FACE RINGS OF SIMPLICIAL COMPLEXES

J. K. VERMA

1. **Hilbert series of face rings**

Let Δ be an abstract finite simplicial complex with vertices $X_1, \dots, X_n$. Let k be a field throughout this chapter. Let R denote the polynomial ring $k[X_1, X_2, \dots, X_n]$, where, by abuse of notation, we regard the vertices $X_1, X_2, \dots, X_n$ as indeterminates over k. Let I_Δ be the ideal of R generated by the monomials $X_{i_1} \dots X_{i_r},\ i_1 < i_2 < \dots < i_r$ such that $\{X_{i_1}, \dots, X_{i_r}\}$ is not a face of Δ. The face ring of Δ is the quotient ring $k[\Delta] := R/I_\Delta$. Since I_Δ is a homogeneous ideal, $k[\Delta]$ is a graded ring. In this section we will prove Stanley's formula for the Hilbert series of $k[\Delta]$. In some sense, this formula opened up the connection of Commutative Algebra with Combinatorics. We will exhibit the power of Hilbert series methods by giving an elementary proof of Dehn-Sommerville equations towards the end of this section. We begin by establishing the primary decomposition of I_Δ.

(1.1) Definition. Let F be a face of a simplicial complex Δ. Let P_F denote the prime ideal of R generated by the indeterminates not in F. We call P_F the *face ideal* of F.

(1.2) Proposition. *The primary decomposition of I_Δ is given by $I_\Delta = P_{F_1} \cap P_{F_2} \cap \dots \cap P_{F_m}$ where $F_1, F_2, \dots F_m$ are the facets of Δ.*

Proof. Let J be any ideal generated by monomials of R and a and b be relatively prime monomials. Then it is easily seen that $(ab, J) = (a, J) \cap (b, J)$. By using this equation succesively we see that I_Δ is an intersection of ideals generated by subsets of $V = \{X_1, X_2, \dots, X_n\}$. Thus I_Δ is a radical ideal. If $(X_{i_1}, X_{i_2}, \dots, X_{i_r}) \supseteq I_\Delta$ then the set $V \setminus \{X_{i_1}, X_{i_2}, \dots, X_{i_n}\}$ is a face of Δ. Conversely if $F \in \Delta$ then $P_F \supset I_\Delta$. Hence I_Δ is the intersection of the face ideals corresponding to the facets of Δ.

(1.3) Proposition. *For any simplicial complex Δ,*

$$\dim k[\Delta] = \dim \Delta + 1.$$

Proof.

$$\begin{aligned} \dim k[\Delta] &= \max\{\dim k[\Delta]/P \mid P \text{ is a minimal prime of } k[\Delta]\} \\ &= \max\{\dim R/P_F \mid F \text{ is a facet of } \Delta\} \\ &= \max\{card(F) \mid F \text{ is a facet of } \Delta\} \\ &= \dim \Delta + 1. \end{aligned}$$

(1.4) Proposition. *If $k[\Delta]$ is a Cohen-Macaulay then Δ is pure, i.e., all facets have same dimension.*

Proof. For any Cohen-Macaulay graded ring S, $dim\ S/P = \dim S$ for all minimal primes P of S. Hence $\dim R/P_F = \dim F + 1 = \dim \Delta + 1$ for all facets F of Δ. Hence Δ is pure.

Notes of four lectures delivered in the Instructional Conference *Combinatorial Topology and Algebra* sponsored by the *National Board for Higher mathematics* and held at IIT Bombay during Dec. 5-24, 1993.

Next we calculate the Hilbert series of $k[\Delta]$. For this purpose, we use "fine grading" of $k[\Delta]$. We say that a ring R is graded by an abelian group G if $R = \bigoplus_{g \in G} R_g$ where R_g are the additive subgroups of R and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. The polynomial ring $R = k[X_1, \ldots, X_n]$ is $\mathbb{Z}^n$ graded. For any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, set $R_a = 0$ if some $a_i < 0$ and $R_a = kX_1^{a_1} \ldots X_n^{a_n}$ if all a_i are non-negative. Then $R = \bigoplus_{a \in \mathbb{Z}^n} R_a$. The $\mathbb{Z}^n-$graded ideals of R are simply the ideals generated by the monomials. The face ring $k[\Delta]$ is $\mathbb{Z}^n-$graded. Indeed for any $a \in \mathbb{Z}^n$, $k[\Delta]_a = R_a/(I_\Delta)_a$. Since R_a is 1-dimensional vector space, $k[\Delta]_a = 0$ or $k[\Delta]_a = R_a$. In fact, if $a = (a_1, \ldots, a_n)$, then $k[\Delta]_a = 0$ iff $X_1^{a_1} \ldots X_n^{a_n} \in I_\Delta$. For the monomial $M = X_1^{a_1} \ldots X_n^{a_n}$, put supp $M = \{X_i \mid a_i > 0\}$. Thus $k[\Delta]_a = 0$ iff supp $M \notin \Delta$.

(1.5) Theorem (Stanley). *Let Δ be a simplicial complex of dimension $d-1$. Let f_i be the number of i-dimensional faces of Δ, $i = 0, \ldots, d-1$. Then*

$$H(k[\Delta], t) := \sum_{a \in \mathbb{Z}^n} \dim k[\Delta]_a t^a = \sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t} \right)^i,$$

where t stands for $(t_1, \ldots, t_n)$ and $t^a = t_1^{a_1} \cdots t_n^{a_n}$ for $a = (a_1, \ldots, a_n)$.

Proof. By the discussion preceding (1.5)

$$\begin{aligned} H(k[\Delta], t) &= \sum_{a \in \mathbb{N}^n, \, \text{supp } (X^a) \in \Delta} t^a \\ &= \sum_{F \in \Delta} \left(\sum_{\text{supp } (X^a) = F} t^a \right) \\ &= 1 + \sum_{\phi \neq F \in \Delta} \left(\sum_{\text{supp } (X^a) = F} t^a \right) \\ &= 1 + \sum_{\phi \neq F \in \Delta} \left(\prod_{X_i \in F} \frac{t_i}{(1-t_i)}. \right) \end{aligned}$$

The ring $k[\Delta]$ is $\mathbb{N}-$graded and for any $r \in \mathbb{N}$

$$k[\Delta]_r = \bigoplus_{a_1 + \ldots + a_n = r} k[\Delta]_a$$

Therefore the Hilbert series of $k[\Delta]$, as an $\mathbb{N}-$graded ring is given by

$$\begin{aligned} H(k[\Delta], t) &= \sum_{i=0}^{\infty} \dim k[\Delta]_i t^i \\ &= \sum_{i=0}^{\infty} \left\{ \sum_{a_1 + \ldots + a_n = i} \dim k[\Delta]_i \right\} t^i \\ &= H(k[\Delta], t, t, \ldots, t) \\ &= 1 + \sum_{\phi \neq F \in \Delta} \left(\frac{t}{1-t} \right)^{|F|} \\ &= 1 + f_0 \left(\frac{t}{1-t} \right) + f_1 \left(\frac{t}{1-t} \right)^2 + \ldots + f_{d-1} \left(\frac{t}{1-t} \right)^d. \end{aligned}$$

(1.6) Corollary *The Hilbert function of $k[\Delta]$ is given by*

$$H(k[\Delta],n)=\dim k[\Delta]_n=\begin{cases}1 & \text{if } n=0\\ \sum_{i=0}^{d-1} f_i\binom{n-1}{i} & \text{if } n>0\end{cases}$$

Proof. It is clear that $H(k[\Delta],0)=1$. Let $n>0$. Then $\dim k[\Delta]_n$ is the coefficient of t^n in the power series

$$1+f_0\left(\frac{t}{1-t}\right)+\ldots+f_{d-1}\left(\frac{t}{1-t}\right)^d.$$

Using the identity $(1-t)^{-(s+1)}=\sum_{i=0}^{\infty}\binom{s+i}{i}t^i$, we get

$$\begin{aligned}\dim k[\Delta]_n &= f_0+f_1\binom{1+n-2}{n-2}+f_2\binom{2+n-3}{n-3}+f_{d-1}\binom{d-1+n-d}{n-d}\\ &= \sum_{i=0}^{d-1} f_i\binom{n-1}{i}.\end{aligned}$$

(1.7) Remark. The above formula for the Hilbert polynomial shows that $H(k[\Delta],n)$ is a polynomial for all $n>0$. Evaluating the polynomial

$$\sum_{i=0}^{d-1} f_i\binom{n-1}{i}$$

at $n=0$ gives $\chi(\Delta)=f_0-f_1+\ldots+(-1)^{d-1}f_{d-1}$, the Euler characteristic of Δ. Thus the Hilbert polynomial of $k[\Delta]$ and its Hilbert function coincide for all $n\geq 0$ iff $\chi(\Delta)=1$. The order of the pole of the Hilbert series of $k[\Delta]$ is d, hence $\dim k[\Delta]=d$.

Recall the definition of the $h-$vector of a simplicial complex Δ. Suppose the $f-$vector is $(1=f_{-1},f_0,f_1,\ldots,f_{d-1})$. Then the $h-$vector is defined by the equation

$$(x-1)^d+f_0(x-1)^{d-1}+f_1(x-1)^{d-2}+\ldots+f_{d-1}=h_0x^d+h_1x^{d-1}+h_2x^{d-2}+\ldots+h_d.$$

(1.8) Proposition.

$$H(k[\Delta],t)=\frac{h_0+h_1t+\ldots+h_dt^d}{(1-t)^d}.$$

Proof.

$$\begin{aligned}H(k[\Delta],t) &= 1+\frac{f_0t}{1-t}+\frac{f_1t^2}{(1-t)^2}+\ldots+\frac{f_{d-1}t^d}{(1-t)^d}\\ &= \frac{(1-t)^d+tf_0(1-t)^{d-1}+t^2f_1(1-t)^{d-2}+\ldots+f_{d-1}t^d}{(1-t)^d}.\end{aligned}$$

From the definition of $h-$vector it follows that

$$\begin{aligned}h_j &= \sum_{i=0}^{j}(-1)^{j-i}\binom{d-i}{j-i}f_{i-1} \text{ for } j=0,1,\ldots,d\\ &= \text{coefficient of } t^j \text{ in } \sum_{i=0}^{d} f_{i-1}t^i(1-t)^{d-i}.\end{aligned}$$

(1.9) Corollary. $h_0=1,\ h_1=f_0-d,\ h_d=(-1)^{d-1}(\chi(\Delta)-1)$ *and* $f_{d-1}=\sum_{i=0}^{d}h_i$.

(1.10) Theorem. *Let Δ be a $d-1$ dimensional Cohen-Macaulay complex, (i.e., $k[\Delta]$ is Cohen-Macaulay) with n vertices and the $h-$vector $(h_0, h_1, \dots, h_d)$. Then*

$$0 \leq h_i \leq \binom{n-d+i-1}{i}, \quad 0 \leq i \leq d.$$

Proof. We may assume that k is infinite. Since $k[\Delta]$ is Cohen-Macaulay there exists a regular sequence $x_1, \dots, x_d$ of degree one elements. Then $S := k[\Delta]/(x_1, \dots, x_d)$ is zero-dimensional. Thus $h_i = H(S, i) \geq 0$ for all i. Now S is a graded ring generated by $n-d$ elements of degree one. Hence the Hilbert function of S is bounded above by the Hilbert function of the polynomial ring over k in $n-d$ variables. This yields the upper bound for h_i.

(1.11) Example. We will illustrate the above results by considering the simplicial complex Δ :

Put $R = k[X_1, \dots, X_5]$. Then $I_\Delta = (X_2, X_3) \cap (X_4, X_5)$. By Stanley's formula, the Hilbert series of $k[\Delta]$ is

$$\begin{aligned} H(k[\Delta], t) &= 1 + \frac{5t}{1-t} + \frac{6t^2}{(1-t)^2} + \frac{2t^3}{(1-t)^3} \\ &= \frac{1+2t-t^2}{(1-t)^3}. \end{aligned}$$

Since $h_2 < 0$, $k[\Delta]$ is not Cohen-Macaulay by (1.10). We give an independent verification of the Hilbert series by considering the exact sequence

$$0 \longrightarrow k[\Delta] \longrightarrow \frac{R}{(X_2, X_3)} \bigoplus \frac{R}{(X_4, X_5)} \longrightarrow \frac{R}{(X_2, X_3) + (X_4, X_5)} \longrightarrow 0$$

which gives

$$\begin{aligned} H(k[\Delta], t) &= H(\frac{R}{(X_2, X_3)}, t) + H(\frac{R}{(X_4, X_5)}, t) - H(k[X_1], t) \\ &= \frac{1}{(1-t)^3} + \frac{1}{(1-t)^3} - \frac{1}{1-t} \\ &= \frac{1+2t-t^2}{(1-t)^3}. \end{aligned}$$

We end this section by deducing the Dehn-Sommerville equations by employing the Hilbert series of the face ring.

(1.12) Definition. A simplicial complex Δ is called *Eulerian* if it is pure and $\tilde{\chi}(Lk\,F) = (-1)^{\dim(Lk\,F)}$ for all $F \in \Delta$.

(1.13) Theorem (Dehn–Sommerville Equations). *Let Δ be a $(d-1)-$dimensional Eulerian complex with $h-$vector $(h_0, h_1, \dots, h_d)$. Then $h_{d-i} = h_i$ for all $i = 0, 1, \dots, d$.*

Proof. The equations $h_{d-i} = h_i$ for $i = 0, 1, \ldots, d$ are clearly equivalent to the equality

$$H(k[\Delta], t) = (-1)^d H(k[\Delta], t^{-1}).$$

To establish this equality we substitute t_i by t_i^{-1} in the fine Hilbert series of $k[\Delta]$.

$$\begin{aligned}
H(k[\Delta], t_1^{-1}, \ldots, t_n^{-1}) &= \sum_{F\in\Delta} \left\{ \prod_{x_i\in F} (1 + \frac{t_i}{1-t_i}) \right\} (-1)^{\dim F+1} \\
&= \sum_{F\in\Delta} (-1)^{\dim F+1} \left\{ \sum_{G\subseteq F} \left(\prod_{x_i\in G} \frac{t_i}{1-t_i} \right) \right\} \\
&= \sum_{G\in\Delta} \left\{ \sum_{F\in\Delta, G\subseteq F} (-1)^{\dim F+1} \prod_{x_i\in G} \frac{t_i}{1-t_i} \right\} \\
&= \sum_{G\in\Delta} \left\{ \sum_{F\in Lk\, G} (-1)^{\dim F-\dim G} \right\} \prod_{x_i\in G} \frac{t_i}{1-t_i} \\
&= \sum_{G\in\Delta} \left\{ (-1)^{\dim G} \tilde{\chi}(Lk\, G) \right\} \prod_{x_i\in G} \frac{t_i}{1-t_i} \\
&= (-1)^d H(k[\Delta], t_1, \ldots, t_n).
\end{aligned}$$

Substituting $t_1 = \cdots = t_n = t$ we get $H(k[\Delta], t) = (-1)^d H(k[\Delta], t^{-1})$ which yields the Dehn-Sommerville equations.

2. **Shellable Complexes are Cohen-Macaulay**

Our objective in this section is to show that the face ring of a shellable simplicial complex is Cohen-Macaulay. Let Δ be a simplicial complex. Suppose $F_1, F_2, \ldots, F_m$ are certain faces of Δ. We denote by $(F_1, F_2, \ldots, F_m)$ the subcomplex of Δ consisting of those faces G of Δ such that $G \subseteq F_i$ for some i. Recall that a pure simplicial complex Δ is called *shellable* if there is an ordering $F_1, F_2, \ldots, F_m$ of all facets of Δ such that $(F_1, F_2, \ldots, F_i) \cap (F_{i+1})$ is generated by a non-empty subset of maximal proper faces of F_{i+1} for $i = 1, \ldots, m-1$. The order complexes of a large number of posets in combinatorics are shellable. Hence it is important to analyse the face rings of such complexes. Besides proving that face rings of shellable simplicial complexes are Cohen-Macaulay we shall also give an algebraic proof of McMullen's interpretation of the $h-$vector of a shellable simplicial complex. This interpretation played an important role in his solution of the upper bound conjecture for simplicial polytopes. We begin with several preparatory propositions.

(2.1) Proposition. *Let I and J be ideals of a ring R. Then the sequence of $R-$modules*

$$0 \longrightarrow \frac{R}{I\cap J} \xrightarrow{f} \frac{R}{I} \bigoplus \frac{R}{J} \xrightarrow{g} \frac{R}{I+J} \longrightarrow 0$$

is exact. Here f and g are defined by $f(\bar{r}) = (\bar{r}, -\bar{r})$ and $g(\bar{a}, \bar{b}) = \bar{a} + \bar{b}$.

Proof. Exercise.

(2.2) Proposition (Depth Lemma). *Let A, B, C be finitely generated modules over a Noetherian ring R. Let I be an ideal of R with $IA \neq A$, $IB \neq B$ and $IC \neq C$. Suppose*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence. Then one of the following inequalities occurs:

(i) $d(I,A) = d(I,B) < d(I,C)$.

(ii) $d(I,B) = d(I,C) \leq d(I,A)$.

(iii) $d(I,A) - 1 = d(I,C) < d(I,B)$.

Proof. Suppose that $d(I,B)$ and $d(I,C)$ are both positive. Then I is not contained in the associated primes of B and C. By prime avoidance lemma we can find an x in I avoiding all the associated primes of B and C. Hence x is regular on B and C. Consider the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "x"'] & B \arrow[r] \arrow[d, "x"'] & C \arrow[r] \arrow[d, "x"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are multiplication by x.
By snake lemma we get the exact sequence

$$0 \longrightarrow 0 \underset{A}{:} x \longrightarrow 0 \underset{B}{:} x \longrightarrow 0 \underset{C}{:} x \longrightarrow \frac{A}{xA} \longrightarrow \frac{B}{xB} \longrightarrow \frac{C}{xC} \longrightarrow 0.$$

Since x is a nonzero divisor on C, $0 \underset{C}{:} x = 0$. Hence we get the exact sequence

$$0 \longrightarrow \frac{A}{xA} \longrightarrow \frac{B}{xB} \longrightarrow \frac{C}{xC} \longrightarrow 0.$$

Since $d(I, B/xB) = d(I,B) - 1$, we are reduced to the case when $d(I,B) = 0$ or $d(I,C) = 0$.

Suppose that $d(I,B) = 0$. For simplicity we assume that f is the inclusion map. Let $b \in B$ such that $Ib = 0$. Case (iii) does not arise and case (ii) clearly holds. As for case(i) let $d(I,C) > 0$. Then $Ib = 0$ gives $Ig(b) = 0$. Hence $g(b) = 0$. Therefore $b \in A$ and consequently $d(I,A) = 0$.

Now let $d(I,B) > 0$ and $d(I,C) = 0$. Then we need to show that (iii) occurs. For this we show that $d(I,A) = 1$. Since $d(I,C) = 0$ there is a non-zero c such that $Ic = 0$. Let $b \in B$ be the preimage of c. Let $x \in I$ be a non-zerodivisor on B. We show that $d(I, A/xA) = 0$. If $xb \in xA$ then $xb = xa$ for some $a \in A$ and $x(b-a) = 0$. Since x is a non-zerodivisor on B, $b = a \in A$. But then $g(b) = c = 0$ which is a contradiction. Thus xb is not in xA. Now $g(Ixb) = Ixg(b) = Ixc = 0$. Hence $Ixb \subseteq xA$. Thus I consists of only zero divisors on A/xA. Hence $d(I,A) = 1$.

(2.3) Corollary. *Let I, J be homogeneous ideals of the polynomial ring $R = k[X_1, \ldots, X_n]$. Suppose that R/I and R/J are Cohen-Macaulay rings of dimension d and $R/(I+J)$ is a Cohen-Macaulay ring of dimension $d-1$. Then $R/(I \cap J)$ is Cohen-Macaulay of dimension d.*

Proof. Since the set of primes which contain $I \cap J$ is the union of the primes which contain I or J, it follows that $dim R/(I \cap J) = d$. Now use the depth lemma and (2.1) to complete the proof.

(2.4) Proposition. *Suppose Δ is a simplicial complex with n vertices and $F_1, \ldots, F_m$ are the facets of Δ. Let $\Delta_t = (F_1, \ldots, F_t)$ where $t \leq m$. Then*

$$k[\Delta_t] \simeq \frac{k[X_1, \ldots, X_n]}{\bigcap_{i=1}^{t} P_{F_i}}.$$

Proof. Suppose that the vertex set of Δ_t is $X_1, \ldots, X_r$. Let Q_{F_i} denote the face ideal of F_i in $S = k[X_1, \ldots, X_r]$. Then $P_{F_i} = (Q_{F_i}, X_{r+1}, \ldots, X_n)R$.

$$\bigcap_{i=1}^{t} P_{F_i} = \bigcap_{i=1}^{t} (Q_{F_i}, X_{r+1}, \ldots, X_n)R.$$
$$= (\bigcap_{i=1}^{t} Q_{F_i})R + (X_{r+1}, \ldots, X_n)R.$$

Therefore

$$k[\Delta_t] = \frac{k[X_1, \ldots, X_r]}{\bigcap_{i=1}^{t} Q_{F_i}} \simeq \frac{k[X_1, \ldots, X_n]}{\bigcap_{i=1}^{t} P_{F_i}}.$$

(2.5) Proposition. *Suppose Δ_1 and Δ_2 are subcomplexes of Δ. Then* $I_{\Delta_1 \cap \Delta_2} = I_{\Delta_1} + I_{\Delta_2}$.

Proof. Exercise.

(2.6) Theorem. *Let Δ be $(d-1)-$dimensional shellable simplicial complex with shelling $F_1, \ldots, F_m$ of its facets. Then $k[\Delta]$ is Cohen-Macaulay for any field k.*

Proof. We apply induction on the number of facets. If $m = 1$ then $\Delta = (F_1)$ and $k[\Delta]$ is a polynomial ring which is Cohen-Macaulay. Suppose the theorem is valid for $m - 1$. Put $I = P_{F_1} \cap \ldots \cap P_{F_{m-1}}$ and $J = P_{F_m}$. Suppose Δ has n vertices $X_1, \ldots, X_n$. Put $k[X] = k[X_1, \ldots, X_n]$. Consider the exact sequence

$$0 \longrightarrow \frac{k[X]}{I \cap J} \longrightarrow \frac{k[X]}{I} \bigoplus \frac{k[X]}{J} \longrightarrow \frac{k[X]}{I + J} \longrightarrow 0.$$

Since Δ is pure $\dim k[X]/I = \dim k[X]/J = d$. By induction $k[X]/I$ is Cohen-Macaulay. Since $k[X]/J$ is a poynomial ring in d variables over k, it is Cohen-Macaulay. The ring $k[X]/(I + J)$ is the face ring of the simplicial complex $\Delta' = (F_1, \ldots, F_{m-1}) \cap (F_m)$ by (2.5). Suppose Δ', after renumbering has vertices $X_1, \ldots, X_d$. By shellability Δ' is generated by a nonempty set of maximal proper faces of F_m say $G_1, \ldots, G_q$. Then each of the face ideals $P_{G_1}, \ldots, P_{G_q}$ in $T := k[X_1, \ldots, X_d]$ is generated by a single variable. Hence $k[\Delta'] = T/f$ where f is a product of q distinct indeterminates. It is easy to see that T/f is Cohen-Macaulay of dimension $d-1$. By the depth lemma $k[X]/(I \cap J)$ is Cohen-Macaulay.

(2.7) Example. It is natural to ask if there is a non- shellable simplicial complex Δ which is Cohen-Macaulay. There do exist triangulations of spheres which are not shellable. But they are difficult to describe. Instead, we take a clue from (2.6) which asserts that for any field k and any shellable simplicial complex Δ, $k[\Delta]$ is Cohen-Macaulay. Thus if we find a Δ so that Cohen-Macaulayness of $k[\Delta]$ depends on k it will follow that Δ is not shellable. We will see in the next chapter that $k[\Delta]$ is Cohen-Macaulay iff the reduced homology groups of links of all faces of Δ vanish below the top dimension (Reisner's Theorem). We know that $\tilde{H}_1(\mathbf{P}^2, k) = 0$ if $\operatorname{char} k \neq 2$ and $\tilde{H}_1(\mathbf{P}^2, k) = k$ if $\operatorname{char} k = 2$. Hence if Δ any triangulation of $\mathbf{P}^2$ then for fields k with $\operatorname{char} k = 2$, $k[\Delta]$ is not Cohen-Macaulay. Thus Δ is not shellable. By Reisner's Theorem $k[\Delta]$ is Cohen-Macaulay for k with $\operatorname{char} k \neq 2$.

We close this section by deducing McMullen's interpretation of the h–vector of a shellable simplicial complex. Suppose that Δ is shellable with shelling $F_1, \ldots, F_j$. Let r_j be the number of facets of $\Delta_{j-1} \cap (F_j)$ for $j = 2, \ldots, m$ and set $r_1 = 0$.

(2.8) Theorem. *Let Δ be a $(d-1)$-dimensional shellable complex with shelling $F_1,\dots,F_m$. Then $h_i = card\{j \mid r_j = i\}$ for $i = 0,\dots d$. In other words, $1 + t^{r_2} + \cdots + t^{r_m} = h_0 + h_1 t + \cdots + h_d t^d$.*

Proof. Put $H(k[\Delta_j], t) = Q_j(t)/(1-t)^d$. We have the exact sequence

$$0 \longrightarrow \frac{R}{I_{\Delta_{j-1}} \cap P_{F_j}} \longrightarrow \frac{R}{I_{\Delta_{j-1}}} \bigoplus \frac{R}{P_{F_j}} \longrightarrow \frac{R}{I_{\Delta_{j-1}} + P_{F_j}} \longrightarrow 0$$

where $R = k[X_1,\dots,X_n]$, $n =$ number of vertices of Δ. As before $R/(I_{\Delta_{j-1}} + P_{F_j})$ is a quotient of a polynomial ring in d variables over k by a square free monomial of degree r_j. Hence

$$H(\frac{R}{I_{\Delta_{j-1}} + P_{F_j}}, t) = \frac{(1-t^{r_j})}{(1-t)^d}.$$

By the exact sequence above

$$\begin{aligned} \frac{Q_j(t)}{(1-t)^d} &= \frac{Q_{j-1}(t)}{(1-t)^d} + \frac{1}{(1-t)^d} - \frac{(1-t^{r_j})}{(1-t)^d} \\ &= \frac{Q_{j-1}(t) + t^{r_j}}{(1-t)^d} \end{aligned}$$

Since $Q_1(t) = 1$ we obtain $Q_m(t) = 1 + t^{r_2} + \cdots + t^{r_m} = h_0 + h_1 + \cdots + h_d t^d$, which implies the theorem.

Exercises

(1) Find the face ring and Hilbert Series of the simplicial complex Δ:

Find the $h-$vector of Δ and verify McMullen's interpretation of it.

(2) Give an example of a simplicial complex which fails the Dehn-Sommerville equation.

(3) Find a triangulation Δ of $I\!RP^2$ and calculate the depth of $\mathbb{Z}_2[\Delta]$.

(4) Find an example of a simplicial complex Δ which fails the inequality $h_i \leq \binom{n-d+i-1}{i}$ where $d-1 = \dim\Delta$ and $n = f_0(\Delta)$ and $h_i \geq 0$ for all i.

(5) Let Δ be a $(d-1)$-dimensional simplicial complex. For r, $0 \leq r \leq d-1$, define the $r-$skeleton of Δ to be $\Delta_r = \{F \in \Delta \mid \dim F \leq r\}$. Find $h(\Delta_r)$ in terms of $h(\Delta)$.

(6) Let Δ_1 and Δ_2 be simplicial complexes. Find $h(\Delta_1 \star \Delta_2)$ in terms of $h(\Delta_1)$ and $h(\Delta_2)$.

(7) Let Δ be simlicial complex. Δ is called disconnected if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is called connected. Show if Δ is disconnected, then depth $k[\Delta] = 1$.