STANLEY'S SOLUTION OF THE ADG-CONJECTURE

J. K. VERMA

1. Introduction

Our objective in these notes is to present Stanley's solution of the Anand-Dumir-Gupta (ADG) conjecture concerning enumeration of doubly stochastic matrices or magic squares. Let $\mathbb{N}$ denote the set of nonnegative integers and let $\mathbb{P}$ denote the set of positive integers. An $n \times n$ matrix M is called a magic square if its entries are in $\mathbb{N}$ and the sum of entries in any row or column is a given integer r. The number r is called the line sum of M. It is clear that

$$H_1(r) = 1 \text{ and } H_2(r) = r + 1.$$

MacMahon [2] and independently Anand-Dumir-Gupta [1] showed that the number of 3×3 magic squares with line sum r is given by

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Inspired by these formulas they proposed the following conjectures in 1966. [1]:

Conjecture 1.1 (Anand-Dumir-Gupta). *Fix $n \geq 1$. Then*

(1) $H_n(r) \in \mathbb{C}[r]$.
(2) $\deg H_n(r) = (n-1)^2$.
(3) $H_n(i) = 0$ *for* $i = -1, -2, \ldots, -(n-1)$.
(4) $H_n(-n-r) = (-1)^{n-1} H_n(r)$ *for all* r.

We will see that the above four assertions about $H_n(r)$ are equivalent to the following:

$$\sum_{r=0}^{\infty} H_n(r)\lambda^r = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^{(n-1)^2+1}},$$

where $h_0, h_1, \ldots, h_d$ are integers, $d = (n-1)^2 + 1 - n$, $h_0 + h_1 + \cdots + h_d \neq 0$ and $h_{d-i} = h_i$ for $i = 0, 1, \ldots, d$. Stanley made the additional conjectures that
(5) $h_i \geq 0$ for all i and
(6) $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$.

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Stanley settled (1)-(5) in 1973 [4]. A geometric proof based on Ehrhart polynomials of integral polytopes appears in Stanley's Red Book [5]. The conjecture (6) is still open.

2. **Linear homogeneous Diophantine equations**

Let $x_{ij};\ i,j=1,2,\ldots,n$ be indeterminates. The entries of an $n\times n$ magic square are solutions to the following system of linear homogeneous Diophantine equations:

$$\begin{aligned} x_{11}+x_{12}+\cdots+x_{1n} &= \sum_{j=1}^{n} x_{ij} \text{ for } i=2,3\ldots,n. \qquad (1)\\ x_{11}+x_{12}+\cdots+x_{1n} &= \sum_{i=1}^{n} x_{ij} \text{ for } j=2,3\ldots,n. \end{aligned}$$

Thus the problem of counting magic squares is a special case of counting nonnegative integer solutions of a system of linear Diophantine equations. Let Φ be an $r\times n$ $\mathbb{Z}$-matrix. Let $x_1,x_2,\ldots,x_n$ be indeterminates. Let X denote the column vector $(x_1,x_2,\ldots,x_n)^t$. We are interested in the $\mathbb{N}$-solutions to the system $\Phi X=0$. We gather all the solutions in the semigroup

$$E_\Phi=\{\beta\in\mathbb{N}^n:\Phi\beta=0\}.$$

Let k be any field. For $\beta=(\beta_1,\beta_2,\ldots,\beta_n)^t$, put $x^\beta=x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}$. With E_Φ we can associate the semigroup ring

$$R_\Phi=k[x^\beta:\beta\in E_\Phi].$$

Stanley studied the semigroup ring R_μ where μ is the $(2n-2)\times n^2$ coefficient matrix of the system (1). In particular he showed that the ring R_μ is Gorenstein and calculated its canonical module and thus its a-invariant. We shall see that these observations are enough to settle the conjectures (1)-(5). Let us begin by observing the

Theorem 2.1. *The semigroup ring R_Φ is a finitely generated k-algebra.*

Proof. Let I denote the ideal in $R=k[x_1,x_2,\ldots,x_n]$ generated by the set

$$P=\{x^\beta:0\neq\beta\in E_\Phi\}.$$

Since R is Noetherian, I is generated by a finite subset $G=\{x^{\delta_1},x^{\delta_2},\ldots,x^{\delta_t}\}$ of P. We claim that

$$R_\Phi=k[x^\delta:x^\delta\in G].$$

Indeed, Any $x^\beta\in R_\Phi$ can be written as $x^\beta=x^{\delta_i}x^\gamma$ for some i and $x^\gamma\in R$. Thus $\gamma=\beta-\delta_i\in E_\Phi$. The argument can be repeated for x^γ, eventually yielding an expression for x^β in terms of x^{δ_i} for $i=1,2,\ldots,t$. □

As far as the structure of R_μ is concerned, we have more precise information due to

Theorem 2.2 (Birkhoff-von Neumann Theorem). *Every $n \times n$ magic square is an $\mathbb{N}$-linear combination of the $n \times n$ permutation matrices.*

Thus R_μ is generated by $n!$ degree n monomials. Let $[R_\mu]_r$ denote the k-subspace of R_μ generated by the monomials of degree nr. These monomials are in one-to-one correspondence with magic squares of line sum r. Moreover, $R_\mu = \oplus_{r=0}^{\infty}[R_\mu]_r$. Thus

$$H(R_\mu, r) = \dim_k [R_\mu]_r = H_n(r).$$

This observation will eventually lead to the conclusion that $H_n(r)$ is a polynomial in r for all r. But for the time being we can see that it is so for all large values of r in view of the Hilbert-Serre theorem.

Lemma 2.3. *If $\Phi X = 0$ has a positive solution, then $\dim R_\Phi = n - \operatorname{rank} \Phi$.*

Proof. We show that the vectors $\beta_1, \beta_2, \ldots, \beta_d \in E_\Phi$ are $\mathbb{Q}$-linearly independent if and only if $x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_d}$ are algebraically independent over k. Suppose $\beta_1, \beta_2, \ldots, \beta_d \in E_\Phi$ are linearly independent over $\mathbb{Q}$. Let

$$\sum_{\alpha} a_\alpha (x^{\beta_1})^{\alpha_1} (x^{\beta_2})^{\alpha_2} \ldots (x^{\beta_d})^{\alpha_d} = 0,$$

for certain $a_\alpha \in k$ and distinct vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. Since $\beta_1, \beta_2, \ldots, \beta_d$ are linearly independent over $\mathbb{Q}$, the vectors $\alpha_1\beta_1 + \cdots + \alpha_d\beta_d$ are distinct. Hence $a_\alpha = 0$ for all α.

Conversely let $x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_d}$ be algebraically independent over k. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}$ such that $\alpha_1\beta_1 + \cdots + \alpha_d\beta_d = 0$. Without loss of generality we may assume that $\alpha_1, \alpha_2, \ldots, \alpha_p > 0$ and $\alpha_{p+1}, \ldots, \alpha_d < 0$. Then

$$\alpha_1\beta_1 + \cdots + \alpha_p\beta_p = \alpha_{p+1}\beta_{p+1} + \cdots + \alpha_d\beta_d.$$

This yields the algebraic dependency relation $x^{\alpha_1\beta_1} \cdots x^{\alpha_p\beta_p} = x^{\alpha_{p+1}\beta_{p+1}} \cdots x^{\alpha_d\beta_d}$.

Let $\alpha \in \mathbb{P}^n \cap E_\Phi$. Let $d = n - \operatorname{rank} \Phi$. Pick linearly independent solutions $\beta_1, \beta_2, \ldots, \beta_d \in \mathbb{Z}^n$ of $\Phi X = 0$. Let $t \in \mathbb{Q}_+$. If $\alpha - t\beta_1, \alpha - t\beta_2, \ldots, \alpha - t\beta_d$ are linearly dependent over $\mathbb{Q}$, then there exist $a_1, a_2, \ldots, a_d \in \mathbb{Z}$, not all zero such that

$$a_1(\alpha - t\beta_1) + a_2(\alpha - t\beta_2) + \cdots + a_d(\alpha - t\beta_d) = 0.$$

We have unique rational numbers $b_1, b_2, \ldots, b_d$ such that $\alpha = b_1\beta_1 + b_2\beta_2 + \cdots + b_d\beta_d$. Put $a = \sum_{i=1}^d a_i$. Then $\sum_{i=1}^d (ab_i - ta_i)\beta_i = 0$. Let $a_p \neq 0$. Then $t = ab_p/a_p$. Hence by selecting $t \in \mathbb{Q}_+$ sufficiently small, we get a contradiction. This proves that $\delta_1 = \alpha - t\beta_1, \delta_2 = \alpha - t\beta_2, \ldots, \delta_d = \alpha - t\beta_d$ are linearly independent solutions in $\mathbb{P}^n$. Hence $x^{\delta_1}, \ldots, x^{\delta_d}$ are algebraically independent elements of R_Φ. □

Corollary 2.4. *The function $H_n(r)$ is a polynomial in r for large r of degree $(n-1)^2$.*

Proof. The ring R_μ is a standard graded k-algebra. The r^{th} graded component of it is generated by monomials of degree rn corresponding to magic squares of line sum r. Hence $H_n(r)$ is a polynomial for large r. We show that

$$\dim R_\mu = (n-1)^2 + 1.$$

By the above lemma, $\dim R_\mu = \text{nullity}\,\mu$. Note that to construct a magic square, we may assign any nonnegative values to the variables x_{ij}, for $i, j = 1, 2, \ldots, (n-1)$ and a value for x_{1n} will determine the rest of the entries. Thus $\text{nullity}\,\mu = (n-1)^2 + 1$.

□

Proposition 2.5 (**MacMahon** [2], **Anand-Dumir-Gupta** [1]). *The number of* 3×3 *magic squares with line sum* r *is given by*

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Proof. By the above corollary, the dimension of the semigroup ring R generated over a field k by the monomials corresponding to the six 3×3 permutation matrices is $(n-1)^2 + 1 = 5$. Let $S = k[y_1, y_2, \ldots, y_6]$. Put

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that $M_1 + M_2 + M_3 = M_4 + M_5 + M_6$. Let $f = y_1y_2y_3 - y_4y_5y_6$. Hence $S/(f) \simeq R$. Therefore

$$H(S/(f), \lambda) = (1-\lambda^3)/(1-\lambda)^6 = (1+\lambda+\lambda^2)/(1-\lambda)^5.$$

This yields the desired formula.

□

3. Cohen-Macaulay Property of R_Φ

In this section we show that R_Φ is a Cohen-Macaulay ring. This is done by showing that it is a ring of invariants of an algebraic torus acting linearly on a polynomial ring. A well-known theorem of Hochster then implies that it is Cohen-Macaulay .

Write the $r \times n$ matrix $\Phi = [\gamma_1, \gamma_2, \ldots, \gamma_n]$ where γ_i is the i^{th} column vector of Φ. Let k^* denote the multiplicative group of k. Consider the algebraic torus

$$T = \{\text{diag}(u^{\gamma_1}, u^{\gamma_2}, \ldots, u^{\gamma_n}) : u = (u_1, u_2, \ldots, u_r) \in (k^*)^r\}.$$

T acts on $R = k[x_1, x_2, \ldots, x_n]$ via the automorphisms $\tau_u : x_i \longrightarrow u^{\gamma_i} x_i, \quad i = 1, 2, \ldots, n$. Let $\beta \in \mathbb{N}^n$. Then

$$\begin{aligned} \tau_u(x^\beta) &= (u^{\gamma_1}x_1)^{\beta_1}(u^{\gamma_2}x_2)^{\beta_2} \cdots (u^{\gamma_n}x_n)^{\beta_n} \\ &= u^{\beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_n\gamma_n} x^\beta \end{aligned}$$

Hence $\tau_u(x^\beta) = x^\beta$ if and only if $\beta \in E_\Phi$. Hence R_Φ is the ring of invariants of the torus T acting linearly on R. By Hochster's theorem [3] R_Φ is Cohen-Macaulay .

We can now dispose the conjecture (5) of Stanley. Since R_μ is Cohen-Macaulay homogeneous ring of dimension $d = (n-1)^2 + 1$, there exists an hsop $\mathbf{a}$ for R_μ of elements of degree one. Hence

$$F(R_\mu, \lambda) = \frac{F(R_\mu/(\mathbf{a}), \lambda)}{(1-\lambda)^d}.$$

Hence the numerator of the above Hilbert series is a polynomial with positive coeficients.

4. Macaulay's theorem for Gorenstein graded rings

The purpose of this section is to recall the basic definitions and facts about Gorenstein graded rings and provide a proof of Macaulay's theorem concerning their Hilbert series.

Let R be an $\mathbb{N}$-graded ring. Let $\mathcal{M}$ be the category of $\mathbb{Z}$-graded R-modules. Let $M = \oplus M_n$ and $N = \oplus N_n \in \mathcal{M}$. An R-linear map $f : M \longrightarrow N$ is a morphism in $\mathcal{M}$ if $f(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. By $M(n)$ we mean the module M with grading defined by $[M(n)]_m = M_{m+n}$ for all $m \in \mathbb{Z}$. Put

$${}^*\operatorname{Hom}(M,N)_n = \{f : M \longrightarrow N(n)\} \text{ and } {}^*\operatorname{Hom}(M,N) = \bigoplus_{n\in\mathbb{Z}} {}^*\operatorname{Hom}(M,N)_n.$$

It is easy to check that if M is finitely generated then ${}^*\operatorname{Hom}(M,N) = \operatorname{Hom}(M,N)$.

Proposition 4.1. *Let $A = k[x_1, x_2, \ldots, x_s]$ be polynomial ring over a field k. Let I be a homogeneous ideal of A. Let A/I be Cohen-Macaulay . Then*

$$\operatorname{Ext}^i(A/I, A) \neq 0 \iff i = h = ht(I).$$

Proof. By Auslander-Buchsbaum formula $\operatorname{pd}(A/I) = \operatorname{depth} A - \dim A/I = s - (s-h) = h$. Write a graded minimal resolution of A/I as an A-module:

$$0 \longrightarrow A^{\beta_h} \longrightarrow A^{\beta_{h-1}} \longrightarrow \cdots \longrightarrow A^{\beta_1} \longrightarrow A \longrightarrow A/I \longrightarrow 0.$$

Thus $\operatorname{Ext}^i(A/I, A) = 0$ for $i > h$. Since A is Cohen-Macaulay , $\operatorname{Ext}^i(A/I, A) = 0$ for $i < h$. □

Definition 4.2. *The A-module $K_{A/I} = \operatorname{Ext}^h(A/I, A)$ is called the* **canonical module** *of A/I. The ring A/I is called* **Gorenstein** *if $K_{A/I} \simeq A/I(a)$. for some $a \in \mathbb{Z}$. The integer a is called the a-invariant of A/I.*

Theorem 4.3. *Put $R = A/I$ and $d = \dim(R)$. Let the degree of $x_i = e_i \in \mathbb{P}$ for $i = 1, 2, \ldots, s$. Then as rational functions of λ*

$$F(K_R, \lambda) = (-1)^d \; F(R, 1/\lambda) \; \lambda^{-\sum_{i=1}^s e_i}.$$

Proof. Write a minimal free resolution of R as an A-module:

$$0 \longrightarrow M_h \xrightarrow{\phi_h} M_{h-1} \xrightarrow{\phi_{h-1}} \cdots \longrightarrow M_1 \xrightarrow{\phi_1} M_0 \xrightarrow{\phi_0} R \longrightarrow 0.$$

Apply $\mathrm{Hom}(-, A)$ to the above resolution to get the complex:

$$0 \longrightarrow \mathrm{Hom}(M_0, A) \xrightarrow{\phi_0^*} \mathrm{Hom}(M_1, A) \xrightarrow{\phi_1^*} \cdots \xrightarrow{\phi_h^*} \mathrm{Hom}(M_h, A) \longrightarrow 0.$$

Thus $K_R \simeq \mathrm{Hom}(M_h, A)/\mathrm{Im}(\phi_h^*)$. Hence we have the following minimal free resolution for K_R as an A-module:

$$0 \longrightarrow \mathrm{Hom}(M_0, A) \xrightarrow{\phi_0^*} \cdots \xrightarrow{\phi_h^*} \mathrm{Hom}(M_h, A) \longrightarrow K_R \longrightarrow 0.$$

It is easy to see that for integers m and n,

$$F(M(n), \lambda) = \lambda^{-n} F(M, \lambda) \text{ and } \mathrm{Hom}(A(m), A) \simeq A(-m).$$

Let $\mathrm{rank}(M_i) = \beta_i$, and $M_i = \oplus_{j=1}^{\beta_i} A(-g_{ij})$ for $i = 0, 1, \ldots, h$. Put $D(\lambda) = \prod_{p=1}^{s}(1 - \lambda^{e_p})$ and $N_i(\lambda) = \sum_{j=1}^{\beta_i} \lambda^{g_{ij}}$. Now we calculate the Hilbert series of R and K_R from their minimal free resolutions written above. Put $e = \sum_{i=1}^{s} e_i$.

$$F(M_i, \lambda) = \sum_{j=1}^{\beta_i} F(A(-g_{ij}), \lambda) = \frac{\sum_{j=1}^{\beta_i} \lambda^{g_{ij}}}{\prod_{p=1}^{s}(1 - \lambda^{e_p})} = \frac{N_i(\lambda)}{D(\lambda)}.$$

Hence $F(R, \lambda) = \sum_{i=0}^{h} N_i(\lambda)/D(\lambda)(-1)^i$. To find $F(K_R, \lambda)$, note that

$$F(K_R, \lambda) = \sum_{i=0}^{h} (-1)^{i+h} F(M_i^*, \lambda) = \sum_{i=0}^{h} (-1)^{i+h} F(A(g_{ij}), \lambda) = \sum_{i=0}^{h} (-1)^{i+h} N_i(\lambda^{-1})/D(\lambda).$$

Since $D(\lambda^{-1}) = (-1)^s D(\lambda) \lambda^{-e}$, we get

$$F(R, 1/\lambda) = \sum_{i=0}^{h} (-1)^i \frac{N_i(\lambda^{-1})}{D(\lambda^{-1})} = (-1)^{s-h} \lambda^e F(K_R, \lambda) = (-1)^d \lambda^e F(K_R, \lambda).$$

□

Corollary 4.4 (Macaulay's Theorem). *If the ring $R = A/I$ is Gorenstein of dimension d then for some $\sigma \in \mathbb{Z}$,*

$$F(R, 1/\lambda) = (-1)^d \lambda^\sigma F(R, \lambda).$$

If R is standard Gorenstein with $F(R, \lambda) = (h_0 + h_1\lambda + \cdots + h_g\lambda^g)/(1 - \lambda)^d$, and $h_g \neq 0$, then

(1) *$h_i = h_{g-i}$, for all $i = 0, 1, \ldots, g$.*
(2) *$\sigma = d - g$.*
(3) *If $\sigma \geq 1$, then $H(n) = \dim R_n$ is a polynomial $P(n)$ for all n,*
    (a) *$P(-i) = 0$ for all $i = 1, 2, \ldots, (\sigma - 1)$, and*
    (b) *$P(n) = (-1)^{d-1} P(-\sigma - n)$ for all $n \in \mathbb{Z}$.*

Proof. (1) and (2): Put $e = \sum_{i=1}^{s} e_i$. Suppose R is Gorenstein. Then $K_R \simeq R(a)$, for some $a \in \mathbb{Z}$. Hence

$$F(K_R, \lambda) = \lambda^{-a} F(R, \lambda) = (-1)^d \lambda^{-e} F(R, 1/\lambda).$$

Hence $F(R, \lambda) = \lambda^{a-e}(-1)^d F(R, 1/\lambda)$. Now let R be standard Gorenstein. Write

$$F(R, \lambda) = (h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_g\lambda^g)/(1 - \lambda)^d$$

where $h_g \neq 0$. Then

$$F(R,1/\lambda) = (-1)^d \lambda^{d-g}(h_0\lambda^g + h_1\lambda^{g-1} + \cdots + h_g)/(1-\lambda)^d = \lambda^{e-a}(-1)^d F(R,\lambda).$$

Hence $d-g = e-a = \sigma$ and $h_i = h_{g-i}$ for all $i = 0,1,\ldots,g$.

(3) We know that if $\sigma \geq 1$, then $H(n)$ is a polynomial for all $n \in \mathbb{Z}$ and as $\dim R_n = 0$ for all $n < 0$, $P(n) = 0$ for all $n = -1,-2,\ldots,-(\sigma-1)$, and $P(-\sigma) \neq 0$. We have for all $n \geq -(\sigma-1)$,

$$P(n) = h_0\binom{n+d-1}{d-1} + h_1\binom{n+d-2}{d-1} + \cdots + h_g\binom{d-1+n-g}{d-1}.$$

Now use the fact that $h_i = h_{g-i}$ for all $i = 1,2,\ldots,g$, and $\binom{n}{p} = (-1)^p\binom{p-n-1}{p}$, as polynomials,

$$\begin{aligned}
P(n) &= \sum_{i=0}^{g} h_i \binom{d-1+n-i}{d-1} \\
&= \sum_{i=0}^{g} h_{g-i}\binom{d-1-(d-1+n-i)-1}{d-1}(-1)^{d-1} \\
&= \sum_{i=0}^{g} h_i\binom{g-i-n-1}{d-1}(-1)^{d-1} \\
&= \sum_{i=0}^{g} h_i\binom{d-\sigma-i-n-1}{d-1}(-1)^{d-1} \\
&= (-1)^{d-1}P(-\sigma-n).
\end{aligned}$$

□

Definition 4.5. *The vector* $(h_0, h_1, \ldots, h_g)$ *is called the* h**-vector** *of the standard graded algebra* R. *If the condition* $h_i = h_{g-i}$ *is satisfied for all* $i = 0, i, \ldots, g$ *then we say that the* h*-vector of* R *is* **symmetric.**

Example 4.6. The symmetry of the h-vector of a standard graded Cohen-Macaulay algebra R does not imply that R is Gorenstein. We construct an example. Consider the ideal $I = (xyz, xw, zw)$ of the polynomial ring $A = k[x,y,z,w]$. The ideal I is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} 0 & z & x \\ -w & -yz & 0 \end{bmatrix}.$$

A resolution of $R = A/I$ as an A-module is:

$$0 \longrightarrow A(-3)\oplus A(-4) \xrightarrow{f} A(-3)\oplus A(-2)^2 \xrightarrow{g} A \longrightarrow R \longrightarrow 0,$$

where the maps f and g are defined as

$$f([r,s]) = [r,s]M \ \text{ and } \ g([r,s,t]) = rxyz - sxw + tzw.$$

It can be shown easily that the above sequence is a minimal resolution of R. Hence by Auslander-Buchsbaum formula, $\operatorname{depth} R = \operatorname{depth} A - \operatorname{pd} R = 4-2 = 2 = \dim R$. Hence R is Cohen-Macaulay. However it is not Gorenstein as the above resolution shows that $\operatorname{rank} K_R = 2$. The Hilbert series of R can be found from the resolution and it turns out to be $(1+2\lambda+\lambda^2)/(1-\lambda)^2$. Hence the h-vector of R is symmetric, although it is not Gorenstein.

Remark: The principal result of [6] shows that the symmetry of the h-vector implies Gorenstein property provided R is a Cohen-Macaulay domain.

5. A sketch of Stanley's solution

By Corollary 4.4, we need to show that the degree of the Hilbert series of R_μ is $-n$ and it is Gorenstein. By the Grothehdieck-Serre difference formula, the degree of the Hilbert series of R_Φ is the integer $a(R_\Phi) = \max\{n : H^d(R_\Phi)_n \neq 0\}$.

Theorem 5.1 (**Stanley**, [4]). (1) *$H^d(R_\Phi) = k[x^\beta : \beta \in E_\Phi,$ and $\beta < 0]$.*
(2) *$K_{R_\Phi} = k[x^\beta : \beta \in E_\Phi,$ and $\beta > 0]$.*
(3) *If $\gamma = (1, 1, \ldots, 1) \in E_\Phi$, then $K_{R_\Phi} = x^\gamma R_\Phi$. Hence in this case, R_Φ is Gorenstein.*

For the case of magic squares, the $n \times n$ magic square J_n whose each entry is 1 is the smallest positive solution and by the description of $H^d(R_\Phi)$, the a-invariant of R_μ is $-n$. Hence The degree of its Hilbert series is $-n$. It proves that $H_n(r)$ is a polynomial for all $r > -n$. Moreover $H_n(-n) \neq 0$ and

$$H_n(-1) = H_n(-2) = \cdots = H_n(-(n-1)) = 0.$$

By Corollary 4.4, we conclude that

$$H_n(r) = (-1)^{(n-1)^2} H_n(-r-n) = (-1)^{n-1} H_n(-r-n).$$

for all n.

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Department of Mathematics, IIT Bombay, Powai, Mumbai 400 076.

E-mail address: jkv@math.iitb.ac.in