EFFICIENT GENERATION OF IDEALS IN A DISCRETE HODGE ALGEBRA

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ABSTRACT. Let R be a commutative Noetherian ring and D be a discrete Hodge algebra over R of dimension $d > \dim(R)$. Then we show that

(i) the top Euler class group $E^d(D)$ of D is trivial.

(ii) if $d > \dim(R) + 1$, then $(d-1)$-st Euler class group $E^{d-1}(D)$ of D is trivial.

1. INTRODUCTION

Let R be a commutative Noetherian ring. An R-algebra D is called a *discrete Hodge algebra over* R if $D = R[X_1, \cdots, X_n]/\mathcal{I}$, where $\mathcal{I}$ is an ideal of $R[X_1, \cdots, X_n]$ generated by monomials. Typical examples are $R[X_1, \cdots, X_n]$, $R[X, Y](XY)$ etc. In [V], Vorst studied the behaviour of projective modules over discrete Hodge algebras. He proved [V, Theorem 3.2] that *every finitely generated projective D-module is extended from R if for all k, every finitely generated projective $R[X_1, \cdots, X_k]$-module is extended from R.*

Later Mandal [M 2] and Wiemers [Wi] studied projective modules over discrete Hodge algebra D. In [Wi], Wiemers proved the following significant result. *Let P be a projective D-module of rank $\geq$ dim$(R) + 1$. Then (i) $P \simeq Q \oplus D$ for some D-module Q and (ii) P is cancellative, i.e. $P \oplus D \simeq P' \oplus D$ implies $P \simeq P'$.*

When $D = R[X, Y](XY)$, above results of Wiemers are due to Bhatwadekar and Roy [B-R]. Very recent, inspired by results of Bhatwadekar and Roy, Das and Zinna [D-Z 3] studied the behaviour of ideals in $R[X, Y]/(XY)$ and proved the following result on efficient generation of ideals. *Assume dim$(R) \geq 1$, $D = R[X, Y]/(XY)$ and $I \subset D$ is an ideal of height $n =$ dim(D). Assume I/I^2 is generated by n elements. Then any given set of n generators of I/I^2 can be lifted to a set of n generators of I. In particular, the top Euler class group $E^n(D)$ of D is trivial.*

As $R[X, Y]/(XY)$ is the simplest example of a discrete Hodge algebra over R, motivated by above discussions, one can ask the following question.

Question 1.1. Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n > \dim(R)$. Let $I \subset D$ be an ideal of height n. Suppose that $I = (f_1, \cdots, f_n) + I^2$. Do there exist $g_1, \cdots, g_n \in I$ such that

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$I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$? In other words, Is the top Euler class group $E^n(D)$ of D trivial? (For definition of Euler class groups, see [B-RS 2] and [B-RS 3].)

We answer Question 1.1 affirmatively and prove the following more general result ((3.1) below).

Proposition 1.2. *Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n >$ dim(R). Let P be a projective D-module of rank n which is extended from R and I be an ideal in D of height ≥ 2. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$. In particular, the n-th Euler class group $E^n(D)$ of D is trivial.*

The above result can be extended to any rank n projective D-module when R contains $\mathbb{Q}$ (3.8). Here is the precise statement.

Theorem 1.3. *Let R be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension ≥ 2 and D be a discrete Hodge algebra over R of dimension $n >$ dim(R). Let I be an ideal in D of height ≥ 3 and P be any rank n projective D-module. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

After studying the top rank case, one is tempted to go one step further and inquire the following question.

Question 1.4. Let R be a commutative Noetherian ring of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d \geq \dim(R) + 2$. Let I be an ideal in D of height $d - 1$ and P be a projective D-module of rank $d - 1$. Suppose that $\alpha : P/IP \twoheadrightarrow I/I^2$ is a surjection. Can α be lifted to a surjection $\beta : P \twoheadrightarrow I$?

We answer Question 1.4 affirmatively when R contains $\mathbb{Q}$ ((4.3) below) as follows.

Theorem 1.5. *Let R be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d >$ dim(R). Let I be an ideal in D of height ≥ 4 and P be a projective D-module of rank $n \geq max\{$dim$(R) + 1, d - 1\}$. Suppose that $\alpha : P/IP \twoheadrightarrow I/I^2$ is a surjection. Then there exists a surjection $\beta : P \twoheadrightarrow I$ which lifts α. As a consequence, if $d \geq$ dim$(R) + 2$, then $(d - 1)$-st Euler class group $E^{d-1}(D)$ of D is trivial.*

Finally we derive an interesting consequence of above result as follows (see (4.6)).

Theorem 1.6. *Let R be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d >$ dim(R). Let I be a locally complete intersection ideal in D of height $n \geq max\{$dim$(R) + 1, d - 1\}$. Then I is set theoretically generated by n elements.*

In Section 5, we give some partial answer to the following question.

Question 1.7. Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $> \dim(R)$. Let $I \subset D$ be an ideal of height $> \dim(R)$. Suppose that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq \dim(D/I) + 2$. Do there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$?

The above question has been settled in the affirmative by Mandal in [M 1] when D is a polynomial algebra over R. Recently Fasel [Fa] has settled a conjecture of Murthy and proved the following result. *Let k be an infinite field of characteristic $\neq 2$ and $I \subset k[T_1, \cdots, T_m]$ be an ideal. Then we have $\mu(I) = \mu(I/I^2)$.*

Therefore, we may ask the following natural question.

Question 1.8. Let k be an infinite field of characteristic $\neq 2$ and D be a discrete Hodge algebra over k. Let $I \subset D$ be an ideal. Is $\mu(I) = \mu(I/I^2)$?

2. Preliminaries

Assumptions. Throughout this paper, rings are assumed to be commutative Noetherian and projective modules are finitely generated and of constant rank. For a ring A, $\dim(A)$ will denote the Krull dimension of A.

We start with the following definition.

Definition 2.1. An R-algebra D is said to be a *discrete Hodge algebra over R* if D is isomorphic to $R[X_1, \cdots, X_n]/J$, where J is an ideal of $R[X_1, \cdots, X_n]$ generated by monomials. A discrete Hodge algebra over R is called *trivial* if it is a polynomial algebra over R. Otherwise, it is called a *non-trivial* discrete Hodge algebra.

Definition 2.2. We call an ideal I of a ring R to be *efficiently generated* if $\mu(I) = \mu(I/I^2)$, where $\mu(I)$ (resp. $\mu(I/I^2)$) stands for the minimal number of generators of I (resp. I/I^2) as an R-module (resp. R/I-module).

Definition 2.3. Let I be an ideal of a ring R. We say that I is *set theoretically generated by k elements $f_1, \cdots, f_k$* in R if $\sqrt{(f_1, \cdots, f_k)} = \sqrt{I}$.

The next two results are standard. For proofs the reader may consult [B-RS 2].

Lemma 2.4. [B-RS 2, 2.11] *Let R be a ring and J be an ideal of R. Let $K \subset J$ and $L \subset J^2$ be two ideals of R such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ with $e(1-e) \in K$ and $K = J \cap J'$, where $J' + L = R$.*

Lemma 2.5. [B-RS 2, 2.13] *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that $\operatorname{ht}(I_a) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$, then $\operatorname{ht} I \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then $\operatorname{ht} I = n$.*

The following lemma is proved in [D-K, Lemma 3.1].

Lemma 2.6. *Let R be a ring and $J \subset R$ be an ideal. Let P be a projective R-module of rank $n \geq dim(R/J) + 1$ and let $\alpha : P/JP \twoheadrightarrow J/J^2f$ be a surjection for some $f \in R$. Given any ideal $K \subset R$ with $dim(R/K) \leq n-1$, the map α can be lifted to a surjection $\beta : P \twoheadrightarrow J''$ such that:*

(1) $J'' + (J^2 \cap K)f = J$,
(2) $J'' = J \cap J'$ *and* $ht(J') \geq n$,
(3) $(J^2 \cap K)f + J' = R$.

The following theorem is due to Mandal [M 3, Theorem 2.1].

Theorem 2.7. *Let R be a ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let P be a projective R-module of rank $n \geq dim(R[T]/I) + 2$. Suppose that there exists a surjection $\phi : P[T] \twoheadrightarrow I/(I^2T)$. Then, there exists a surjection $\psi : P[T] \twoheadrightarrow I$ which lifts ϕ.*

We improve [D-Z 3, Lemma 2.9] in the following form to suit our needs. The proof is similar to the one given in [D, Lemma 4.9].

Lemma 2.8. *Let R be a ring and I, J be two ideals in R such that $J \subset I^2$. Let P be a projective R-module and $K \subset R$ be an ideal. Suppose that we are given surjections $\alpha : P \twoheadrightarrow I/J$ and $\beta : P \twoheadrightarrow \overline{I}$ such that $\alpha \equiv \beta$ mod $\overline{J}$, where bar denotes reduction modulo the ideal K. Then α can be lifted to surjection $\phi : P \twoheadrightarrow I/(JK)$.*

The following result is implicit in the proof of [V, Theorem 3.2].

Theorem 2.9. *Let R be a ring and $r > 0$ be an integer. Assume that all projective modules of rank r over polynomial extensions of R are extended from R. Then all projective modules of rank r over discrete Hodge R-algebras are extended from R.*

The following result is due to Das and Zinna [D-Z 1, Theorem 3.12].

Theorem 2.10. *Let R be a ring of dimension $n \geq 2$. Let $R \hookrightarrow S$ be a subintegral extension and L be a projective R-module of rank one. Then, the natural map $E^n(R, L) \longrightarrow E^n(S, L \otimes_R S)$ is an isomorphism.*

The following result follows from [Sw, Lemma 3.2].

Lemma 2.11. *Let $R \hookrightarrow S$ be a subintegral extension and $\mathcal{J} \subset R[X_1, \cdots, X_m]$ be an ideal generated by monomials. Then $R[X_1, \cdots, X_m]/\mathcal{J} \hookrightarrow S[X_1, \cdots, X_m]/\mathcal{J}$ is also subintegral.*

The following result is from [D-Z 2, Proposition 2.13] for $d \geq 2$. By patching argument, it can be proved for $d = 1$.

Proposition 2.12. *Let A be a ring of dimension $d \geq 1$. Let I be an ideal of $A[T]$ of height ≥ 2 and P be a projective $A[T]$-module of rank $n \geq d+1$. Suppose that there exists a surjection $\phi : P/IP \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\Psi : P \twoheadrightarrow I$.*

The following result is due to Wiemers [Wi, Corollary 4.3].

Theorem 2.13. *Let R be a ring of dimension d and D be a discrete Hodge algebra over R. Let P be a projective D-module of rank $> d$. Then*
(1) $P = D \oplus Q$ for some projective D-module Q.
(2) P is cancellative, i.e. if $P \oplus D \xrightarrow{\sim} P' \oplus D$, then $P \xrightarrow{\sim} P'$.

It is not hard to see that, adapting the same proof of [D-RS, Theorem 4.2], we can extend [D-RS, Theorem 4.2] in the following form.

Theorem 2.14. *Let R be a ring containing $\mathbb{Q}$ with dim$(R) = n \geq 3$ and $I \subseteq R[T]$ be an ideal of height ≥ 3. Let L be a projective R-module of rank 1 and P be a projective $R[T]$-module of rank n whose determinant is $L[T]$. Assume that we are given a surjection $\psi : P \twoheadrightarrow I/(I^2T)$. Assume further that $\psi \otimes R(T)$ can be lifted to a surjection $\psi' : P \otimes R(T) \twoheadrightarrow IR(T)$. Then, there exists a surjection $\Psi : P \twoheadrightarrow I$ such that Ψ is a lift of ψ .*

3. Main Theorems: Codimension Zero Case

We begin with the following result which is motivated by [D-Z 3, Theorem 4.2].

Proposition 3.1. *Let R be a ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n >$ dim(R). Let P be a projective D-module of rank n which is extended from R and I be an ideal in D of height ≥ 2. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

Proof. If D is a trivial discrete Hodge algebra over R, then we are done by (2.12). So we assume that R is a non-trivial discrete Hodge algebra over R. Let 'prime' denote reduction modulo the nil radical N of D. Assume $\alpha \otimes D'$ can be lifted to a surjection $\alpha_1 : P \otimes D' \twoheadrightarrow I \otimes D'$. Then α_1 can be lifted to a surjection $\alpha_2 : P_{1+N} \twoheadrightarrow I_{1+N}$. Since $1 + N$ consists of units of D, α_2 is a lift of α. Therefore, we may assume that D is reduced.

Let $D = R[X_1, \cdots, X_m]/\mathcal{J}$, where $\mathcal{J}$ is an ideal of $R[X_1, \cdots, X_m]$ generated by square-free monomials. We prove the result using induction on the number of variables m. If $m = 1$, then D is just $R[X_1]$ and the result follows from (2.12).

Let us assume that $m \geq 2$. We can assume that $\mathcal{J} = \mathcal{K} + X_m\mathcal{L}$, where $\mathcal{K}$ and $\mathcal{L}$ are monomial ideals of $R[X_1, \cdots, X_{m-1}]$. Then $D = R[X_1, \cdots, X_m]/(\mathcal{K}, X_m\mathcal{L})$.

Case 1. $n \geq 3$. Given $\alpha : P/IP \twoheadrightarrow I/I^2$, applying (2.6), α can be lifted to a surjection $\gamma_1 : P \twoheadrightarrow I'$ such that (1) $I' = I \cap J$, (2) $I + J = D$, (3) ht$(J) \geq n$.

If $\text{ht}(J) > n$, then $J = D$ and we are done. So assume $\text{ht}(J) = n$. Let $\gamma : P \twoheadrightarrow J/J^2$ be the surjection induced from γ_1.

Let x_m and L be the images of X_m and $\mathcal{L}$ in D, respectively. We shall use 'tilde' when we move modulo (x_m) and 'bar' when we move modulo L. We first go modulo x_m and consider the surjection $\widetilde{\gamma} : \widetilde{P} \twoheadrightarrow \widetilde{J}/\widetilde{J}^2$. Note that $\widetilde{J}$ is an ideal of $\widetilde{D} = R[X_1, \cdots, X_{m-1}]/\mathcal{K}$ of height equal to dimension of $\widetilde{D}$. For this, we observe that

$$\dim(\widetilde{D}[X_m]) + \text{ht}(\widehat{X_m\mathcal{L}}) = \dim(D),$$

where $\widehat{X_m\mathcal{L}}$ is the image of $X_m\mathcal{L}$ in $\widetilde{D}[X_m]$.

By induction hypothesis on m, there exists a surjection $\phi : \widetilde{P} \twoheadrightarrow \widetilde{J}$ which is a lift of $\widetilde{\gamma}$. Therefore, it follows from (2.8) that γ can be lifted to a surjection $\psi : P \twoheadrightarrow J/(J^2x_m)$.

We now move to the ring $\overline{D} = \dfrac{R[X_1, \cdots, X_{m-1}]}{(\mathcal{K}, \mathcal{L})}[X_m]$ (i.e., go modulo L) and consider the surjection

$$\overline{\psi} : \overline{P} \twoheadrightarrow \overline{J}/(\overline{J}^2 X_m)$$

Now observe that J is of the form $J'/(X_m\mathcal{L})$ for some ideal J' in $\frac{R[X_1,\cdots,X_{m-1}]}{\mathcal{K}}[X_m]$ containing $X_m\mathcal{L}$. Observe that $\text{ht}(J') = \dim(\frac{R[X_1,\cdots,X_{m-1}]}{\mathcal{K}}[X_m])$. Therefore we may assume that J' contains a monic polynomial in X_m. Since $\overline{J} = J/L \cap J = J'/L \cap J'$, it follows that $\overline{J}$ contains a monic in X_m. Also $n \geq \dim(\overline{D}/\overline{J}) + 2 (= 2)$. By (2.7), there exists a surjection $\theta : \overline{P} \twoheadrightarrow \overline{J}$ which lifts $\overline{\psi}$.

Therefore, it follows from (2.8) that there exists a surjection $\delta : P \twoheadrightarrow J/(J^2x_mL)$ which is a lift of ψ. As $x_mL = 0$ in D, we obtain $\delta : P \twoheadrightarrow J$ is a surjection which lifts γ. Now we have

(1) $\gamma_1 : P \twoheadrightarrow I \cap J$ such that $\gamma_1 \otimes D/I = \alpha \otimes D/I$,
(2) $\delta : P \twoheadrightarrow J$ with $\delta \otimes D/J = \gamma_1 \otimes D/J = \gamma$.

Now by (2.13), $P = D \oplus P'$. Also it follows that $n \geq \dim(D/I) + 2$ and $n + \text{ht}(J) \geq \dim(D) + 3$. We can now use the subtraction principle [D-K, Proposition 3.2] to find a surjection $\beta : P \twoheadrightarrow I$ which lifts α. This completes the proof in case $n \geq 3$.

Case 2. $n = 2$. In this case $\dim(R) = 1$ and hence by (2.13), $P \simeq L \oplus D$ for some rank one projective D-module L.

We have $I = \alpha(P) + I^2$. Applying (2.4), we can find $f \in I$ such that $I = (\alpha(P), f)$ with $f(1-f) \in \alpha(P)$ and therefore we have a surjection $\alpha_{1-f} : P_{1-f} \twoheadrightarrow I_{1-f}$. Let $\pi : P_f = L_f \oplus D_f \twoheadrightarrow D_f = I_f$ be the projection onto the second factor. Now consider the following surjections:

$$\alpha_{f(1-f)} : P_{f(1-f)} \twoheadrightarrow I_{f(1-f)} = D_{f(1-f)}$$

$$\pi_{1-f} : P_{f(1-f)} \twoheadrightarrow I_{f(1-f)} = D_{f(1-f)}$$

Now it is not hard to show that there exists $\tau \in SL(P_{f(1-f)})$ such that $\alpha_{f(1-f)}\tau = \pi_{1-f}$. Therefore standard patching argument implies that there is a projective D-module Q of rank 2 such that Q maps onto I. By (2.13), $Q = \wedge^2(Q) \oplus D$. Also note that Q has determinant L and hence $Q \simeq L \oplus D$.

By (2.13), $L \oplus D$ is cancellative. We can now apply [B, Lemma 3.2] to find a surjection $\beta : P \twoheadrightarrow I$ which lifts α. $\square$

Corollary 3.2. *Let R be a ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n >$ dim(R). Let I be an ideal in D of height ≥ 2. Suppose that $I = (f_1, \cdots, f_n) + I^2$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $i = 1, \cdots, n$.*

Corollary 3.3. *Let R be a ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n >$ dim(R). Let L be any rank one projective D-module. Then the n-th Euler class group $E^n(D, L)$ is trivial.*

Proof. Let $D = R[X_1, \cdots, X_m]/\mathcal{J}$. Without loss of generality we can assume that D is reduced (see [B-RS 2, Corollary 4.6]). In particular, R is reduced. Let S be the seminormalization of R in its total quotient ring. Since S is seminormal, by [Sw, Theorem 6.1], every rank one projective $S[X_1, \cdots, X_k]$-module is extended from S for all k. Therefore, it follows from (2.9) that $L \otimes_R S$ is extended from S.

Let us denote $S[X_1, \cdots, X_m]/\mathcal{J}$ by D_1. Since $R \hookrightarrow S$ is a subintegral extension, by (2.11), $D \hookrightarrow D_1$ is also subintegral. As $L \otimes_R S$ is extended from S, by (3.1), it follows that $E^n(D_1, L \otimes_R S)$ is trivial. Finally, using (2.10), we have $E^n(D, L)$ is trivial. $\square$

The following result is due to Katz [Ka].

Theorem 3.4. *Let R be a ring and $I \subset R$ be an ideal. Let d be the maximum of the heights of maximal ideals containing I, and suppose that $d < \infty$. Then some power of I admits a reduction J satisfying $\mu(J/J^2) \leq d$.*

A result of Mandal from [M 2], can now be deduced.

Corollary 3.5. *Let R be a ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n >$ dim(R). Let $I \subset D$ be an ideal of height ≥ 2. Then I is set theoretically generated by n elements.*

Proof. Using Katz (3.4), there exists $k > 0$ such that I^k has a reduction J with $\mu(J/J^2) \leq n$. If $\mu(J/J^2) \leq n-1$, then clearly J is generated by at most n elements. Therefore we assume that $\mu(J/J^2) = n$. Since J is a reduction of I^k, it is easy to see that $\sqrt{I} = \sqrt{I^k} = \sqrt{J}$ and ht$(I) =$ ht(J). Applying (3.2), we see that J is generated by n elements. Therefore, I is set-theoretically generated by n elements. $\square$

We have the following variant of (3.1) for rings containing $\mathbb{Q}$.

Proposition 3.6. *Let R be a ring containing $\mathbb{Q}$ of dimension ≥ 2 and D be a discrete Hodge algebra over R of dimension $n > dim(R)$. Let I be an ideal in D of height ≥ 3 and P be any rank n projective D-module whose determinant is extended from R. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

Proof. We follow the proof of (3.1). The only thing which we need to show is that $\overline{\psi} : \overline{P} \twoheadrightarrow \overline{J}/(\overline{J}^2 X_m)$ can be lifted to a surjection $\theta : \overline{P} \twoheadrightarrow \overline{J}$. Rest of the proof is same. To show this, we use (2.14) in place of (2.7). By (2.14), it is enough to show that $\overline{\psi} \otimes R(X_m)$ can be lifted to a surjection from $\overline{P} \otimes R(X_m) \twoheadrightarrow \overline{J} \otimes R(X_m)$. This is clearly true, since $\overline{J}$ contains a monic polynomial in X_m and $\overline{P} = \overline{D} \oplus P'$ by (2.13). $\square$

The following lemma is very crucial to generalize above result.

Lemma 3.7. *Let R be a reduced ring and D be a discrete Hodge algebra over R. Let L be a rank one projective D-module. Then there exists a ring S such that*

(1) *$R \hookrightarrow S \hookrightarrow Q(R)$,*
(2) *S is a finite R-module,*
(3) *$R \hookrightarrow S$ is subintegral and*
(4) *$L \otimes_R S$ is extended from S.*

Proof. Let $R \hookrightarrow B \hookrightarrow Q(R)$ be the seminormalization of R. By Swan's result [Sw, Theorem 6.1], rank one projective modules over polynomial extensions of B are extended from B. Hence by (2.9), rank one projective modules over discrete Hodge algebras over B are extended from B. In particular $L \otimes_R B$ is extended from B. By [Sw, Theorem 2.8], B is direct limit of B_λ, where $R \hookrightarrow B_\lambda$ is finite and subintegral extension. Since L is finitely generated, we can find a subring $S = B_\lambda$ for some λ satisfying conditions $(1-4)$. $\square$

We now prove the general case of (3.6).

Theorem 3.8. *Let R be a ring containing $\mathbb{Q}$ of dimension ≥ 2 and D be a discrete Hodge algebra over R of dimension $n > dim(R)$. Let I be an ideal in D of height ≥ 3 and P be any rank n projective D-module. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

Proof. Without loss of generality, we may assume that D is reduced. In particular, R is reduced. Let $D = R[X_1, \cdots, X_m]/\mathcal{J}$, where $\mathcal{J}$ is an ideal of $R[X_1, \cdots, X_m]$ generated by square free monomials. By (3.7), there exists an extension $R \hookrightarrow S$ such that

(1) $R \hookrightarrow S \hookrightarrow Q(R)$,
(2) S is a finite R-module,

(3) $R \hookrightarrow S$ is subintegral and
(4) $\wedge^n(P) \otimes_R S$ is extended from S.

Let $E = S[X_1, \cdots, X_m]/\mathcal{J}$. Since $\wedge^n(P) \otimes_R S$ is extended from S, by (3.6), the induced surjection $\alpha^* : P \otimes E \twoheadrightarrow IE/I^2E$ can be lifted to a surjection $\phi : P \otimes E \twoheadrightarrow IE$. By (2.13), $P = D \oplus Q$. In case $P = \wedge^n(P) \oplus D^{n-1}$, the rest of the proof is given in [D-Z 1, Theorem 3.12]. The proof of [D-Z 1, Theorem 3.12] works for $P = D \oplus Q$ also. Hence we are done. $\square$

4. Main Theorems: Codimension One Case:

The aim of this section is to give an affirmative answer to Question 1.4 mentioned in the introduction. We start with the following lemma which generalizes (2.12).

Lemma 4.1. *Let R be a ring containing $\mathbb{Q}$ of dimension ≥ 2 and I be an ideal of $R[X, Y]$ of height ≥ 3. Let P be a projective $R[X, Y]$-module of rank $\geq \dim(R) + 1$ whose determinant is extended from $R[X]$. Suppose that there exists a surjection $\phi : P \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\psi : P \twoheadrightarrow I$.*

Proof. If rank of P is $> \dim(R) + 1$, then we are done by (2.12). So assume rank of $P = \dim(R) + 1$. Since R contains $\mathbb{Q}$, using [B-RS 1, Lemma 3.3] and replacing Y by $Y - \lambda$ for some $\lambda \in \mathbb{Q}$, we can assume that either $I(0) = R[X]$ or ht$(I(0)) =$ ht(I). If $I(0) = R[X]$, then by (2.8), we can lift ϕ to a surjection $\alpha : P \twoheadrightarrow I/(I^2Y)$.

Now assume that ht$(I(0)) =$ ht$(I) \geq 3$. Let "bar" denote the reduction modulo Y and consider $\overline{\phi} : \overline{P} \twoheadrightarrow \overline{I}/\overline{I}^2$. By (2.12), there exists a surjection $\beta : \overline{P} \twoheadrightarrow \overline{I}$ which lifts $\overline{\phi}$. Therefore, again by (2.8), we can lift ϕ to a surjection $\alpha : P \twoheadrightarrow I/(I^2Y)$. Therefore, in any case, we can lift ϕ to a surjection $\alpha : P \twoheadrightarrow I/(I^2Y)$.

Consider the surjection $\alpha \otimes R(Y) : P \otimes R(Y) \twoheadrightarrow I \otimes R(Y)/I^2 \otimes R(Y)$. Since $\dim(R(Y)) =$ $\dim(R)$, by (2.12), $\alpha \otimes R(Y)$ can be lifted to a surjection $\delta : P \otimes R(Y) \twoheadrightarrow I \otimes R(Y)$. Using (2.14), we get a surjection $\psi : P \twoheadrightarrow I$ which lifts α and hence lifts ϕ. $\square$

Proposition 4.2. *Let R be a ring containing $\mathbb{Q}$ of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d >$ dim(R). Let I be an ideal in D of height ≥ 4 and P be a projective D-module of rank $n \geq max\{dim(R) + 1, d - 1\}$ whose determinant is extended from R. Suppose that $\alpha : P \twoheadrightarrow I/I^2$ is a surjection. Then there exists a surjection $\beta : P \twoheadrightarrow I$ that lifts α.*

Proof. As in the proof of (3.1), we can assume that R is reduced and $D = R[X_1, \cdots, X_m]/\mathcal{I}$, where $\mathcal{I}$ is an ideal of $R[X_1, \cdots, X_m]$ generated by square free monomials. where $X_{i_1}^{l_1} \cdots X_{i_k}^{l_k} \in \mathcal{I}$ and $l_i \geq 1$ We prove the result using induction on m. If $m = 1$, then $D = R[X_1]$ and the result follows from (2.12).

Let us assume that $m \geq 2$. If D is a polynomial ring over R, then we are done by (4.1). Now suppose that D is a non-trivial discrete Hodge algebra. Then we can assume that $\mathcal{I} = (\mathcal{K}, X_m\mathcal{L})$, where $\mathcal{K}$ and $\mathcal{L}$ are monomial ideals in $R[X_1, \ldots, X_{m-1}]$. Then $D = R[X_1, \cdots, X_m]/(\mathcal{K}, X_m\mathcal{L})$.

Let x_m and L be the images of X_m and $\mathcal{L}$ in D respectively. We shall use "tilde" when we move modulo (x_m) and "bar" when we move modulo L. We first go modulo (x_m), i.e. to the discrete Hodge algebra $\widetilde{D} = R[X_1, \cdots, X_{m-1}]/\mathcal{K}$ and consider the surjection $\widetilde{\alpha} : \widetilde{P} \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$. Note that $\widetilde{I}$ is an ideal of $\widetilde{D}$ of height $\geq \dim(\widetilde{D}) - 1$. By induction hypothesis on m, there exists a surjection $\phi : \widetilde{P} \twoheadrightarrow \widetilde{I}$ which is a lift of $\widetilde{\alpha}$. Therefore, using (2.8), we can lift α to a surjection $\psi : P \twoheadrightarrow I/(I^2x_m)$.

We now move modulo L, i.e. $\overline{D} = \frac{R[X_1,\cdots,X_{m-1}]}{(\mathcal{K},\mathcal{L})}[X_m] := D_0[X_m]$ and consider the surjection

$$\overline{\psi} : \overline{P} \twoheadrightarrow \overline{I}/(\overline{I}^2 X_m).$$

Observe that ht$(\overline{I}) \geq$ dim$(R) \geq 3$. If dim$(D_0) < n$, then by (2.12), $\overline{\psi}$ can be lifted to a surjection $\theta : \overline{P} \twoheadrightarrow \overline{I}$. So assume dim$(D_0) = n$. Since dim$(R(X_m)) =$ dim(R) and $\overline{D} \otimes R(X_m) = \frac{R(X_m)[X_1,\cdots,X_{m-1}]}{(\mathcal{K},\mathcal{L})}$, by (3.6), the surjection $\overline{\psi} \otimes R(X_m) : \overline{P} \otimes R(X_m) \twoheadrightarrow \overline{I} \otimes R(X_m)/(\overline{I}^2 \otimes R(X_m))$ can be lifted to a surjection $\eta : \overline{P} \otimes R(X_m) \twoheadrightarrow \overline{I} \otimes R(X_m)$. By (2.14), there exists a surjection $\theta : \overline{P} \twoheadrightarrow \overline{I}$ which lifts $\overline{\psi}$.

Finally it follows from (2.8) that there exists a surjection $\beta : P \twoheadrightarrow I/(I^2x_mL)$ which lifts ψ. As $x_mL = 0$ in D, we obtain a surjection $\beta : P \twoheadrightarrow I$ which lifts α. $\square$

Now we will answer Question 1.4.

Theorem 4.3. *Let R be a ring of dimension ≥ 3 containing $\mathbb{Q}$ and D be a discrete Hodge algebra over R of dimension $d >$ dim(R). Let I be an ideal in D of height ≥ 4 and P be a projective D-module of rank $n \geq max\{dim(R) + 1, d - 1\}$. Suppose that $\alpha : P \twoheadrightarrow I/I^2$ is a surjection. Then there exists a surjection $\beta : P \twoheadrightarrow I$ which lifts α.*

Proof. Without loss of generality we may assume that D is reduced. Using (2.6), we can lift α to a surjection $\alpha' : P \twoheadrightarrow I \cap I_1$ such that $I + I_1 = D$ and ht$(I_1) \geq n$.

If ht$(I_1) > n$, then $I_1 = D$ and hence α' is the required surjective lift of α. Assume ht$(I_1) = n$. The map α' induces a surjection $\alpha_1 : P \twoheadrightarrow I_1/I_1^2$. If we can show that α_1 can be lifted to a surjection $\Delta : P \twoheadrightarrow I_1$, then by subtraction principle [D-K, Proposition 3.2], we can find a surjection $\Delta_1 : P \twoheadrightarrow I$ which lifts α. Therefore it is enough to show that α_1 has a surjective lift Δ. Now replacing I_1 by I and α_1 by α, we assume that ht$(I) = n$.

By (3.7), there exists an extension $R \hookrightarrow S$ such that

(1) $R \hookrightarrow S \hookrightarrow Q(R)$,
(2) S is a finite R-module,

(3) $R \hookrightarrow S$ is subintegral and
(4) $\wedge^n(P) \otimes_R S$ is extended from S.

Let C be the conductor ideal of R in S. Then $\text{ht}(C) \geq 1$. Since $\text{ht}(I) = n \geq max\{\dim(R)+1, d-1\}$ and $\text{ht}(C) \geq 1$, it follows that $\text{ht}(I^2 \cap C) \geq 1$. Therefore, we can choose an element $b \in I^2 \cap C$ such that $\text{ht}(b) = 1$. Let "bar" denote reduction modulo the ideal (b). Consider the surjection $\overline{\alpha} : \overline{P} \twoheadrightarrow \overline{I}/\overline{I}^2$ and note that $\dim(\overline{R}) < \dim(R)$.

Now applying (3.8), we can find a surjection $\gamma' : \overline{P} \twoheadrightarrow \overline{I}$ which lifts $\overline{\alpha}$. Choose a lift $\gamma : P \longrightarrow I$ of γ'. Since $b \in I^2$, γ is a lift of α and hence $(\gamma(P), b) = I$. Since $\text{hh}(I) = n$ and $b \in I^2$, applying (2.5) and replacing γ by $\gamma + b\delta$ for some $\delta \in P^*$, we can assume that $\text{ht}(\gamma(P)) = n$.

Applying (2.4), there exists an ideal I' of height $\geq n$ such that $I' + bD = D$ and $\gamma(P) = I \cap I'$. If $\text{ht}(I') > n$, then $I' = D$ and hence γ is the required surjective lift of α. Assume that $\text{ht}(I') = n$ and consider the surjection $\theta : P \twoheadrightarrow I'/I'^2$ induced from $\gamma : P \twoheadrightarrow I \cap I'$.

Consider the surjection $\theta \otimes_R S : P \otimes S \twoheadrightarrow I' \otimes S/I'^2 \otimes S$. Since $\wedge^n(P \otimes_R S)$ is extended from S, by (4.2), $\theta \otimes S$ can be lifted to a surjection $\Theta : P \otimes S \twoheadrightarrow I' \otimes S$. Now we need to show that we get a surjection $\eta : P \twoheadrightarrow I'$ which lifts θ. In the case of $P = \wedge^n(P) \oplus D^{n-1}$, this is proved in [D-Z 2, Lemma 5.1]. Note that $P = D \oplus P'$, by (2.13). The proof of [D-Z 2, Lemma 5.1] works in this case also, so we do not repeat it here. Therefore we have a surjection $\eta : P \twoheadrightarrow I'$ which lifts θ. Applying subtraction principle [D-K, Proposition 3.2], we can find a surjection $\beta : P \twoheadrightarrow I$ which lifts α. $\square$

The following result is immediate from (4.3).

Corollary 4.4. *Let R be a ring of dimension $d \geq 3$ containing $\mathbb{Q}$ and $D = \frac{R[X_1,X_2,X_3]}{\mathcal{I}}$ be a discrete Hodge algebra over R. Let I be an ideal in D of height ≥ 4 and P be a projective D-module of rank $n \geq \dim(R) + 1$. Suppose that $\alpha : P \twoheadrightarrow I/I^2$ be a surjection. Then there exists a surjection $\beta : P \twoheadrightarrow I$ which lifts α.*

The following theorem is due to Ferrand and Szpiro [Sz].

Theorem 4.5. *Let R be a ring and $I \subset R$ be a locally complete intersection ideal of height $r \geq 2$ and $\dim(R/I) \leq 1$. Then there is a locally complete intersection ideal $J \subset R$ of height r such that*

(1) $\sqrt{I} = \sqrt{J}$ *and*
(2) J/J^2 *is free R/J-module of rank r.*

As an application of (4.3), we improve a result of Mandal [M 2, Corollary 2.2], albeit with a stronger hypothesis on ideals.

Theorem 4.6. *Let R be a ring of dimension ≥ 3 containing $\mathbb{Q}$ and D be a discrete Hodge algebra over R with dim$(D) = d >$ dim(R). Let I be a locally complete intersection ideal in D of height $n = max\{dim(R)+1, d-1\}$. Then there exist $f_1, \cdots, f_n \in I$ such that $\sqrt{I} = \sqrt{(f_1, \cdots, f_n)}$. In other words, I is set theoretically generated by n elements.*

Proof. By (4.5), there is a locally complete intersection ideal J such that $\sqrt{I} = \sqrt{J}$ and J/J^2 is a free R/J-module of rank n. Applying (4.3), we see that J is generated by n elements. Therefore, I is set theoretically generated by n elements. $\square$

5. Some Auxiliary Results

After answering Question 1.1 and Question 1.4, it is natural to ask the following more general question.

Question 5.1. Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $>$ dim(R). Let $I \subset D$ be an ideal of height $>$ dim(R). Suppose that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq$ dim$(D/I) + 2$. Do there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$?

The above question has been answered affirmatively by Mandal when D is a polynomial algebra over R ([M 1]). Using [D-RS, Theorem 4.2] and following the proofs of (3.1) and (4.2), we can obtain the following result which gives a partial answer to the above question.

Theorem 5.2. *Let R be a ring of dimension $d \geq 2$ containing $\mathbb{Q}$ and $D = \frac{R[X_1, \cdots, X_m]}{(J_1, X_m J_2)}$, where J_1, J_2 are two ideals of $R[X_1, \cdots, X_{m-1}]$ generated by monomials. Let I be an ideal in D of height $> d$. Suppose that $I = (f_1, \cdots, f_n) + I^2$ with $n \geq dim(D/I) + 2$. Then $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ in each of the following cases:*

(1) *$n \geq max\{dim(D/J_1), dim(D/J_2)\}$ and ht$(\frac{I+J_2}{J_2}) \geq 2$.*
(2) *$n = max\{dim(D/J_1) - 1, dim(D/J_2) - 1\}$ and ht$(\frac{I+J_2}{J_2}) \geq 3$.*

As an application of (5.2), we give some explicit examples.

Example 5.3. Let R be a ring of dimension $d \geq 4$ containing $\mathbb{Q}$ and $D = \frac{R[X_1, \cdots, X_4]}{(X_4 J)}$ where $J = (X_1X_2, X_2X_3, X_1X_3)$. Let $I \subset D$ be an ideal of height $n \geq d+1$. Suppose that $I = (f_1, \cdots, f_n) + I^2$. Then there exist $g_1, \cdots g_n \in I$ such that $I = (g_1, \cdots g_n)$ with $f_i - g_i \in I^2$. In other words, the n-th Euler class group $E^n(D)$ is trivial.

Proof. Using (3.1) and (4.2), we can assume that $n = d+1$. We have dim$(D/J) = d+2$, i.e., $n = d+1 =$ dim$(D/J) - 1$ and ht$(\frac{I+J}{J}) \geq 3$. Also note that $n = d+1 \geq 5 \geq$ dim$(D/I) + 2$. Now the result follows from (5.2(2)). $\square$

The following result follows from (5.2).

Example 5.4. Let R be a ring of dimension $d \geq 3$ containing $\mathbb{Q}$ and $D = \frac{R[X_1,\cdots,X_m]}{(X_m J)}$ where $J = (X_i X_j | 1 \leq i \neq j \leq m-1)$. Let $I \subset D$ be an ideal such that $\text{ht}(\frac{I+J}{J}) \geq 3$. Suppose that $I = (f_1, \cdots, f_n) + I^2$ with $n \geq max\{d+1, \dim(D/I)+2\}$. Then there exist $g_1, \cdots g_n \in I$ such that $I = (g_1, \cdots g_n)$ with $f_i - g_i \in I^2$. $\square$

Now using (4.4) and following the proof of (5.2), we can derive the following.

Example 5.5. Let R be a ring of dimension $d \geq 4$ containing $\mathbb{Q}$ and $D = \frac{R[X_1,\cdots,X_4]}{(J_1, X_4 J_2)}$ where J_1, J_2 are two ideals in $R[X_1, X_2, X_3]$ generated by monomials and $\text{ht}(J_1 + J_2) \geq 2$. Let $I \subset D$ be an ideal of height $n \geq d+1$. Suppose that $I = (f_1, \cdots, f_n) + I^2$. Then there exist $g_1, \cdots g_n \in I$ such that $I = (g_1, \cdots g_n)$ with $f_i - g_i \in I^2$. In other words, the n-th Euler class group $E^n(D)$ is trivial.

Proof. Since $\dim(D) \leq d+3$, the case $n \geq d+2$ is covered by (3.1) and (4.2). Let us assume that $n = d+1$. Then $n = d+1 = \dim(D/J_2) - 1$ and $2n \geq \dim(D) + 2$. Now the result follows from (5.2). $\square$

The following result follows from (5.2).

Example 5.6. Let R be a ring of dimension $d \geq 4$ containing $\mathbb{Q}$ and $D = \frac{R[X_1,\cdots,X_5]}{(X_5 J)}$ where $J = (X_1X_2X_3, X_1X_2X_4, X_2X_3X_4)$. Let $I \subset D$ be an ideal such that $\text{ht}(I) = n \geq d+1$. Suppose that $I = (f_1, \cdots, f_n) + I^2$ with $n \geq d+2$. Then there exist $g_1, \cdots g_n \in I$ such that $I = (g_1, \cdots g_n)$ with $f_i - g_i \in I^2$. In other words, the n-th Euler class group $E^n(D)$ is trivial. $\square$

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