $y^3 - x^3, y^2 - x^2 - 9$
2. $x^2 - y^2 + z^2 - 16, xyz = 4, x^3 + y^2xz - 3$
3. $x^3 - y^3 - 8, x^2 + y - 8x - 10y + 41$

A5. Existence of common factor

This experiment shows how one can apply theory of resultant to say if the given polynomials have a common factor.

```
i2 : k=ZZ/32003

o2 = k

o2 : QuotientRing

i3 : R=k[x,y,MonomialOrder => Lex]

o3 = R

o3 : PolynomialRing

i4 : f = x^2 - 2*x*y + y^2

      2            2
o4 = x  - 2*x*y + y

o4 : R

i5 : g = x^2 - y^2

      2    2
o5 = x  - y
```

```
o5 : R

i6 : I = ideal(f, g)

              2            2   2    2
o6 = ideal (x  - 2*x*y + y , x  - y )

o6 : Ideal of R

i7 : K = gb I

o7 = {0} | x2-y2 xy-y2 |

o7 : GroebnerBasis
```

This shows that K intersection k[y] is zero. Therefore by the theory of resultant the polynomials f and g have a common factor.

```
i8 : f1 = x^2 - 2*x*y + y^2

      2            2
o8 = x  - 2*x*y + y

o8 : R

i9 : g1 = x^2 + 2*x*y + y^2

      2            2
o9 = x  + 2*x*y + y

o9 : R

i10 : J = ideal(f1, g1)

               2            2   2            2
o10 = ideal (x  - 2*x*y + y , x  + 2*x*y + y )

o10 : Ideal of R

i11 : L = gb J

o11 = {0} | x2+y2 xy y3 |

o11 : GroebnerBasis
```

This shows that L intersection k[y] is non-empty, and thus the polynomials f1 and g1 does not have a common factor.

Exercise :

1. $x^4 - 2x^3y - 3x^2y^2 + xy^3 + y^4,\ x^5 - 5x^4y + 6x^3y^2 + x^2y^3 - 2xy^4$
2. $x^4 + y^4 + x^2y^2,\ x^4 + 2x^3y - 2xy^3 - y^4$

A6. Intersection of ideals

We have the following theorem : Let I, J be ideals in $k[x_1, ..., x_n]$ and w be a new variable. Consider the ideal(wI, (1-w)J) in $k[x_1, ..., x_n, w]$. Then $I \cap J = (wI, (1-w)J) \cap k[x_1, ..., x_n]$.

We demonstrate the use of GB theory to compute intersection of two ideals.

```
i2 :  R = QQ[w,x,y,MonomialOrder => Eliminate 1]

o2 = R

o2 : PolynomialRing

i3 : I= ideal(x^2+y^3-1, x-y*x+3)

             3    2
o3 = ideal (y  + x  - 1, - x*y + x + 3)

o3 : Ideal of R

i4 :  J = ideal(x^2*y-1)

            2
o4 = ideal(x y - 1)

o4 : Ideal of R


i5 : IwJ = w * I + (1-w) * J

               3      2                                 2         2
o5 = ideal (w*y  + w*x  - w, - w*x*y + w*x + 3*w, - w*x y + w + x y - 1)

o5 : Ideal of R

i6 : gb IwJ

o5 = {0} | x3y2-x3y-3x2y-xy+x+3 x2y4+x4y-x2y-y3-x2+1 x5y-3924/1189x2y4-
3924/1189x4y+3x2y3+3x2y2-x3-7491/-1189x2y+3924/1189y3+3924/1189x2-3y2-
3y+7491/-1189 w+119/12853x2y5-237/12853x2y4+9/12853x2y3-1091/12853x2y2-
119/12853y4-2389/-12853x2y+237/12853y3-9/12853y2+1091/12853y+2389/-12853 |

o6 : GroebnerBasis
```

¿From the above set we ignore the polynomials involving variable w. Then we get the generators of $I \cap J$ as $(x^3y^2 - x^3y - 3x^2y - xy + x + 3, x^2y^4 + x^4y - x^2y - y^3 - x^2 + 1, x^5y - 3924/1189x^2y^4 - 3924/1189x^4y + 3x^2y^3 + 3x^2y^2 - x^3 + 7491/1189x^2y + 3924/1189y^3 + 3924/1189x^2 - 3y^2 - 3y - 7491/1189)$

Exercise : Compute the intersection of the following ideals.

1. $I = (xy+1, x^2-1),\ J = (x^2+y, y+x)$
2. $I = (x, y),\ J = (x-1, y),\ K = (x-2, y-1)$
3. $I = (x^2y - z - 1, xy + y + 1),\ J = (x-y, z^2 - x)$

Use the following fact to compute the GCD.
Fact : For $f, g \in k[x_1, \ldots, x_n]$, both non-zero, we have

$$(f) \cap (g) = (\mathrm{lcm}(f, g))$$

Exercise : Compute the gcd of the following polynomials

1. $f = x^2y^2 - y^2 + x^2 - 1,\ g = xy^2 - y^2 - x + 1$
2. $f = x^3 - x^2y - 3x^2 + xy - y^2 - 3y,\ g = x^3y + xy^2 + 2x^2 + 2y$

A7. Colon of two ideals

We apply the following two lemmas to compute the colon of two ideals :
Lemma 1: Let $I = (f_1, \ldots, f_s)$ and J be ideals in $k[x_1, \ldots, x_n]$. Then

$$J : I = \bigcap_{i=1}^{s} J : (f_i)$$

Lemma 2: Let J be an ideal and $f \neq 0$ be a polynomial in $k[x_1, \ldots, x_n]$, then

$$J : (f) = \frac{1}{f}(J \cap (f))$$

We make use of the previous experiment to compute the intersection of the ideals.

```
i2 : k=ZZ/32003

o2 = k

o2 : QuotientRing

i3 : R=k[w,x,y,MonomialOrder => Lex]

o3 = R

o3 : PolynomialRing

i4 : f1 = x^2

      2
o4 = x

o4 : R
```

```
i5 : f2 = x+y

o5 = x + y

o5 : R

i6 : g1 = x*(x+y)^2

      3      2       2
o6 = x  + 2*x y + x*y

o6 : R

i7 : g2 = y

o7 = y

o7 : R

i8 : I = ideal(f1, f2)

             2
o8 = ideal (x , x + y)

o8 : Ideal of R

i9 : J = ideal(g1,g2)

             3      2       2
o9 = ideal (x  + 2*x y + x*y , y)

o9 : Ideal of R
```

By Lemma 1, $J : I = J : f1 \cap J : f2$ and by Lemma 2 $J : fi = (1/fi)(J \cap fi)$. We calculate $J \cap f1$ and $J \cap f2$ by the previous method.

```
i17 : Jwf1 = w * J + ideal((1-w) * f1)

                3        2         2             2    2
o17 = ideal (w*x  + 2*w*x y + w*x*y , w*y, - w*x  + x )

o17 : Ideal of R

i18 : Jwf2 =  w * J + ideal((1-w) * f2)

                3        2         2
o18 = ideal (w*x  + 2*w*x y + w*x*y , w*y, - w*x - w*y + x + y)

o18 : Ideal of R

i19 : gb Jwf1

o19 = {0} | x2y x3 wy wx2-x2 |
```

```
o19 : GroebnerBasis

i20 : gb Jwf2

o20 = {0} | xy+y2 x3+y3 wy wx-x-y |

o20 : GroebnerBasis
```

Therefore by elimination theorem GB of $Jwf1 \cap k[x,y]$ is $\{x^2y, x^3\}$ and GB of $Jwf2 \cap k[x,y]$ is $\{xy+y^2, x^3+y^3\}$. Now we find $\frac{1}{f1}(J:f1)$ and $\frac{1}{f2}(J:f2)$

```
i21 : (x^2*y)/f1

o21 = y

o21 : frac R

i22 : (x^3)/f1

o22 = x

o22 : frac R

i23 : (x*y+y^2)/f2

o23 = y

o23 : frac R

i24 : (x^3+y^3)/f2

       2          2
o24 = x  - x*y + y

o24 : frac R
```

Thus $\frac{1}{f1}(J \cap f1) = (x,y)$ and $\frac{1}{f2}(J \cap f2) = (x^2 - xy + y^2, y)$. Now we find their intersection to get $J : I$

```
i25 : J1 = ideal(x,y)

o25 = ideal (x, y)

o25 : Ideal of R

i26 : J2 = ideal (x^2-x*y+y^2, y)

              2          2
o26 = ideal (x  - x*y + y , y)

o26 : Ideal of R

i27 : J1wJ2 = w * J1 + (1-w) * J2

                           2              2    2          2
```

```
o27 = ideal (w*x, w*y, - w*x  + w*x*y - w*y  + x  - x*y + y , - w*y + y)

o27 : Ideal of R

i28 : gb J1wJ2

o28 = {0} | y x2 wx |
```

Therefore GB of $J1wJ2 \cap k[x,y]$ is (x^2, y) which is the ideal $J : I$.

Exercise : Compute $J : I$ for the following ideals

1. $I = (x^2y - y + x, xy^2 - x),\ J = (x^2 + y^2, x^3 + y)$
2. $I = (y - x^2, x^2 - y^2),\ J = (2xy - 2x - y^2 + y, y^3 - 3y^2 + 2y, x^3 - 2x^2 - x - y^2 + 3y)$

A8. Saturation of ideals

In this experiment we calculate, for given two ideals I and J, the ideal $\cap_{i\geq 0} I : J^i$, called saturation of I w.r.t J.

```
i1 : k=ZZ/32003

o1 = k

o1 : QuotientRing

i2 : R=k[w,x,y,z,MonomialOrder => Lex]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(x^4*y^2+z^2-4*x*y^3*z-2*y^5*z, x^2+2*x*y^2+y^4)

             4 2        3       5     2   2        2    4
o3 = ideal (x y  - 4*x*y z - 2*y z + z , x  + 2*x*y  + y )

o3 : Ideal of R

i4 : J = ideal(y*z-x^3)

              3
o4 = ideal(- x  + y*z)

o4 : Ideal of R

i10 : Jw = ideal(1-w*y*z+w*x^3)
```

```
                3
o10 = ideal(w*x  - w*y*z + 1)

o10 : Ideal of R

i11 : IwJ = I + Jw

              4 2         3       5     2   2         2    4     3
o11 = ideal (x y  - 4*x*y z - 2*y z + z , x  + 2*x*y  + y , w*x  - w*y*z + 1)

o11 : Ideal of R

i12 : gb IwJ

o12 = {0} | 1 |

o12 : GroebnerBasis
```

Exercise : Find $I : J^\infty$ for the following ideals.

1.

A9. Ideal Membership Problem

Ideal membership algorithm : Given an ideal $I = (f_1, \ldots, f_s)$, we can decided whether a given polynomial f lies in I as follows. Find a GB $G = \{g_1, \ldots, g_t\}$ for I. Then

$$f \in I \text{ iff } f \,\%\, G = 0$$

where $f \,\%\, G$ denotes a remainder obtained in the process of dividing f by G.

```
i1 : R = QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : I = ideal(x*z - y^2, x^3-z^2)

               2         3    2
o2 = ideal (- y  + x*z, x  - z )

o2 : Ideal of R

i3 : f = -4*x^2*y^2*z^2+y^6+3*z^5

      6      2 2 2      5
o3 = y  - 4*x y z  + 3*z
```

```
o3 : R

i4 : g = x*y-5*z^2+x

                 2
o4 = x*y - 5*z  + x

o4 : R

i5 : K = gb I

o5 = {0} | y2-xz x3-z2 |

o5 : GroebnerBasis

i6 : f % K

o6 = 0

o6 : R

i7 : g % K

                 2
o7 = x*y - 5*z  + x

o7 : R
```

This shows that $f \in I$ and $g \notin I$.

Exercise : Determine whether the following polynomial f is in the ideal I.

1. $f = xy^3 - z^2 + y^5 - z^3,\ I = (-x^3 + y, x^2y - z)$
2. $f = x^3z - 2y^2,\ I = (xz - y, xy + 2z^2, y - z)$

Show that the polynomial $f = y - x^2$ is in the ideal $I = (x - z, y - z^2)$. Can you express f as $g_1(x - z) + g_2(y - z^2)$ for appropriate g_i's.

A10. Radical Membership Problem

Let $I = (f_1, \ldots, f_s)$ be an ideal in $k[x_1, \ldots, x_n]$. Then $f \in \sqrt{I}$ iff $1 \in J = (f_1, \ldots, f_s, 1 - wf) \subset k[x_1, \ldots, x_n, w]$. Note that $1 \in J$ iff GB of $J = \{1\}$.

```
i1 : R = QQ[x,y,w]

o1 = R
```

```
o1 : PolynomialRing

i2 : I = ideal(x*y^2+2*y^2, x^4-2*x^2+1)

               2      2   4      2
o2 = ideal (x*y  + 2*y , x  - 2*x  + 1)

o2 : Ideal of R

i3 : f = y-x^2+1

        2
o3 = - x  + y + 1

o3 : R

i4 : fw = ideal(1-w*f)

            2
o4 = ideal(x w - y*w - w + 1)

o4 : Ideal of R

i5 : J = I + fw

               2      2   4      2       2
o5 = ideal (x*y  + 2*y , x  - 2*x  + 1, x w - y*w - w + 1)

o5 : Ideal of R

i6 : gb J

o6 = {0} | 1 |

o6 : GroebnerBasis
```

Therefore $f \in \sqrt{I}$. Can you find the power of f which is in I ?

Exercise : Determine whether the following polynomials lie in the following ideals, if the answer is yes, what is the smallest power of the polynomial that lies in the ideal ?

1. $f = x + y,\ I = (x^3, y^3, xy(x+y))$
2. $f = x^2 + 3xz,\ I = (x+z, x^2y, x-z^2)$
3. $f = yz - x^3,\ I = (x^4y^2 + z^2 - 4xy^3z - 2y^5z, x^2 + 2xy^2 + y^4)$

B1. Computation of HOM

Let $R = Q[x,y,z]$ with lex order with $x > y > z$. Consider modules $M =< f_1, f_2, f_3, f_4 >\subseteq R^3$ where $f_1 = [xy, xz, yz]^T, f_2 = [y, x, y]^T, f_3 = [0, x^3 - x^2z, x^2y - xyz]^T$ and let $N =< g_1, g_2, g_3 >\subseteq R^2$ where $g_1 = [x^2y^2]^T, g_2 = [x^2, yz]^T, g_3 = [x^2z, xy^2 - yz^2]^T$

i1 : $R = QQ[x, y, z,$ MonomialOrder => GLex]

o1 = R

o1 : PolynomialRing

i2 : m =transpose matrix $\{\{x*y, x*z, y*z\}, \{y, x, y\}, \{0, x\hat{}3-x\hat{}2*z, x\hat{}2*y-x*y*z\}, \{y*z, x\hat{}2, x*y\}\}$

$$o2 = \begin{array}{c} \{-2\} \\ \{-3\} \\ \{-3\} \end{array} \left| \begin{array}{cccc} xy & y & 0 & yz \\ xz & x & x3-x2z & x2 \\ yz & y & x2y-xyz & xy \end{array} \right|$$

o2 : Matrix $R^3 <---R^4$

i3 : n =transpose matrix $\{\{x\hat{}2, y\hat{}2\}, \{x\hat{}2, y*z\}, \{x\hat{}2*z, x*y\hat{}2+y*z\hat{}2\}\}$

$$o3 = \begin{array}{c} \{-3\} \\ \{-3\} \end{array} \left| \begin{array}{ccc} x2 & x2 & x2z \\ y2 & yz & xy2+yz2 \end{array} \right|$$

o3 : Matrix $R^2 <---R^3$

We first need to compute presentations of M and N. To do this let $L = Syz(f_1, f_2, f_3, f_4)$ and compute L to be: $< (-1, x+z, 0, 1), (-z, z^2, -1, x-z) > \subseteq R^3$.

i4 : L = gb syz m

$$o4 = \begin{array}{c} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{array} \left| \begin{array}{cc} -1 & -z \\ x+z & z2 \\ 0 & -1 \\ -1 & x-z \end{array} \right|$$

o4 : GroebnerBasis

We also let $K =, Syz(g_1, g_2, g_3)$ to be: $K =< (yx, -yx+yz-z^2, -y+z) > \subseteq R^2$.

i5 : K = gb syz n

$$o5 = \begin{array}{c} \{0\} \\ \{0\} \\ \{0\} \end{array} \left| \begin{array}{c} -xy \\ xy-yz+z2 \\ y-z \end{array} \right|$$

o5 : Matrix $R^3 <---R^1$

So we have presentations $M \cong A^4/L$ and $N \cong A^3/N$. We now consider the matrix T:

$$\begin{aligned} i6 : T \quad = \quad & \text{transpose (matrix } \{ \\ & \{-1, 0, 0, x+z, 0, 0, 0, 0, -1, 0, 0\}, \\ & \{0, -1, 0, 0, x+z, 0, 0, 0, 0, 0, -1, 0\}, \\ & \{0, 0, -1, 0, 0, x+z, 0, 0, 0, 0, 0, -1\}, \\ & \{-z, 0, 0, z^2, 0, 0, -1, 0, 0, x-z, 0, 0\}, \\ & \{0, -z, 0, 0, z^2, 0, 0, -1, 0, 0, x-z, 0\}, \\ & \{0, 0, -z, 0, 0, z^2, 0, 0, -1, 0, 0, x-z\}\}) \end{aligned}$$

$$o6 = \begin{array}{c} \{-1\} \\ \{-1\} \\ \{-1\} \\ \{-2\} \\ \{-2\} \\ \{-2\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{-1\} \\ \{-1\} \\ \{-1\} \end{array} \left| \begin{array}{cccccc} -1 & 0 & 0 & -z & 0 & 0 \\ 0 & -1 & 0 & 0 & -z & 0 \\ 0 & 0 & -1 & 0 & 0 & -z \\ x+z & 0 & 0 & z2 & 0 & 0 \\ 0 & x+z & 0 & 0 & z2 & 0 \\ 0 & 0 & x+z & 0 & 0 & z2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & x-z & 0 & 0 \\ 0 & -1 & 0 & 0 & x-z & 0 \\ 0 & 0 & -1 & 0 & 0 & x-z \end{array} \right|$$

$o6$: Matrix $R^{12} <---R^{6}$

To compute the kernel of the map $R^{12} \longrightarrow R^{6}/(K \oplus K)$ given by T composed with the projection $R^{6} \longrightarrow R^{6}/(K \oplus K)$, we compute the syzygy module if the columns of T and of $K \oplus K$.

i7 : $K =$ gens K

$$o7 = \begin{array}{c} \{0\} \\ \{0\} \\ \{0\} \end{array} \left| \begin{array}{c} -xy \\ xy-yz+z2 \\ y-z \end{array} \right|$$

o7 : Matrix $R^{3} <---R^{1}$

i8 : delta = transpose$(K++K)$

$$o8 = \begin{array}{c} \{-2\} \\ \{-2\} \end{array} \left| \begin{array}{cccccc} -xy & xy-yz+z2 & y-z & 0 & 0 & 0 \\ 0 & 0 & 0 & -xy & xy-yz+z2 & y-z \end{array} \right|$$

o8 : Matrix $R^{2} <---R^{6}$

The first 12 entries of the generators of the syzygy module are the columns of the matrix U, where

i9: $U =$ generators(gb (syz transpose (T || delta)))

$$o9 = \begin{array}{c} \{1\} \\ \{1\} \\ \{1\} \\ \{2\} \\ \{2\} \\ \{2\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{1\} \\ \{1\} \\ \{1\} \\ \{2\} \\ \{2\} \end{array} \left| \begin{array}{cccccccccc} y & 0 & x & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -y & yz-z2 & 0 & x & 0 & 0 & -1 & 0 & 0 & yz-z2 \\ 0 & -y+z & 0 & 0 & x & 0 & 0 & -1 & 0 & -y+z \\ 0 & -y & 1 & 0 & 0 & 0 & 0 & 0 & -xy & -y \\ 0 & y & 0 & 1 & 0 & 0 & 0 & 0 & xy-yz+z2 & y \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & y-z & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ yz-z2 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ -y+z & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ -y & -yz & z & 0 & 0 & 1 & 0 & 0 & 0 & -xy-yz \\ y & yz & 0 & z & 0 & 0 & 1 & 0 & 0 & xy+z2 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 1 & 0 & y-z \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -x-z & 0 \\ -1 & -z & 0 & 0 & 0 & 0 & 0 & 0 & -z2 & -x \end{array} \right|$$

o9 : Matrix $R^{14} <---R^{10}$

The following 12 entries are the generators of the syzygy module:

i10 : $U1 =$ submatrix$(U, \{0..11\}, \{0..7\})$

$$o10 = \begin{array}{c} \{1\} \\ \{1\} \\ \{1\} \\ \{2\} \\ \{2\} \\ \{2\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{1\} \\ \{1\} \\ \{1\} \end{array} \left| \begin{array}{cccccccc} y & 0 & x & 0 & 0 & -1 & 0 & 0 \\ -y & yz-z2 & 0 & x & 0 & 0 & -1 & 0 \\ 0 & -y+z & 0 & 0 & x & 0 & 0 & -1 \\ 0 & -y & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ yz-z2 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ -y+z & 0 & 0 & 0 & 0 & 0 & 0 & x \\ -y & -yz & z & 0 & 0 & 1 & 0 & 0 \\ y & yz & 0 & z & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 1 \end{array} \right|$$

o10 : Matrix $R^{12} <---R^{8}$

We have

$$\mathrm{H}om(M,N) \cong \mathrm{H}om(R^{4}/L, R^{3}/K) \cong < U > / < K \oplus K \oplus K \oplus K >$$

Finally to compute a presentation of $< U1 > / < K \oplus K \oplus K \oplus K >$ we conpute the syzygy module of the eight columns of $U1$ and the four columns of $< K \oplus K \oplus K \oplus \oplus K >$.

i11 : $D = \text{transpose}\ (K + +K + +K + +K)$

$$o11 = \begin{array}{c|cccccccccccc|}
\{-2\} & -xy & xy-yz+z2 & y-z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{-2\} & 0 & 0 & 0 & -xy & xy-yz+z2 & y-z & 0 & 0 & 0 & 0 & 0 & 0 \\
\{-2\} & 0 & 0 & 0 & 0 & 0 & 0 & -xy & xy-yz+z2 & y-z & 0 & 0 & 0 \\
\{-2\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -xy & xy-yz+z2 & y-z
\end{array}$$

o11 : Matrix $R^4 < - - - R^{12}$

i12 : $U2 = \text{transpose}\ (U1||D)$

$$o12 = \begin{array}{c|cccccccccccc|}
\{0\} & y & 0 & x & 0 & 0 & -1 & 0 & 0 & -xy & 0 & 0 & 0 \\
\{0\} & -y & yz-z2 & 0 & x & 0 & 0 & -1 & 0 & xy-yz+z2 & 0 & 0 & 0 \\
\{0\} & 0 & -y+z & 0 & 0 & x & 0 & 0 & -1 & y-z & 0 & 0 & 0 \\
\{0\} & 0 & -y & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -xy & 0 & 0 \\
\{0\} & 0 & y & 0 & 1 & 0 & 0 & 0 & 0 & 0 & xy-yz+z2 & 0 & 0 \\
\{0\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & y-z & 0 & 0 \\
\{0\} & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & -xy & 0 \\
\{0\} & yz-z2 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & xy-yz+z2 & 0 \\
\{0\} & -y+z & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & y-z & 0 \\
\{0\} & -y & -yz & z & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -xy \\
\{0\} & y & yz & 0 & z & 0 & 0 & 1 & 0 & 0 & 0 & 0 & xy-yz+z2 \\
\{0\} & 0 & 0 & 0 & 0 & z & 0 & 0 & 1 & 0 & 0 & 0 & y-z
\end{array}$$

o12 : Matrix $R^{12} < - - - R^{12}$

i13 : $S = \text{syz}\ U2$

$$o13 = \begin{array}{c|cccc|}
\{2\} & 1 & 0 & x & 0 \\
\{3\} & 0 & 1 & -1 & -x \\
\{2\} & 0 & y & -y & 0 \\
\{2\} & 0 & -y & y & yz-z2 \\
\{2\} & 0 & 0 & 0 & -y+z \\
\{1\} & y & 0 & 0 & 0 \\
\{1\} & -y & 0 & -yz+z2 & 0 \\
\{1\} & 0 & 0 & y-z & 0 \\
\{2\} & 0 & 1 & 0 & 0 \\
\{2\} & 0 & 0 & 0 & 1 \\
\{2\} & 1 & 0 & 0 & 0 \\
\{2\} & 0 & 0 & -1 & z
\end{array}$$

o13 : Matrix $R^{12} < - - - R^4$

There are 4 generators for this syzygy module and we need the “first” 8 coordinates of each of them, which give

Therefore,

$$\text{Hom(M,N)} \cong \text{Hom}(A^4/L, A^3/K) \cong A^8/ < t_1, t_2, t_3, t_4 >$$

Exercise 1 Let $A = Q[x, y, z]$, $M = A^3/L$, $N = A^4/K$ where

$$\begin{aligned}
L &= < (x, x+y, z), (x, x^2, x-z), (-xz, x-z, y-z) > \text{ and} \\
K &= < (x+y, x^2, xy+yz, y-z), (xy, x-z, y-z, xz^2), (xy, y-z, x-z, yz^2) >
\end{aligned}$$

Compute the generators for, and the presentation of $\text{Hom}(M, N)$.

Exercise 2 Consider the following modules

$$\begin{aligned} M &= \ < (0,y,x),(y,0,y),(x,-y,xy),(0,x,x) > \subseteq (Q[x,y])^3 \\ N &= \ < (xy,y,y,y),(x^2,x,x,x) > \subseteq (Q[x,y])^4 \end{aligned}$$

Compute the generators for, and the presentation of $\mathrm{Hom}(M,N)$.

B2. Implicitization

Example 1. Tangent surface of the twisted cubic

We know that the parametric representation of the twisted cubic is given by $x = t, y = t^2, z = t^3$ ad that of the tangent surface is $x = t + u, y = t^2 + 2tu, z = t^3 + 3t^2u$.

We now solve the implicitzation problem for the tangent surface of the twisted cubic.

i1 : $R = QQ[t, u, x, y, z,$ MonomialOrder =>Eliminate 2]

o1 = R

o1 : PolynomialRing

We now define the ideal:

i2 : $I =$ ideal $(x - t - u, y - t\hat{}2 - 2 * t * u, z - t\hat{}3 - 3 * t\hat{}2 * u)$

o2 = ideal $(-t - u + x, -t^2 - 2t * u + y, -t^3 - 3t^2u + z)$

o2 : Ideal of R

i3 : $J =$ selectInSubring(1,gens gb I)

$$o3 = \quad \{0\} \quad | \quad x2y2 + 4/-3x3z + 4/-3y3 + 2xyz - 1/3z2 \quad |$$

o3 : Matrix $R^1 <---R^1$ This gives the equation of the tangent surface of the twisted cubic.

Example 2. Given a parametric representation find the defining equation.

In our final expression we would like to eliminate the variables u, t.

i1 : $R = QQ[t, x, y,$ MonomialOrder => Eliminate 1]

o1 = R

o1 : PolynomialRing

i2 : $I =$ ideal $(x - 1 - t, y - 1 - t\hat{}2)$

o2 = ideal $(-t + x - 1, -t^2 + y - 1)$

o2 : Ideal of R

i3 : $J =$ selectInSubring(1, gens gb I)

$$o3 = \quad \{0\} \quad | \quad x2 - 2x - y + 2 \quad |$$

o3 : Matrix $R^1 <---R^1$

Exercise 1. The Whitney Umbrella surface is parametrically given by: $x = uv, y = v, z = u^2$. Find the equations of the smallest variety parameterizing this surface.

Exercise 2 The Enneper surface is defined parametrically by the $x = 3u+3uv^2-u^3$, $y = 3v+3u^2v-v^3$, $z = 3u^2 - 3v^2$ Find the equation of the smallest variety containing the Enneper surface. It will be a very complicated equation.

Exercise 3 Let S be the parametric surface $x = uv$,$y = uv^2$, $z = u^2$. Find the equation of the smallest variety that contains S.

Exercise 4 The most interesting example is to find the defining ideal of the projective curve given parametrically on an affine piece by $t| - -- > (t^{a1}, ..., t^{an})$.

B3 Ring of Invariants

i2 : $T = QQ[a, b,$ MonomialOrder => Lex]

o2 = T

o2 : PolynomialRing

i3 : M= matrix{{a, b }}

$$o3 = \quad \{0\} \quad | \quad a \quad b \quad |$$

o3 : Matrix $T^1 <---T^2$

i4 : I = matrix {{1,0}, {0,1}}

$$o4 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

o4 : Matrix $ZZ^2 <---ZZ^2$

i5 : $I = \text{substitute}(I, T)$

$$o5 = \begin{array}{c} \{0\} \\ \{0\} \end{array} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

o5 : Matrix $T^2 <---T^2$

i6 : I2 = substitute(matrix {{-1,0},{0,-1}}, T)

$$o6 = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$$

o6 : Matrix $T^2 <---T^2$

i7 : $I3 = \text{substitute}(\text{matrix } \{\{0, 1\}, \{-1, 0\}\}, T$

$$o7 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

o7 : Matrix $T^2 <---T^2$

i8 : $I4 = \text{substitute}(\text{matrix } \{\{0, -1\}, \{1, 0\}\}, T)$

$$o8 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

o8 : Matrix $T^2 <---T^2$

i9 : $J1 = M * I$

$$o9 = \quad \{0\} \quad | \quad a \quad b \quad |$$

o9 : Matrix $T^1 <---T^2$

i10 : $J2 = M * I2$

$$o10 = \quad \{0\} \quad | \quad -a \quad -b \quad |$$

o10 : Matrix $T^1 <---T^2$

i11 : $J3 = M * I3$

$$o11 = \quad \{0\} \quad | \quad -b \quad a \quad |$$

o11 : Matrix $T^2 <---T^2$

i12 : $J4 = M * I4$

$$o12 = \quad \{0\} \quad | \quad b \quad -a \quad |$$

o12 : Matrix $T^1 <---T^2$

i13 : $f1 = \text{map}(T, , J1)$

o13 = map$(T, T, \{a, b\})$

o13 : RingMap $T < - - -T$

i14 : $f2 = \text{map}(T, T, J2)$

o14 = map$(T, T, \{-a, -b\})$

o14 : RingMap $T < - - -T$

i15 : $f3 = \text{map}(T, T, J3)$

o15 = map$(T, T, \{-b, a\})$

o15 : RingMap $T < - - -T$

i16 : $f4 = \text{map}(T, T, J4)$

o16 = map$(T, T, \{b, -a\})$

o16 : RingMap $T < - - -T$

We now define a function which gives the ithe symmetric power of the variables of T, i.e. $s(i) = \text{gens } m^i$.

i16 : $s = (i)- >$ symmetricPower $(i,$ vars $T)$

o16 = s

o16 : Function

For example

i40 : $s(1)$

$$o40 = 0|ab|$$

o40 : Matrix $T^1 < - - -T^2$

We now define the function $g : Q[a, b] \longrightarrow Q[a, b]^G$

i24 : $g = (i)- >$ ideal (gens gb $(f1(s(i)) + f2(s(i)) + f3(s(i)) + f4(s(i))))$

o24 = g

o24 : Function

i24 : $g(1)$

o24 = 0

o24 : Ideal of T

i25 : $g(2)$

o25 = ideal$(a^2 + b^2\ )$

o25 : Ideal of T

i26 : $g(3)$

o26 = 0

o26 : Ideal of T

i27 : $g(4)$

o27 = ideal $(a^2b^2, a^3b - a * b^3, a^4 + b^4, a * b^4, b^5\ )$

o27 : Ideal of T

It is easy to se that $a^2 + b^2, a^2b^2, a^3b - a * b^3$ are contained in the ring of invariants. We will now compute the Hilbert Series of $Q[a^2 + b^2, a^2b^2, a^3b - a * b^3]$.

i28 : $W = QQ[t1, t2, t3, \text{Degrees } => \{2, 4, 4\}]$

o28 = W

o28 : PolynomialRing

i29 : $f = \text{map } (T, W, \text{ matrix}\{\{a\hat{\ }2 + b\hat{\ }2, a\hat{\ }2 * b\hat{\ }2, a\hat{\ }3 * b - a * b\hat{\ }3\}\})$

o29 = map$(T, W, \{a^2 + b^2, a^2b^2, a^3b - a * b^3\})$

o29 : RingMap $T < - - -W$

i30 : $K = $ ideal(gens gb (kernel f))

o30 = ideal$(t1^2t2 - 4t2^2 - t3^2\ )$

o30 : Ideal of W

i31 : $H = \text{hilbertSeries}\ (W/K)$

$$o31 = \quad \frac{(\$T^7 + \$T^6 + \$T^5 + \$T^4 + \$T^3 + \$T^2 + \$T + 1)(-\$T + 1)}{(-\$T^2 + 1)(-\$T^4 + 1)^2}$$

o31 : Divide

i32 : value H

$$o32 = \quad \frac{-\$T^4 - 1}{-\$T^6 + \$T^4 + \$T^2 - 1}$$

o32 : frac $ZZ[ZZ^1]$

We will compute the Molein Series to verify that $Q[a^2 + b^2, a^2b^2, a^3b - a * b^3]$ is the ring of invariants under the action of the group G on the ring $Q[a, b]$.

```
i20 :  t = quote t

o20 = t

o20 : Symbol

i21 : A = (coefficientRing T)[t]

o21 = A

o21 : PolynomialRing

i22 : m1 := substitute(I1, A)

o22 = {0} | 1 0 |
      {0} | 0 1 |

               2       2
o22 : Matrix A  <--- A

i23 : m2 := substitute(I2, A)

o23 = {0} | -1 0  |
      {0} | 0  -1 |

               2       2
o23 : Matrix A  <--- A

i24 : m3 := substitute(I3, A)

o24 = {0} | 0  1 |
      {0} | -1 0 |

               2       2
o24 : Matrix A  <--- A

i25 : m4 := substitute(I4, A)

o25 = {0} | 0 -1 |
      {0} | 1 0  |

               2       2
o25 : Matrix A  <--- A
```

```
i26 :
      d1 = 1 / det (1 - A_0* m1)

            1
o26 = -----------
       2
      t  - 2t + 1

o26 : frac A

i27 : d2 = 1 / det (1 - A_0* m2)

            1
o27 = -----------
       2
      t  + 2t + 1

o27 : frac A

i28 : d3 = 1 / det (1 - A_0* m3)

        1
o28 = ------
       2
      t  + 1

o28 : frac A

i29 : d4 = 1 / det (1 - A_0* m4)

        1
o29 = ------
       2
      t  + 1

o29 : frac A

i30 :
      Molein = 1/4 * ( d1 + d2 + d3 + d4)

             4
            t  + 1
o30 = ----------------
       6    4    2
      t  - t  - t  + 1

o30 : frac A
```

Exercise 1 Consider the finite matrix group C_4 generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

a. Show that C_4 is a cyclic group of order 4.
b. Determine $Q[x,y]^{C_4}$

Exercise 2 Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

a. Show that A generates C_3 a group of order 3 and $-A$ generates C_6a group of order 6.
b. Find $k[x,y]^{C_3}$ and $k[x,y]^{C_6}$.

Exercise 3 Consider the action of the dihedral group

$$D_6 = \{id, \delta, \delta^2, \delta^3, \delta^4, \delta^5, \sigma, \sigma\delta, \sigma\delta^2, \sigma\delta^3, \sigma\delta^4, \sigma\delta^5\}$$

on $Q[x,y,z]$ which is defined by the matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \delta = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 4 Consider the finite matrix group

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that A and $-A$ generate a group G of order 8. Find $k[x,y]^G$.

Exercise 5 Find the Molien Series of all the exercises considered above.

B5. Computation of Syzygy of a module

Example

Let $f_1 = y^2x^2 - yx^3$
$f_2 = y^3x - y^2x^2$
$f_3 = y^4 - x^3$. Let $F = [f_1 \quad f_2 \quad f_3]$.

i1 : $R = QQ[y, x,$ MonomialOrder => Lex]

o1 = R

o1 : PolynomialRing

i2 : $F =$ transpose matrix $\{\{x\verb|^|2 * y\verb|^|2 - x\verb|^|3 * y\}, \{x * y\verb|^|3 - x\verb|^|2 * y\verb|^|2\}, \{y\verb|^|4 - x\verb|^|3\}\}$

$$o2 = \quad \{-4\} \quad | \quad y2x2 - yx3 \quad y3x - y2x2 \quad y4 - x3 \quad |$$

o2 : Matrix $R^1 <---R^3$

We first compute the Grobner basis G with respect to the lex term ordering with $x < y$ for $< f_1, f_2, f_3 >$.

i3 : $n =$ generators gb m

$$o3 = \quad \{-4\} \quad | \quad x5 - x4 \quad yx4 - x4 \quad y2x2 - yx3 \quad y3x - yx3 \quad y4 - x3 \quad |$$

o3 : Matrix $R^1 <---R^5$

We find $G = [g_1 \quad g_2 \quad g_3 \quad g_4 \quad g_5]$ where g_1's are defined below.

i4 : apply$(i = 1..5, g = (i)-> \det(\text{submatrix}(n, 0, 4-(i-1))))$

o4 = $(x^5 - x^4,\ y*x^4 - x^4\ ,\ y^2x^2 - y*x^3,\ y^3x - y*x^3\ ,\ y^2x^2 - y*x^3,\ y^4 - x^3\ )$

o4 : Sequence

We next compute $x_i =$ leading power product of g_i.

i5 : apply$(i = 1..5, x = (i)->$ leadTerm $g(i))$

o5 = $(x^5, y*x^4, y^2x^2, y^3x, y^4\ )$

o5 : Sequence

We next compute

$$y_i := x_{i,i+1} = lcm(x_i, x_{i+1}) = x_i x_{i+1}/\text{gcd}(x_i, x_{i+1}).$$

It is enough to compute

$$S(g_1, g_j) = \frac{x_{ij}}{x_i} * g_i - \frac{x_{ij}}{x_j} * g_j$$

for $(i,j) \in= (1,2),(2,3),(3,4),(4,5)$.

i6 : $\text{apply}(i=1..4, y=(i)->x(i)*x(i+1)/(\gcd(x(i),x(i+1))))$

o6 = $(y*x^5,\ y^2x^4,\ y^3x^2,\ y^4x)$

o6 = Sequence

i7 : $\text{apply}(i=1..4, S=(i)->\text{substitute}(((y(i)/x(i))*g(i))-((y(i)/x(i+1))*g(i+1)),R))$

o7 = $(-y*x^4+x^5,\ y*x^5-y*x^4,\ -y^2x^3+y*x^4,\ -y^2x^3+x^4\ )$

o7 : Sequence

We now compute the polynomials $h_{ij\nu}$ where

$$S(g_i,g_j)=\sum_{\nu=1}^{t} h_{ij\nu} g_\nu$$

The polynomials $h_{ij\nu}$ are obtained by Division Algorithm.

i8 : $\text{apply}(i=1..4, h=(i)->S(i)//n)$

$$o8 = \left(\begin{array}{c|c|}\{1\} & 1\\ \{1\} & -1\\ \{0\} & 0\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{1\} & y\\ \{1\} & 0\\ \{0\} & 0\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{1\} & 0\\ \{1\} & 0\\ \{0\} & -x\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{1\} & 0\\ \{1\} & -1\\ \{0\} & -x\\ \{0\} & 0\\ \{0\} & 0\end{array}\right)$$

o8 : Sequence

i9 : $L=\text{id}_(R^5)$

$$o9 = \begin{array}{c|ccccc|}\{0\} & 1 & 0 & 0 & 0 & 0\\ \{0\} & 0 & 1 & 0 & 0 & 0\\ \{0\} & 0 & 0 & 1 & 0 & 0\\ \{0\} & 0 & 0 & 0 & 1 & 0\\ \{0\} & 0 & 0 & 0 & 0 & 1\end{array}$$

o9 : Matrix $R^5<---R^5$

$$\begin{aligned} i10:\text{apply}(i=1..4, t=(i)-> \quad & \text{transpose}(\text{matrixsubstitute}(y(i)/x(i),R)*L^{i-1}\\ - \quad & \text{matrixsubstitute}(y(i)/x(i+1),R)*L^{i})) \end{aligned}$$

$$o10 = \left(\begin{array}{c|c|}\{0\} & y\\ \{0\} & -x\\ \{0\} & 0\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{0\} & 0\\ \{0\} & y\\ \{0\} & -x2\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{0\} & 0\\ \{0\} & 0\\ \{0\} & y\\ \{0\} & -x\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{0\} & 0\\ \{0\} & 0\\ \{0\} & 0\\ \{0\} & y\\ \{0\} & -x\end{array}\right)$$

o10 : Sequence

i11 : $\text{apply}(i=1..4, s=(i)->t(i)-h(i))$

$$o11 = \left(\begin{array}{c|c|}\{0\} & y-1\\ \{0\} & -x+1\\ \{0\} & 0\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{0\} & -y\\ \{0\} & y\\ \{0\} & -x2\\ \{0\} & 0\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{0\} & 0\\ \{0\} & 0\\ \{0\} & y+x\\ \{0\} & -x\\ \{0\} & 0\end{array},\ \begin{array}{c|c|}\{0\} & 0\\ \{0\} & 1\\ \{0\} & x\\ \{0\} & y\\ \{0\} & -x\end{array}\right)$$

o11 : Sequence

****

Exercise 1 Let R = Q[x,y,z,w] and let $I=(x^2-yw, xy-wz, y^2-xz)$

(i) Show that with respect to degrlex ordering with $x>y>z>w$, the reduced Grobner basis $g_1=x^2-yw$, $g2=xy-wz$, $g3=y^2-xz$

(ii) $X_1=x^2$, $X_2=xy$, $X_3=y^2$, $X_{12}=x^2y$, $X_{23}=xy^2$, $X_{13}=x^2y^2$

(iii) $S(g_1,g_2)=-wy^2=xwz=-wg_3$, $S(g_1,g_3)=x^3z-y^3w=xzg_1-ywg_3$, $S(g_2,g_3)=x^2z-yzw=zg_1$.
This will show that
$s_{1,2}=(y,-x,-w)$, $s_{1,3}=(y^2-xz,0,-x^2+yw)$, $s_{2,3}=(-z,y,x)$.

Exercise 2 Compute the generators for $\text{Syz}(f_1,\ldots,f_s)$ for $f_1==x^2y+z$, $f_2=xz+y\in Q[x,y,z]$.

B6. Singular Locus

Example Let $f_1, \ldots, f_r \in k[x_1, \cdots, x_n]$. Let $J(f_1, \ldots, f_r)$ be the $r \times n$ matrix of partial derivatives. Compute $\partial/\partial x_i(f_1, \cdots, f_r)$.

i1 : $R = QQ[x, y, z]$

o1 = R

o1 : PolynomialRing

i2 : I = transpose(matrix $\{\{x + y + z\}, \{x\text{\^{}}2 - y^2 * z\text{\^{}}2 - z\text{\^{}}3\}\}$)

$$o2 = \begin{array}{ccc|} \{-1\} & | & x+y+z \\ \{-4\} & | & -y2z2 - z3 + x2 \end{array}$$

o2 : Matrix $R^2 <---R^1$

i3 : J = vars R

$$o3 = \quad \{0\} \quad | \quad x \quad y \quad z \quad |$$

o3 : Matrix $R^1 <---R^6$

We now compute the $J(f_1, \ldots, f_r)$

i4 : $D = \text{diff}(J, I)$

$$o4 = \begin{array}{cccccc} \{0\} & | & 1 & 1 & 1 & | \\ \{0\} & | & 2x & -2yz2 & -2y2z - 3z2 & | \end{array}$$

o4 : Matrix $R^2 <---R^3$

The 2×2 minors of the jacobian are:

i6 : $M = \text{minors}(2, D)$

o6 = ideal $(-2y * z^2 - 2x, -2y^2z - 3z^2 - 2x, -2y^2z + 2y * z^2 - 3z^2\ )$

o6 : Ideal of R

i7 : SL = gens gb (I+M)

$$o7 = \quad \{0\} \quad | \quad z2 \quad y + 3/ - 2z4 + z3 + 3/ - 2z2 + z \quad x + y + z \quad |$$

o7 : Matrix $R^1 <---R^3$

Exercise Find all singular points of curves in k^2/k^3. Can you guess the figures.

$$\begin{array}{ll} 1.x^2 = x^4 + y^4 & 2.xy = x^6 + y^6 \\ 3.x^3 = y^2 + x^4 + y^4 & 4.x^2y + xy^2 = x^4 + y^4 \\ 5.xy^2 = z^2 & 6.x^2 + y^2 = z^2 \\ 7.xy + x^3 + y^3 = 0 & \end{array}$$

Exercise Show that $V(y - x^2 + z^2, 4x - y^2 + w^3) \subseteq C[x, y, z, w]$ is nonempty smooth surface.

B7. Symmetric Polynomials

i1 : $R = QQ[t, x, y, z,$ MonomialOrder => Eliminate 0]

o1 = R

o1 : PolynomialRing

i2 : n := (numgens R) - 1

o2 = 3

i3 : M := matrix$\{\{(\text{apply}(i = 0..n + 1, i- > t\text{\^{}}i))\}\}$

$$o3 = \quad \{0\} \quad | \quad 1 \quad t \quad t2 \quad t3 \quad t4 \quad |$$

o3 : Matrix $R^1 <---R^5$

i4 : J := mingens(ideal (vars R))

$$o4 = \quad \{0\} \quad | \quad z \quad y \quad x \quad t \quad |$$

o4 : Matrix $R^1 <---R^4$

i5 : apply($i = 0..n-1, K = (i)->$ ideal(submatrix($J, \{0\}, \{i\}) - matrix\{\{t\}\}$));

i5 : $N1$:= product($(i = 0..n-1, i-> K(i))$)

o5 = ideal$(-t^3 + t^2x + t^2y - t*x*y + t^2z - t*x*z - t*y*z + x*y*z)$

o8 : Ideal of R

i6 : N := gens gb $N1$

$$o6 = \quad \{0\} \quad | \quad t3 - t2x - t2y + txy - t2z + txz + tyz - xyz \quad |$$

o6 : Matrix $R^1 <---R^1$

i7 : D := diff(M, N)

$$\begin{array}{lll} o7 = \quad \{0\} \quad | & t3 - t2x - t2y + txy - t2z + txz + tyz - xyz \\ & 3t2 - 2tx - 2ty + xy - 2tz + xz + yz \\ & 6t - 2x - 2y - 2z \quad 6 \quad 0 \quad | \end{array}$$

o7 : Matrix $R^1 <---R^5$

i8 : apply($i = 0..n, c = (i)->$ submatrix($D, \{0\}, \{i\}$))

$$\begin{array}{llcl} o8 = & (\{0\} \quad | & t3 - t2x - t2y + txy - t2z + txz + tyz - xyz & |, \\ & \{0\} \quad | & 3t2 - 2tx - 2ty + xy - 2tz + xz + yz & |, \\ & \{0\} \quad | & 6t - 2x - 2y - 2z & |, \\ & \{0\} \quad | & 6 & |) \end{array}$$

o8 : Sequence

i9 : $T := R/$ideal (t)

o9 = T

o9 : QuotientRing

i10 : apply $(i = 0..n-1, s = (i)->$ ideal (substitute($c(i), T$)))

o10 = (ideal -$(x*y*z)$, ideal$(x*y + x*z + y*z)$, ideal$(-2x - 2y - 2z)$)

o10 : Sequence

We now test whether certain polynomials can be written as a sum of symmetric polynomials.

i11 : B = matrix{ {gens gb $s(0)$ }, {gens gb $s(1)$ }, {gens gb $s(2)$}}

$$\begin{array}{llcl} o11 = & \{0\} \quad | & xyz & | \\ & \{0\} \quad | & xy + xz + yz & | \\ & \{0\} \quad | & x + y + z & | \end{array}$$

o11 : Matrix $T^3 <---T^1$

i12 : matrix $\{\{x^3 + y^3\}\}$ %(transpose B)

o12 = 0

o12 : Matrix $T^1 <---T^1$

Hence the polynomial$x^3 + y^2$ is in the ideal generated by elementry symmetric functions

i13 : matrix $\{\{x^3 + x*y^2\}\}$ % (B)

$$o13 = \quad \{0\} \quad | \quad x3 + xy2 \quad |$$

o13 : Matrix $T^1 <---T^1$

i14 : matrix $\{\{x^3 + x*y^2$ }} % (transpose B)

$$o14 = \quad \{0\} \quad | \quad yz2 \quad |$$

o14 : Matrix $T^1 <---T^1$

The polynomial $x^3 + xy^2$ is not a symmetric polynomial.

Exercise Write $f = (x^2 + y^2) * (x^2 + z^2) * (y^2 + z^2) \in QQ[x, y, z]$ as a polynomial in elementary symmetric functions.

Exercise Show that $x_1^k + \cdots x_n^k$ can be written as a sum of elementry symmetric functions in $x_1, \ldots, x_k$.

B8. The Tangent Cone

Example Suppose $V = \mathbf{V}(xy, xz + z(y^2 - z^2))$. Show that the tangent cone at the origin is given by

$$C_0(v) = xy, xz, yz(y^2 - z^2).$$

i1 : $R = QQ[w, x, y, z]$

o1 = R

o1 : PolynomialRing

i2 : $I = ideal(x * y, x * z + z * (y\hat{}2 - z\hat{}2))$

o2 = ideal $(x * y, y^2z - z^3 + x * z)$

o2 : Ideal of R

i3 : J = homogenize(I, w)

o3 = ideal $(x * y, w * x * z + y^2z - z^3\ )$

o3 : Ideal of R

i4 : K = gens gb J

$$o4 = \quad \{0\} \quad |xy \quad wxz + y2z - z3 \quad y3z - yz3 \quad |$$

o4 : Matrix $R^1 < - - - R^3$

i5 : $S = R/$ ideal $(w - 1)$

o5 = S

o5 : QuotientRing

i6 : K = substitute(K, S)

$$o6 = \quad \{0\} \quad | \quad xy \quad y2z - z3 + xz \quad y3z - yz3 \quad |$$

o6 : Matrix $S^1 < - - - S^3$

Take the minimal homogeneous component. Then the tangent cone at the origin is given by

$$C_0(V) = (xy, xz, yz(y^2 - z^2))$$

Exercise Find the tangent cone of the varieties defined by the equations:
a. $< xz, xy > \subset k[x, y, z]$
b. $< x - y^2, x - z^3 > \subset k[x, y, z]$

Exercise Show that the the singular locus of the tangent surface defined by

$$x^3z - (3/4)x^2y^2 - (3/2)xyz + y^3 + (1/4)x^2 = 0$$

is exactly the twisted cubic $\mathbf{V} = (y - x^2, z - x^3)$

Exercise Find the tangent cone and the tangent space at the origin of the varieties defined by the following ideals:
a. $< xz, xy > \subset Q[x, y, z]$
b. $< x - y^2, x - z^3 > \subset Q[x, y, z]$.
(Hint: The tangent space of V at p is the variety)

$$T_p(V) = V(d_p(f) : f \in I(V))$$

where

$$d_p(V) = \frac{\partial f}{\partial x_1}(p) + \cdots + \frac{\partial f}{\partial x_1}(p).$$

B9. The Remove Lowest Dimension

Removing components of dimension less than e

I an ideal or submodule of a free module. Yields the intersection of the primary components of I, excepting those of lowest dimension (and thus returns the ambient free module of I (or unit ideal), if I is pure dimensional).

```
i2  =     R = ZZ/32003[a,b,c]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(a^2,b^2)

             2   2
o3 = ideal (a , b )

o3 : Ideal of R

i4 : J = ideal(a^3,b^3,c^3)

             3   3   3
o4 = ideal (a , b , c )

o4 : Ideal of R

i5 : I = intersect(I,J)

             3   3   2 3   2 3
o5 = ideal (b , a , b c , a c )

o5 : Ideal of R

i6 : M = coker gens I

o6 = cokernel {0} | b3 a3 b2c3 a2c3 |

                               1
o6 : R - module, quotient of R

i7 : R := ring M;
remlow.m2:23:2: warning: local declaration of R shields variable with same name

i8 :     c1 := codim M;

i9 :     p1 := pdim M;

i10 :       while p1 > c1 and codim (E = Ext^p1(M,R)) > p1 do p1 = p1-1;

i11 :     if p1 == c1 then (
             -- M is C.M. and unmixed, so return (1):
             ambient M
             )   else (
             -- use the annihilator of Ext to improve M
             I := ann E;
             coker gens saturate(image presentation M,I))
remlow.m2:33:9: warning: local declaration of I shields variable with same name

o11 = cokernel {0} | b2 a2 |
```

```
                                     1
o11 : R - module, quotient of R
```

EXERCISES FOR B9, B10, B11, B12, B13

Exercise 1

```
R = ZZ/32003[a..d]
I = monomialCurve(R,{1,3,4})
J = ideal(a^3,b^3,c^3-d^3)
I = intersect(I,J)$
```

Exercise 2

```
R = ZZ/101[vars(0..8)]
M = genericMatrix(R,a,3,3)
J = ideal (M^3)
I = ideal(J_0,J_1,J_2)
```

B10. The Top Dimensional part

Yields the intersection of top dimensional primary components of the module or ideal I.

```
i2:=  R = ZZ/32003[a,b,c]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(a^2,b^2)

             2   2
o3 = ideal (a , b )

o3 : Ideal of R

i4 : J = ideal(a^3,b^3,c^3)

             3   3   3
o4 = ideal (a , b , c )

o4 : Ideal of R

i5 : I = intersect(I,J)

             3   3   2 3   2 3
o5 = ideal (b , a , b c , a c )

o5 : Ideal of R

i6 : R = ring I

o6 = R

o6 : PolynomialRing

i7 : c2 = codim I
```

```
o7 = 2

i8 : topdim = ann Ext^c2(coker gens I, R)

             2   2
o8 = ideal (b , a )

o8 : Ideal of R
```

B11. Associated primes of a given codimension

Given a finitely generated module M over a polynomial ring $R = k[x_1, \ldots, x_n]$ find an ideal whose associated primes are the associated primes of R having codimension n.

```
i2     R = ZZ/32003[a,b,c]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(a^2,b^2)

             2   2
o3 = ideal (a , b )

o3 : Ideal of R

i4 : J = ideal(a^3,b^3,c^3)

             3   3   3
o4 = ideal (a , b , c )

o4 : Ideal of R

i5 : I = intersect(I,J)

             3   3   2 3   2 3
o5 = ideal (b , a , b c , a c )

o5 : Ideal of R



i6 :     M = coker gens I

o6 = cokernel {0} | b3 a3 b2c3 a2c3 |

                              1
o6 : R - module, quotient of R

i7 :
     K = (n) -> ann Ext^n(M,R)

o7 = K

o7 : Function
```

```
i8 : J = (n) -> if codim K(n) - n > 0  then (ring M) else ( (top K(n)))

o8 = J

o8 : Function

i9 : J(1)

o9 = R

o9 : PolynomialRing

i10 : J(2)

              2   2
o10 = ideal (b , a )

o10 : Ideal of R

i11 : J(3)

                   3   3   3
o11 = ideal (a*b, c , b , a )

o11 : Ideal of R

i12 : J(4)

o12 = ideal 1

o12 : Ideal of R
```

B12. Radical

```
To compute the radical of $I$ given a regular sequence $J$ contained in $I$

i2 :    R = ZZ/32003[a,b,c]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(a^2,b^2)

             2   2
o3 = ideal (a , b )

o3 : Ideal of R

i4 : J = ideal(a^3,b^3,c^3)

             3   3   3
o4 = ideal (a , b , c )

o4 : Ideal of R

i5 : I = intersect(I,J)
```

```
             3   3   2 3   2 3
o5 = ideal (b , a , b c , a c )

o5 : Ideal of R

i6 :  n = dim R

o6 = 3

i7 : K = jacobian I

o7 = {1} | 0   3a2 0     2ac3  |
     {1} | 3b2 0   2bc3  0     |
     {1} | 0   0   3b2c2 3a2c2 |

             3       4
o7 : Matrix R  <--- R

i8 : L= (t) -> minors(n-t, K)

o8 = L

o8 : Function

i9 : c = codim I

o9 = 2

i10 : I : L(n-c)

o10 = ideal (b, a)

o10 : Ideal of R
```

B13. Reduction of equidimensional radical to complete intersection case

Given ideal $J \subseteq I \subseteq k[x_1, \ldots, x_n]$ where J is known to be a complete intersection and I and J both have the same codimension, compute the equidimensional hull of the radical of I.

```
i2 :    R = ZZ/32003[a,b,c]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(a^2,b^2)

             2   2
o3 = ideal (a , b )

o3 : Ideal of R

i4 : J = ideal(a^3,b^3,c^3)

             3   3   3
```

```
o4 = ideal (a , b , c )

o4 : Ideal of R

i5 : I = intersect(I,J)

             3   3   2 3   2 3
o5 = ideal (b , a , b c , a c )

o5 : Ideal of R

i6 : CI = ideal (a^3, b^3)

             3   3
o6 = ideal (a , b )

o6 : Ideal of R

i7 : codim I

o7 = 2

i8 : codim CI

o8 = 2

i9 : CII = radical CI

o9 = ideal (b, a)

o9 : Ideal of R

i10 : eqdimrad = CII : (CII : I)

o10 = ideal (b, a)

o10 : Ideal of R
```