MA 5102, Basic Algebraic Topology, Quiz-1

Solve all questions. You can get a maximum of 10 marks.

1. Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Show that the induced homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. (2 marks)
2. Let $p : (E, e_0) \to (B, b_0)$ be a covering map and $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ be the lifting correspondence. Let $H = p_*(\pi_1(E, e_0))$ be a subgroup of $\pi_1(B, b_0)$ and $\pi_1(B, b_0)/H = \{H * g : g \in \pi_1(B, b_0)\}$ be the set of all right cosets of H.
   Show that ϕ induces an injective map $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$. (3 marks)
3. Show that a retract of a contractible space is contractible. (2 marks)
4. (a) Assume that $F : X \times I \to X$ gives a deformation retraction of X to a point x_0. Show that for any neighbourhood W of x_0, there exists a neighbourhood U of x_0 such that $F(U \times I) \subset W$. (2 marks)
   (b) Let $X = (0 \times I) \cup (I \times 0) \cup_{n/geq1} (1/n \times I)$ be subspace of $\mathbb{R}^2$ and $x_0 = 0 \times 1$. Show that X is contractible to x_0, but X does not deformation retracts to x_0. (2 marks)
5. Compute the fundamental group of union of three spheres S^2's having one point in common. (2 marks)
6. Show that the space $X = \mathbb{R}^3-$ non-negative x, y, z axes deformation retracts to a figure eight space. (2 marks)

MA 5102, Basic Algebraic Topology
Mid Sem Exam - 25 marks

Solve all questions. D^n **denotes closed unit disc in** $\mathbb{R}^n$.

1. Show that if X is contractible and Y is path connected, then the space of homotopy class of maps $[X, Y]$ has a single element. (2 marks)
2. Find a continuous map $g : T \to S^1$ which is not homotopic to constant, where T is a torus. (2 marks)
3. Show by an example that if $p : E \to B$ is a covering map, and E_0 is a subspace of E, then the restriction $p_{E_0} : E_0 \to p(E_0)$ need not be a covering map. (2 marks)
4. Let $f : S^n \to X$ be a continuous map which is homotopic to a constant map. Show that f can be extended to a continuous map $F : D^{n+1} \to X$. (2 marks)
5. Construct a continuous surjective map $f : S^1 \to S^1$ which is homotopic to a constant map. Extend f to a continuous surjective map $F : D^2 \to D^2$. (3 marks)
6. Show that any continuous map $f : D^2 \to D^2$ has a fixed point. (2 marks)
7. Assume the fact that if $h : S^1 \to S^1$ is continuous and antipode preserving, then h is not null homotopic. Show that if $f : S^2 \to \mathbb{R}^2$ is continuous, then there exist $x \in S^2$ such that $f(x) = f(-x)$. (3 marks)
8. Assume that $F : X \times I \to X$ gives a deformation retraction of X to a point x_0. Show that for any neighbourhood W of x_0, there exists a neighbourhood U of x_0 such that $F(U \times I) \subset W$. (2 marks)
9. Let $X = (0 \times I) \cup (I \times 0) \cup_{n \geq 1} (1/n \times I)$ be subspace of $\mathbb{R}^2$ and $x_0 = 0 \times 1$. Show that X is contractible to x_0, but X does not deformation retracts to x_0. (2 marks)
10. Compute the fundamental group of $\mathbb{R}^3-$ non-negative x, y, z axes. (2 marks)
11. Compute π_1 of sphere S^2 with north and south pole identified. (2 marks)
12. Let $p : S^1 \to P^1$ be the covering map obtained by identifying antipodal points. Find the induced group homomorphism $p_* : \pi_1(S^1) \to \pi_1(P^1)$. (2 marks)
13. Let $X = \bigvee_1^\infty S^1$ be infinite wedge of circles. Compute $\pi_1(X)$. (3 marks)

MA 5102, Basic Algebraic Topology, Quiz 2

Solve all problems. Max marks = 12.

1. Let X be the subspace of $\mathbb{R}^2$ which is union of circles $S_n, n \geq 1$, where S_n is circle of radius n and center at $(n, 0)$. Show that S_n is not wedge of circles $\bigvee_1^\infty S^1$. Compute π_1 of X. (3 marks)
2. Construct a space whose fundamental group is $\mathbb{Z}/3\mathbb{Z}$. (3 marks)
3. Let B, E, E' be path connected and locally path connected spaces, $p : E \to B$ and $p' : E' \to B$ be covering maps with $p(e_0) = b_0 = p'(e_0')$. Assume $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e_0'))$. Then show that there is a homeomorphism $h : E \to E'$ with $h(e_0) = e_0'$ such that $p' \circ h = p$. [You can state and use lifting result.] (3 marks)
4. State appropriate results needed and sketch a proof of the following result: any subgroup of a free group is free. (3 marks)
5. Give an example of a covering space of figure eight which is not a regular covering. (2 marks)

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End Semester Exam, 9:30-12:30 AM.

Solve all questions.

Total marks in the paper = 52.

Max marks you can get = 38.

All spaces are path connected and locally path connected.

1. Let $G = G_1 * G_2$ be free product of non-trivial groups G_1 and G_2. For $x \in G$, let length(x) be the length of the unique reduced word in the elements of G_1 and G_2 that represents x. (2+2)
   (1) Show that if length of x is even > 2, then order of x is infinite.
   (2) Show by an example that if length of x is 3, then order of x may be finite.
2. Find spaces X and Y whose fundamental groups are
   (a) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and (b) $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ respectively. (2+2)
3. Let P_1 and P_2 be regular polygons with 4 and 8 oriented edges respectively.
   Label the edges of P_1 as a_1, a_1, a_2, a_2 and that of P_2 as $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}$, where exponent of label is 1 when the edge is counterclockwise oriented and -1 when the edge is clockwise oriented.
   Let X_1 and X_2 be the quotient of P_1 and P_2 by identifying the edges with same label. Show that X_1 is not homeomorphic to X_2. (4)
4. (a) Show that any continuous map $f : \mathbb{R}P^2 \to S^1$ is null-homotopic. (3)
   (b) What about maps from S^1 to $\mathbb{R}P^2$? Are they all null-homotopic. (3)
5. Give an example of a space X such that X does not have a universal cover. Justify your answer. (4)
6. Let G be a subgroup of $Homeo(X)$. Then G acts on X. (2+2)
   (a) Give an example where the action is fixed-point free but is not properly discontinuous.
   (b) Show that if the action is properly discontinuous, then the quotient map $X \to X/G$ is a covering map.
7. Construct a 3 sheeted covering map of wedge of three circles. (3)

P.T.O.

8. Let X be figure eight space. Then $\pi_1(X)$ is free group on generators a and b. Let H be the subgroup generated by a. (3+1)
   (a) Construct a space E and a covering map $p : E \to X$ such that image of p_* is H.
   (b) Is p a regular covering?
9. Construct a 3 sheeted covering map of wedge of three circles. (3)
10. State appropriate results needed and sketch a proof of the following result: any subgroup of a free group is free. (4)
11. Show that every infinite tree is contractible. (4)
12. Sketch (with statements only) the proof of the fact that $\pi_1(S^1) = \mathbb{Z}$. (3)
13. Let X be figure eight space and Y be theta space. Then X and Y have same homotopy type. Describe maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to each other. (4)
14. Write the steps (statements only) of proof of Jordan curve theorem. (4)