Quiz-1 Algebraic Topology

1. Show that for odd n, the antipodal map and the identity map from S^n to S^n are homotopic.

2. Let X be an Euclidean Neighbourhood Retract space and A a closed subspace of X which is contractible. Show that there exist a continuous map $f : X \to X$ such that f is homotopic to identity and $f(A) =$ point. Further, show that X and X/A have the same homotopy type.

3. Show that $SO(n, \mathbb{R})$ is a retraction of $SL(n, \mathbb{R})$.

4. Compute the fundamental group of the space X which is S^2 together with a line joining north and south poles, with subspace topology from $\mathbb{R}^3$.

Quiz-2 - Algebraic Topology

1. For $n \geq 2$, state the appropriate path lifting result for the canonical map $p : S^n \to \mathbb{R}P^n$ and prove that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$.

2. Let $\pi : E \to B$ be a local trivial fibration with typical fiber F. If F is path connected, then show that the induced map $\pi_\sharp : \pi_1(E, z_0) \to \pi_1(B, x_0)$ is surjective. [You can (state and) use path lifting property for π.]

3. Show that S^3 is a topological group.

4. Show that a locally injective map $f : X \to Y$ has discrete fibers.

Quiz-3 Algebraic Topology

1. Assume that $h : Z \to C(I, Y)$ is a continuous map with compact open topology on $C(I, Y)$. Show that the induced map $H : Z \times I \to Y$, defined as $H(z, t) = h(z)(t)$, is continuous.

2. Find spaces $\widetilde{X}, X$ which are path connected and locally path connected, and a covering map $p : \widetilde{X} \to X$ which is not a regular covering. Conclude that $\pi_1(X)$ is non-abelian.

3. Let X be path connected and locally path connected. Let G be a group of homeomorphisms of X. Show that G acts properly discontinuous on X if and only if the projection map $p : X \to X/G$ is a covering map.

4. (1) Show that every continuous map $f : \mathbb{R}P^2 \to S^1$ is homotopic to constant.

   (2) Find a continuous map $g : T \to S^1$ which is not homotopic to constant, where T is a torus.

Mid Semester Exam
Algebraic Topology - M803

(24th February, 3:30-5:30 pm, 24 marks)

Solve all questions. Each question is of 3 marks.

1. Let $a, b : I \to X$ be two loops in X at the points x_0, y_0 respectively. Show that if a and b are free homotopic, then there exists a path $c : I \to X$ from x_0 to y_0 such that $a \cong (cb)c^{-1}$.

2. Let $f : S^n \to X$ be a continuous map which is homotopic to a constant map. Show that f can be extended to a continuous map $F : B^{n+1} \to X$.

3. Find the fundamental group of $\mathbb{R}^3 - E$, where E is the union of x and y-axis.

4. A subspace Y of X is called a *deformation retract* of X if there exist a homotopy $H : X \times I \to X$ such that $H_0 = id$ and $H_1(X) = A$ with $H_1(a) = a$ for all $a \in A$.

   Clearly, if Y is a deformation retract of X, then Y is a retract of X. Show by an example, that the converse may not be true.

5. Construct a continuous surjective map $F : D^2 \to D^2$ such that $F|_{S^1} = f : S^1 \to S^1$ and f is homotopic to a constant map.

6. Let $a, b : I \to S^1$ be two loops in S^1 (at x_0 and x_1 respectively) with same degree $n(a) = n(b)$. Show that a, b are free homotopic.

7. Show that any continuous map $f : D^2 \to D^2$ has a fixed point.

   [You can use $\pi_1(S^1) = \mathbb{Z}$.]

8. Show that there are no retraction $r : X \to A$, where (X, A) are

   (a) $(\mathbb{R}^3, S^1)$,

   (b) $X = S^1 \times D^1$ is solid torus and $A = S^1 \times S^1$, the boundary of X.

****

End Semester Exam - Algebraic Topology
Instructor: Manoj K. Keshari

(25th April, 2-5 pm, 40 marks)

Solve all the problems. You can get maximum 40 marks out of 45. Give proper justifications.

1. Find whether open interval $(0, 1)$ is a retract of $(0, 2)$. (2)

2. List 4 topological properties which are not presereved under homotopy equivalence. (2)

3. Show that the antipodal map a and the identity map id from $S^3 \to S^3$ are homotopic. (2)

4. Let X be path connected such that $\pi_1(X) = 0$. Show that every map $S^1 \to X$ extends to a map $D^2 \to X$. (2)

5. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map and Z is a connected space. Let $f : (Z, z_0) \to (X, x_0)$ be a continuous map. Assume f has a lift $\widetilde{f} : (Z, z_0) \to (\widetilde{X}, \widetilde{x}_0)$. Show that $\widetilde{f}$ is the unique lift of f. (2)

6. Show that if a path-connected and locally-path-connected space X has a simply connected cover, then X has to be semi-locally simply-connected. (2)

7. Give an example to show that a simply connected space X may not be locally simply connected. (2)

8. Give an example of a locally injective map which is not a local homeomorphism. (2)

9. Give an example of a local homeomorphism which is not a covering map. (2)

P.T.O.

10. Let $Y \subset \mathbb{R}^n$ be a compact space of ENR type. Show that there exist $\delta > 0$ such that if $f, g : X \to Y$ are continuous with $|f(x) - g(x)| < \delta$ for all $x \in X$, then f is homotopic to g. (2)

11. Find a universal cover of the space $\mathbb{R} \wedge S^1$. (3)

12. Let $f : S^1 \to S^1$ be a continuous map which is homotopic to a constant map. Show that there exist $z \in S^1$ such that $f(z) = f(-z)$. (3)

13. Find all two sheeted covering spaces of the figure eight space X. (3)

14. Find a covering map $p : \mathbb{R} \times S^1 \to S^1 \times S^1$ such that the image of $p_\sharp$ is the cyclic subgroup of $\mathbb{Z}^2 = \pi_1(S^1 \times S^1)$ generated by $(1, 1)$. (3)

15. Show that the projection map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a local trivial fibration with typical fiber S^1. (3)

16. Let $p : \widetilde{X} \to X$ be a covering map with $\widetilde{X}$ path connected.

    (i) Show that $\pi_1(X, x)$ acts transitively on the fiber $p^{-1}(x)$. (2)

    (ii) Find the stabilizer of $\widetilde{x} \in p^{-1}(x)$. (2)

    (iii) Show that the group $G(\widetilde{X}/X)$ of covering transformations is a properly discontinuous group of homeomorphisms of $\widetilde{X}$. (2)

17. (i) Show that the fundamental group of any finite graph is a free product of some copies of $\mathbb{Z}$. (2)

    (ii) Using the fact that any covering space of a graph is a graph, show that any subgroup of the group $\mathbb{Z} * \mathbb{Z}$ is free. (2)

End Semester Exam - Algebraic Topology (CBS)

Instructor: Manoj K. Keshari

(25th April, 10:30-13:30 hours, 40 marks)

Solve all the problems. You can get maximum 40 marks out of 47. Give proper justifications.

1. Show that $\mathbb{R}^2 - (0,0)$ has same homotopy type as figure eight. (2)

2. Show that the antipodal map a and the identity map id from $S^3 \to S^3$ are homotopic. (2)

3. Construct a continuous surjective map $F : D^2 \to D^2$ such that $F|_{S^1} = f : S^1 \to S^1$ and f is homotopic to a constant map. (2)

4. Show that a locally injective map $f : X \to Y$ has discrete fibers. (2)

5. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map and Z is a connected space. Let $f : (Z, z_0) \to (X, x_0)$ be a continuous map. Assume f has a lift $\widetilde{f} : (Z, z_0) \to (\widetilde{X}, \widetilde{x}_0)$. Show that $\widetilde{f}$ is the unique lift of f. (2)

6. Assume that $h : Z \to C(I, Y)$ is a continuous map with compact open topology on $C(I, Y)$. Show that the induced map $H : Z \times I \to Y$, defined as $H(z, t) = h(z)(t)$, is continuous. (2)

7. Let X be the figure eight space. Find a path connected and locally path connected space $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$ which is not a regular covering (i.e. image $p_\sharp(\pi_1(\widetilde{X}))$ is not normal in $\pi_1(X)$). (2)

8. Give an example to show that a simply connected space X may not be locally simply connected. (2)

9. Show that if a path-connected and locally-path-connected space X has a simply connected cover, then X has to be semi-locally simply-connected. (2)

P.T.O.

10. Give an example of a locally injective map which is not a local homeomorphism. (2)

11. Give an example of a local homeomorphism which is not a covering map. (2)

12. State Jordan curve theorem and the classification theorem for surfaces. (2)

13. Show that S^3 has a topological group structure. (2)

14. Show that every continuous map $f : \mathbb{R}P^2 \to S^1$ is homotopic to a constant map. (3)

15. Let X be path connected and locally path connected. Let G be a group of homeomorphisms of X. Show that if the projection map $p : X \to X/G$ is a covering map, then G acts properly discontinuously on X . (3)

16. Let $f : S^1 \to S^1$ be a continuous map which is homotopic to constant. Show that there exist $z \in S^1$ such that $f(z) = f(-z)$. (3)

17. For $n \geq 2$, prove that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$. (3)

18. Let $\pi : E \to B$ be a local trivial fibration with typical fiber F. If F is path connected, then show that the induced map $\pi_\sharp : \pi_1(E, z_0) \to \pi_1(B, x_0)$ is surjective. (3)

19. Use generalized Van-Kampen theorem (done in seminar) to find the fundamental group of the subspace X of $\mathbb{R}^2$, where X is union of circles of radius n and center $(n, 0)$, for $n \geq 1$. (3)

20. Find a covering map $p : \mathbb{R} \times S^1 \to S^1 \times S^1$ such that the image of $p_\sharp$ is the cyclic subgroup of $\mathbb{Z}^2 = \pi_1(S^1 \times S^1)$ generated by $(1, 1)$. (3)