QUIZ-1 (Algebra-I)
29th May, 10 marks

1. (1) Show that $\mathbb{Q}(2^{1/3})$ consists of all real numbers of the form $r + s2^{1/3} + t4^{1/3}$ with $r, s, t \in Q$. (3 marks)

   (2) Find the inverse of $1+2^{1/3}$ in $\mathbb{Q}(2^{1/3})$ in the form $r+s2^{1/3}+t4^{1/3}$ with $r, s, t \in Q$. (4 marks)

2. Show that there exist a field with cardinality p^2, where $p > 0$ is a prime. (3 marks)

QUIZ-2 (Algebra-I)
5th June, 10 marks

1. Let K be a perfect field and L/K an algebraic extension. Show that L is perfect. (5 marks)

2. Prove that $X^3 - 2 \in \mathbb{Q}(i)[X]$ is irreducible, where $i = \sqrt{-1}$. (2 marks)

3. Let L be an extension of K of degree 2 with $2 \neq 0$ in K. Show that $L = K(\alpha)$ with $\alpha^2 \in K$. (3 marks)

QUIZ-3 (Algebra-I)
15th June, 10 marks

1. Let $M/L/K$ be field extensions. (2+2 marks)

   (a) Show that M/K normal extension implies M/L is normal.

   (b) Give an example to show that M/K normal does not implies that L/K is normal.

2. Assume L/K is finite extension. Show that the normal closure of L/K is finite extension of K. (2 marks)

3. Let K be a field of char $p > 0$ and L/K finite purely inseparable extension. (2+2 marks)

   (a) Show that $[L : K] = p^e$ for some $e \geq 0$ and $L^{p^e} \subset K$.

   (b) Show that if $L^{p^{e-1}} \not\subset K$, then L/K is a simple extension.

QUIZ-4 (Algebra-I)

22nd June, 10 marks

1. Show that if L/K is normal extension, then $S(L/K)$ is normal over K. (3 marks)

2. Let L/K be an extension of degree 3 which is not normal. Let M be the normal closure of L/K. Find $[M : K]$. (3 marks)

3. Let K be a field of char $p > 0$.

   (a) Show that if L/K is separable, then $L = K \, v \, L^p$.

   (b) If $\alpha \in L$ is separable over K, then $K(\alpha) = K(\alpha^p)$. (2+2 marks)

QUIZ-5 (Algebra-I)

1. Let F be a field and $P \subset F$ a subfield. Show that $Gal(F/P)$ is a Galois group on F. (2 marks)

2. Find whether $\mathbb{Q}$ is an invariant fubfield in $\mathbb{Q}(2^{1/3})$. (2 marks)

3. Assume ξ is transcendental over K and Γ an infinite subgroup of $Gal(K(\xi)/K)$. Show that $Inv(\Gamma) = K$. (3 marks)

4. Let F be a field, $P \subset F$ a subfield and $s \in Aut(F)$. Show that $Gal(F/sP) = sGal(F/P)s^{-1}$. (3 marks)

Final Exam (MA 414, Algebra I)
July 1, 10am-1pm, 50 marks

Solve all the problems and give proper justifications.

1. Let p be a prime. Show that $(p-1)! = -1$ modulo p. (2 marks)
2. Show that every finite field is perfect. (2 marks)
3. Show that if F is a finite field, then F has p^n elements, for some prime p and $n \geq 0$. (2 marks)
4. Assume L/K is a finite extension with char $K = p > 0$. If K is perfect, show that L is perfect. (2 marks)
5. If K is a field, show that every root of unity in $K(X)$ belongs to K. (2 marks)
6. If A is a ring such that $A[X]$ is a PID, show that A is a field. (2 marks)
7. Let p be a prime and n an integer not divisible by p. Show that $X^p - X - n \in Q[X]$ is irreducible. (2 marks)
8. Let L/K be field extension. Show that if $\alpha \in L$ is algebraic over K, then $K[\alpha] = K(\alpha)$. (2 marks)
9. Let K be a field of char $p > 0$. Show that K/K^p is normal extension. (2 marks)
10. Let K be a field of char $p > 0$ and L/K a normal extension. Show that if $\alpha \in L$ is fixed by every element of $Gal(L/K)$, then α is purely inseparable over K. (2 marks)
11. Compute the Galois group of irreduced polynomial $X^3 - 3X + 1 \in \mathbb{Q}[X]$ as a subgroup of S_3. (2 marks)
12. Let Γ be a subgroup of $Aut(F)$ for a field F. Let $\alpha \in F$ such that its Γ-orbit $O_\Gamma(\alpha)$ is a finite set. Show that α is algebraic over $Inv(\Gamma)$. (2 marks)
13. Let K be a field of characteristic $p > 0$. Give an example of a cyclic extension L/K of degree p. (2 marks)
14. Find infinitely many sub-fields of the extension $\mathbb{F}_p(X,Y)/\mathbb{F}_p(X^p, Y^p)$. (3 marks)

P.T.O.

15. Let K be a field, $\overline{K}$ the algebraic closure of K. If L/K is algebraic, show that L can be embedded in $\overline{K}$. (3 marks)

16. Let L/K be field extension of char $p > 0$. Let $\alpha \in L$ be separable over K and $\beta \in L$ be purely inseparable over K. Show that $K(\alpha, \beta) = K(\alpha\beta)$. (3 marks)

    [Hint: Use idea of proof of primitive element theorem]

17. Let L/K be finite separable. Let N be a normal closure of L/K. Show that N/K is a Galois extension. (3 marks)

18. Let L/K be finite Galois extension. For a fixed prime p, assume that if $E \neq K$ is any sub-field of L/K, then p divides $[E : K]$. Show that $[L : K] = p^n$ for some n.

    (3 marks)

19. Prove that $\mathbb{C}$ is algebraically closed using Galois theory. You may assume that $\mathbb{C}$ has no quadratic extension and $\mathbb{R}$ has no extension of odd degree. (3 marks)

20. Give an example of a field F such that $Aut(F)$ is infinite and such that no proper infinite subgroup of $Aut(F)$ is a Galois group in F. (3 marks)

21. Let $\zeta = e^{2\pi i/8}$. Find all subgroups of $Gal(\mathbb{Q}(\zeta)/\mathbb{Q})$ and the corresponding invariant fields. (3 marks)