UNIMODULAR ELEMENTS IN PROJECTIVE MODULES AND AN ANALOGUE OF A RESULT OF MANDAL

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1. Introduction

Throughout the paper, rings are commutative Noetherian and projective modules are finitely generated and of constant rank.

If R is a ring of dimension n, then Serre [Se] proved that projective R-modules of rank $> n$ contain a unimodular element. Plumstead [P] generalized this result and proved that projective $R[X] = R[\mathbb{Z}_+]$-modules of rank $> n$ contain a unimodular element. Bhatwadekar and Roy [B-R 2] generalized this result and proved that projective $R[X_1, \ldots, X_r] = R[\mathbb{Z}_+^r]$-modules of rank $> n$ contain a unimodular element.

In another direction, if A is a ring such that $R[X] \subset A \subset R[X, X^{-1}]$, then Bhatwadekar and Roy [B-R 1] proved that projective A-modules of rank $> n$ contain a unimodular element. Rao [Ra] improved this result and proved that if B is a birational overring of $R[X]$, i.e. $R[X] \subset B \subset S^{-1}R[X]$, where S is the set of non-zerodivisors of $R[X]$, then projective B-modules of rank $> n$ contain a unimodular element. Bhatwadekar, Lindel and Rao [B-L-R, Theorem 5.1, Remark 5.3] generalized this result and proved that projective $B[\mathbb{Z}_+^r]$-modules of rank $> n$ contain a unimodular element when B is seminormal. Bhatwadekar [Bh, Theorem 3.5] removed the hypothesis of seminormality used in [B-L-R].

All the above results are best possible in the sense that projective modules of rank n over above rings need not have a unimodular element. So it is natural to look for obstructions for a projective module of rank n over above rings to contain a unimodular element. We will prove some results in this direction.

Let P be a projective $R[\mathbb{Z}_+^r][T]$-module of rank $n = \dim R$ such that P_f and P/TP contain unimodular elements for some monic polynomial f in the variable T. Then P contains a unimodular element. The proof of this result is implicit in [B-L-R, Theorem 5.1]. We will generalize this result to projective $R[M][T]$-modules of rank n, where $M \subset \mathbb{Z}_+^r$ is a Φ-simplicial monoid in the class $\mathcal{C}(\Phi)$. For this we need the following result whose proof is similar to [B-L-R, Theorem 5.1].

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Proposition 1.1. *Let R be a ring and P be a projective $R[X]$-module. Let $J \subset R$ be an ideal such that P_s is extended from R_s for every $s \in J$. Suppose that*

(a) P/JP contains a unimodular element.

(b) If I is an ideal of $(R/J)[X]$ of height rank$(P)-1$, then there exist $\overline{\sigma} \in \text{Aut}\,((R/J)[X])$ with $\overline{\sigma}(X) = X$ and $\sigma \in \text{Aut}\,(R[X])$ with $\sigma(X) = X$ which is a lift of $\overline{\sigma}$ such that $\overline{\sigma}(I)$ contains a monic polynomial in the variable X.

(c) $EL(P/(X,J)P)$ acts transitively on Um$(P/(X,J)P)$.

(d) There exists a monic polynomial $f \in R[X]$ such that P_f contains a unimodular element.

Then the natural map Um$(P) \to$ Um(P/XP) is surjective. In particular, if P/XP contains a unimodular element, then P contains a unimodular element.

We prove the following result as an application of (1.1).

Theorem 1.2. *Let R be a ring of dimension n and $M \subset \mathbb{Z}_+^r$ a Φ-simplicial monoid in the class $\mathcal{C}(\Phi)$. Let P be a projective $R[M][T]$-module of rank n whose determinant is extended from R. Assume P/TP and P_f contain unimodular elements for some monic polynomial f in the variable T. Then the natural map Um$(P) \to$ Um(P/TP) is surjective. In particular, P contains a unimodular element.*

Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$. If P is a projective $R[X]$-module of rank n, then Das and Zinna [D-Z] have obtained an obstruction for P to have a unimodular element. Let us fix an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n P$, where L is the determinant of P. To the pair (P,χ), they associated an element $e(P,\chi)$ of the Euler class group $E(R[X],L)$ and proved that P has a unimodular element if and only if $e(P,\chi) = 0$ in $E(R[X],L)$ [D-Z].

It is desirable to have such an obstruction for projective $R[X,Y]$-module P of rank n. As an application of (1.2), we obtain such a result. Recall that $R(X)$ denotes the ring obtained from $R[X]$ by inverting all monic polynomials in X. Let L be the determinant of P and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. We define the Euler class group $E(R[X,Y],L)$ of $R[X,Y]$ as the product of Euler class groups $E(R(X)[Y], L \otimes R(X)[Y])$ of $R(X)[Y]$ and $E(R[Y], L \otimes R[Y])$ of $R[Y]$ defined by Das and Zinna [D-Z]. To the pair (P,χ), we associate an element $e(P,\chi)$ in $E(R[X,Y],L)$ and prove the following result (3.4).

Theorem 1.3. *Let the notations be as above. Then $e(P,\chi) = 0$ in $E(R[X,Y],L)$ if and only if P has a unimodular element.*

Let R be a local ring and P be a projective $R[T]$-module. Roitman [Ro, Lemma 10] proved that if the projective $R[T]_f$-module P_f contains a unimodular element for some monic polynomial $f \in R[T]$, then P contains a unimodular element. Roy [Ry, Theorem 1.1] generalized this result and proved that if P and Q are projective $R[T]$-modules with rank$(Q) <$ rank(P) such that Q_f is a direct summand of P_f for some monic polynomial $f \in R[T]$, then Q is a direct summand of P. Mandal [M, Theorem 2.1] extended Roy's result to Laurent polynomial rings.

We prove the following result (4.4) which gives Mandal's [M] in case $A = R[X, X^{-1}]$. Recall that a monic polynomial $f \in R[X]$ is called *special monic* if $f(0) = 1$.

Theorem 1.4. *Let R be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P and Q be two projective A-modules with rank$(Q) <$ rank(P). If Q_f is a direct summand of P_f for some special monic polynomial $f \in R[X]$, then Q is also a direct summand of P.*

2. Preliminaries

Definition 2.1. Let R be a ring and P be a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map $\varphi : P \twoheadrightarrow R$ such that $\varphi(p) = 1$. Note that P has a unimodular element if and only if $P \simeq Q \oplus R$ for some R-module Q. The set of all unimodular elements of P is denoted by $\text{Um}(P)$.

Definition 2.2. Let M be a finitely generated submonoid of $\mathbb{Z}_+^r$ of rank r such that $M \subset \mathbb{Z}_+^r$ is an integral extension, i.e. for any $x \in \mathbb{Z}_+^r$, $nx \in M$ for some integer $n > 0$. Such a monoid M is called a Φ-*simplicial* monoid of rank r [G2].

Definition 2.3. Let $M \subset \mathbb{Z}_+^r$ be a Φ-simplicial monoid of rank r. We say that M belongs to the class $\mathcal{C}(\Phi)$ if M is seminormal (i.e. if $x \in gp(M)$ and $x^2, x^3 \in M$, then $x \in M$) and if we write $\mathbb{Z}_+^r = \{t_1^{s_1} \dots t_r^{s_r} | \, s_i \geq 0\}$, then for $1 \leq m \leq r$, $M_m = M \cap \{t_1^{s_1} \dots t_m^{s_m} | \, s_i \geq 0\}$ satisfies the following properties: Given a positive integer c, there exist integers $c_i > c$ for $i = 1, \dots, m-1$ such that for any ring R, the automorphism $\eta \in Aut_{R[t_m]}(R[t_1, \dots, t_m])$ defined by $\eta(t_i) = t_i + t_m^{c_i}$ for $i = 1, \dots, m-1$, restricts to an R-automorphism of $R[M_m]$. It is easy to see that $M_m \in \mathcal{C}(\Phi)$ and rank $M_m = m$ for $1 \leq m \leq r$.

Example 2.4. The following monoids belong to $\mathcal{C}(\Phi)$ [K-S, Example 3.5, 3.9, 3.10].

(i) If $M \subset \mathbb{Z}_+^2$ is a finitely generated and normal monoid (i.e. $x \in gp(M)$ and $x^n \in M$ for some $n > 1$, then $x \in M$) of rank 2, then $M \in \mathcal{C}(\Phi)$.

(ii) For a fixed integer $n > 0$, if $M \subset \mathbb{Z}_+^r$ is the monoid generated by all monomials in $t_1, \dots, t_r$ of total degree n, then M is a normal monoid of rank r and $M \in \mathcal{C}(\Phi)$. In particular, $\mathbb{Z}_+^r \in \mathcal{C}(\Phi)$ and $< t_1^2, t_2^2, t_3^2, t_1t_2, t_1t_3, t_2t_3 > \in \mathcal{C}(\Phi)$.

(iii) The submonoid M of $\mathbb{Z}_+^3$ generated by $< t_1^2, t_2^2, t_3^2, t_1t_3, t_2t_3 > \in \mathcal{C}(\Phi)$.

Remark 2.5. Let R be a ring and $M \subset \mathbb{Z}_+^r = \{t_1^{m_1} \dots t_r^{m_r} | m_i \geq 0\}$ be a monoid of rank r in the class $\mathcal{C}(\Phi)$. Let I be an ideal of $R[M]$ of height $> \dim R$. Then by [G2, Lemma 6.5] and [K-S, Lemma 3.1], there exists an R-automorphism σ of $R[M]$ such that $\sigma(t_r) = t_r$ and $\sigma(I)$ contains a monic polynomial in t_r with coefficients in $R[M] \cap R[t_1, \ldots, t_{r-1}]$.

We will state some results for later use.

Theorem 2.6. [K-S, Theorem 3.4] *Let R be a ring and M be a Φ-simplicial monoid such that $M \in \mathcal{C}(\Phi)$. Let P be a projective $R[M]$-module of rank $> \dim R$. Then P has a unimodular element.*

Theorem 2.7. [D-K, Theorem 4.5] *Let R be a ring and M be a Φ-simplicial monoid. Let P be a projective $R[M]$-module of rank $\geq max\{\dim R + 1, 2\}$. Then $EL(P \oplus R[M])$ acts transitively on Um$(P \oplus R[M])$.*

The following result is proved in [B-L-R, Criterion-1 and Remark] in case $J = Q(P, R_0)$ is the Quillen ideal of P in R_0. The same proof works in our case.

Theorem 2.8. *Let $R = \oplus_{i \geq 0} R_i$ be a graded ring and P be a projective R-module. Let J be an ideal of R_0 such that J is contained in the Quillen ideal $Q(P, R_0)$. Let $p \in P$ be such that $p_{1+R^+} \in$ Um(P_{1+R^+}) and $p_{1+J} \in$ Um(P_{1+J}), where $R^+ = \oplus_{i \geq 1} R_i$. Then P contains a unimodular element p_1 such that $p = p_1$ modulo R^+P.*

The following result is a consequence of Eisenbud-Evans [E-E], as stated in [P, p. 1420].

Lemma 2.9. *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that $\operatorname{ht}(I_a) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$, then $\operatorname{ht} I \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then $\operatorname{ht} I = n$.*

3. Proofs of (1.1), (1.2) and (1.3)

3.1. **Proof of Proposition 1.1.** Let $p_0 \in \mathrm{Um}(P/JP)$ and $p_1 \in \mathrm{Um}(P/XP)$. Let $\widetilde{p_0}$ and $\widetilde{p_1}$ be the images of p_0 and p_1 in $P/(X,J)P$. By hypothesis (c), there exist $\widetilde{\delta} \in EL(P/(X,J)P)$ such that $\widetilde{\delta}(\widetilde{p_0}) = \widetilde{p_1}$. By [B-R 2, Proposition 4.1], $\widetilde{\delta}$ can be lifted to an automorphism δ of P/JP. Consider the fiber product diagram for rings and modules

$$\begin{array}{ccc} \frac{R[X]}{(XJ)} & \longrightarrow & \frac{R}{J}[X] \\ \downarrow & & \downarrow \\ \frac{R[X]}{(X)} & \longrightarrow & \frac{R[X]}{(X,J)}. \end{array} \qquad \begin{array}{ccc} \frac{P}{(XJ)P} & \longrightarrow & \frac{P}{JP} \\ \downarrow & & \downarrow \\ \frac{P}{XP} & \longrightarrow & \frac{P}{(X,J)P}. \end{array}$$

We can patch $\delta(p_0)$ and p_1 to get a unimodular element $p \in \mathrm{Um}(P/XJP)$ such that $p = \delta(p_0)$ modulo JP and $p = p_1$ module XP. Writing $\delta(p_0)$ by p_0, we assume that $p = p_0$ modulo JP and $p = p_1$ module XP.

Using hypothesis (d), we get an element $q \in P$ such that the order ideal $O_P(q) = \{\phi(q) | \phi \in \text{Hom}_{R[X]}(P, R[X])\}$ contains a power of f. We may assume that $f \in O_P(q)$.

Let "bar" denote reduction modulo the ideal (J). Write $\overline{P} = \overline{R[X]}p_0 \oplus Q$ for some projective $\overline{R[X]}$-module Q and $\overline{q} = (\overline{a}p_0, q')$ for some $q' \in Q$. By Eisenbud-Evans (2.9), there exist $\overline{\tau} \in EL(\overline{P})$ such that $\overline{\tau}(\overline{q}) = (\overline{a}p_0, q'')$ and $ht(O_Q(q''))\overline{R[X]}_{\overline{a}} \geq \text{rank}(P) - 1$. Since $\overline{\tau}$ can be lifted to $\tau \in \text{Aut}(P)$, replacing P by $\tau(P)$, we may assume that $ht(O_Q(q')) \geq \text{rank}(P) - 1$ on the Zariski-open set $D(\overline{a})$ of $\text{Spec}(\overline{R[X]})$.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be minimal prime ideals of $O_Q(q')$ in $\overline{R[X]}$ not containing $\overline{a}$. Then $ht(\cap_1^r \mathfrak{p}_i) \geq \text{rank}(P) - 1$. By hypothesis (b), we can find $\overline{\sigma} \in \text{Aut}(\overline{R[X]})$ with $\overline{\sigma}(X) = X$ and $\sigma \in \text{Aut}(R[X])$ with $\sigma(X) = X$ which is a lift of $\overline{\sigma}$ such that $\overline{\sigma}(\cap_1^r \mathfrak{p}_i)$ contains a monic polynomial in $\overline{R[X]} = \overline{R}[X]$. Note that $\sigma(f)$ is a monic polynomial. Replacing $R[X]$ by $\sigma(R[X])$, we may assume that $\cap_1^r \mathfrak{p}_i$ contains a monic polynomial in $\overline{R}[X]$, and $f \in O_P(q)$ is a monic polynomial.

If $\mathfrak{p}$ is a minimal prime ideals of $O_Q(q')$ in $\overline{R[X]}$ containing $\overline{a}$, then $\mathfrak{p}$ contains $O_{\overline{P}}(\overline{q})$. Since $f \in O_P(q)$, $\mathfrak{p}$ contains the monic polynomial $\overline{f}$. Therefore, all minimal primes of $O_Q(q')$ contains a monic polynomial, hence $O_Q(q')$ contains a monic polynomial, say $\overline{g} \in \overline{R}[X]$. Let $g \in R[X]$ be a monic polynomial which is a lift of $\overline{g}$.

Claim: For large $N > 0$, $p_2 = p + X^N g^N q \in \text{Um}(P_{1+JR})$.

Choose $\phi \in P^*$ such that $\phi(q) = f$. Then $\phi(p_2) = \phi(p) + X^N g^N f$ is a monic polynomial for large N. Since $p = p_0$ module JP, $\overline{p} = p_0$ and $\overline{q} = (\overline{ap}, q')$. Therefore, $\overline{p}_2 = \overline{p} + X^N \overline{g}^N (\overline{ap}, q') = ((1 + T^N \overline{g}^N \overline{a})\overline{p}, X^N \overline{g}^N q')$.

Since $\overline{g} \in O_Q(q') \subset O_{\overline{P}}(\overline{p}_2)$, we get $O_{\overline{P}}(\overline{p}) \subset O_{\overline{P}}(\overline{p}_2)$. Since $\overline{p} \in \text{Um}(\overline{P})$, we get $\overline{p}_2 \in \text{Um}(\overline{P})$ and hence $p_2 \in \text{Um}(P_{1+JR[X]})$. Since $O_P(p_2)$ contains a monic polynomial, by [La, Lemma 1.1, p. 79], $p_2 \in \text{Um}(P_{1+JR})$.

Now $p_2 = p = p_1$ modulo XP, we get $p_2 \in \text{Um}(P/XP)$. By (2.8), there exist $p_3 \in \text{Um}(P)$ such that $p_3 = p_2 = p_1$ modulo XP. This completes the proof. $\square$

3.2. **Proof of Theorem 1.2.** Without loss of generality, we may assume that R is reduced. When $n = 1$, the result follows from well known Quillen [Q] and Suslin [Su]. When $n = 2$, the result follows from Bhatwadekar [Bh, Proposition 3.3] where he proves that if P is a projective $R[T]$-module of rank 2 such that P_f contains a unimodular element for some monic polynomial $f \in R[T]$, then P contains a unimodular element. So now we assume $n \geq 3$.

Write $A = R[M]$. Let $J(A, P) = \{s \in A | P_s \text{ is extended from } A_s\}$ be the *Quillen ideal* of P in A. Let $\widetilde{J} = J(A, P) \cap R$ be the ideal of R and $J = \widetilde{J}R[M]$. We will show that J satisfies the properties of (1.1).

Let $\mathfrak{p} \in \text{Spec}(R)$ with $ht(\mathfrak{p}) = 1$ and $S = R - \mathfrak{p}$. Then $S^{-1}P$ is a projective module over $S^{-1}A[T] = R_{\mathfrak{p}}[M][T]$. Since $\dim(R_{\mathfrak{p}}) = 1$, by (2.6), $S^{-1}P = \wedge^n P_S \oplus S^{-1}A[T]^{n-1}$.

Since determinant of P is extended from R, $\wedge^n P_S = A[T]_S$ and hence $S^{-1}P$ is free. Therefore there exists $s \in R - \mathfrak{p}$ such that P_s is free. Hence $s \in \widetilde{J}$ and so $\text{ht}(\widetilde{J}) \geq 2$.

Since $\dim(R/\widetilde{J}) \leq n-2$ and $A[T]/(J) = (R/\widetilde{J})[M][T]$, by (2.6), P/JP contains a unimodular element.

If I is an ideal of $(A/J)[T] = (R/\widetilde{J})[M][T]$ of height $\geq n-1$, then by (2.5), there exists an $R[T]$-automorphism $\sigma \in \text{Aut}_{R[T]}(A[T])$ such that if $\overline{\sigma}$ denotes the induced automorphism of $(A/J)[T]$, then $\overline{\sigma}(I)$ contains a monic polynomial in T.

By (2.7), $EL(P/(T,J)P)$ acts transitively on $\text{Um}(P/(J,T)P)$.

Therefore, the result now follows from (1.1). $\square$

Corollary 3.1. *Let R be a ring of dimension n, $A = R[X_1, \cdots, X_m]$ a polynomial ring over R and P be a projective $A[T]$-module of rank n. Assume that P/TP and P_f both contain a unimodular element for some monic polynomial $f(T) \in A[T]$. Then P has a unimodular element.*

Proof. If $n = 1$, the result follows from well known Quillen [Q] and Suslin [Su] Theorem. When $n = 2$, the result follows from Bhatwadekar [Bh, Proposition 3.3]. Assume $n \geq 3$. Let L be the determinant of P. If $\widetilde{R}$ is the seminormalization of R, then by Swan [Sw], $L \otimes \widetilde{R}[X_1, \ldots, X_m]$ is extended from $\widetilde{R}$. By (1.2), $P \otimes \widetilde{R}[X_1, \ldots, X_m]$ has a unimodular element. Since $\widetilde{R}[X_1, \ldots, X_n]$ is the seminormalization of A, by Bhatwadekar [Bh, Lemma 3.1], P has a unimodular element. $\square$

3.3. **Obstruction for Projective Modules to have a Unimodular Element.** Let R be a ring of dimension $n \geq 2$ containing $\mathbb{Q}$ and P be a projective $R[X,Y]$-module of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. We call χ an *orientation* of P. In general, we shall use 'hat' when we move to $R(X)[Y]$ and 'bar' when we move modulo the ideal (X). For instance, we have:

(1) $L \otimes R(X)[Y] = \hat{L}$ and $L/XL = \overline{L}$,
(2) $P \otimes R(X)[Y] = \hat{P}$ and $P/XP = \overline{P}$.

Similarly, $\hat{\chi}$ denotes the induced isomorphism $\hat{L} \xrightarrow{\sim} \wedge^n \hat{P}$ and $\overline{\chi}$ denotes the induced isomorphism $\overline{L} \xrightarrow{\sim} \wedge^n \overline{P}$.

We now define the *Euler class* of (P, χ).

Definition 3.2. First we consider the case $n \geq 2$ and $n \neq 3$. Let $E(R(X)[Y], \hat{L})$ be the nth Euler class group of $R(X)[Y]$ with respect to the line bundle $\hat{L}$ over $R(X)[Y]$ and $E(R[Y], \overline{L})$ be the nth Euler class group of $R[Y]$ with respect to the line bundle $\overline{L}$ over $R[Y]$ (see [D-Z, Section 6] for definition). We define the nth *Euler class group* of $R[X,Y]$, denoted by $E(R[X,Y], L)$, as the product $E(R(X)[Y], \hat{L}) \times E(R[Y], \overline{L})$.

To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R[X, Y], L)$, called the *Euler class* of (P, χ), as follows:

$$e(P, \chi) = (e(\hat{P}, \hat{\chi}), e(\overline{P}, \overline{\chi}))$$

where $e(\hat{P}, \hat{\chi}) \in E(R(X)[Y], \hat{L})$ is the Euler class of $(\hat{P}, \hat{\chi})$ and $e(\overline{P}, \overline{\chi}) \in E(R[Y], \overline{L})$ is the Euler class of $(\overline{P}, \overline{\chi})$, defined in [D-Z, Section 6].

Now we treat the case when $n = 3$. Let $\widetilde{E}(R(X)[Y], \hat{L})$ be the nth restricted Euler class group of $R(X)[Y]$ with respect to the line bundle $\hat{L}$ over $R(X)[Y]$ and $\widetilde{E}(R[Y], \overline{L})$ be the nth restricted Euler class group of $R[Y]$ with respect to the line bundle $\overline{L}$ over $R[Y]$ (see [D-Z, Section 7] for definition). We define the *Euler class group* of $R[X, Y]$, again denoted by $E(R[X, Y], L)$, as the product $\widetilde{E}(R(X)[Y], \hat{L}) \times \widetilde{E}(R[Y], \overline{L})$.

To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R[X, Y], L)$, called the *Euler class* of (P, χ), as follows:

$$e(P, \chi) = (e(\hat{P}, \hat{\chi}), e(\overline{P}, \overline{\chi}))$$

where $e(\hat{P}, \hat{\chi}) \in \widetilde{E}(R(X)[Y], \hat{L})$ is the Euler class of $(\hat{P}, \hat{\chi})$ and $e(\overline{P}, \overline{\chi}) \in \widetilde{E}(R[Y], \overline{L})$ is the Euler class of $(\overline{P}, \overline{\chi})$, defined in [D-Z, Section 7].

Remark 3.3. Note that when $n = 2$, the definition of the Euler class group $E(R[T], L)$ is slightly different from the case $n \geq 4$. See [D-Z, Remark 7.8] for details.

Theorem 3.4. *Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$ and P be a projective $R[X, Y]$-module of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Then $e(P, \chi) = 0$ in $E(R[X, Y], L)$ if and only if P has a unimodular element.*

Proof. First we assume that P has a unimodular element. Therefore, $\hat{P}$ and $\overline{P}$ also have unimodular elements. If $n \geq 4$, by [D-Z, Theorem 6.12], we have $e(\hat{P}, \hat{\chi}) = 0$ in $E(R(X)[Y], \hat{L})$ and $e(\overline{P}, \overline{\chi}) = 0$ in $E(R[Y], \overline{L})$. The case $n = 2$ is taken care by [D-Z, Remark 7.8]. Now if $n = 3$, it follows from [D-Z, Theorem 7.4] that $e(\hat{P}, \hat{\chi}) = 0$ in $E(R(X)[Y], \hat{L})$ and $e(\overline{P}, \overline{\chi}) = 0$ in $\widetilde{E}(R[Y], \overline{L})$. Consequently, $e(P, \chi) = 0$.

Conversely, assume that $e(P, \chi) = 0$. Then $e(\hat{P}, \hat{\chi}) = 0$ in $E(R(X)[Y], \hat{L})$ and $e(\overline{P}, \overline{\chi}) = 0$ in $E(R[Y], \overline{L})$. If $n \neq 3$, by [D-Z, Theorem 6.12] and [D-Z, Remark 7.8], $\hat{P}$ and $\overline{P}$ have unimodular elements. If $n = 3$, by [D-Z, Theorem 7.4], $\hat{P}$ and $\overline{P}$ have unimodular elements. Since $\hat{P}$ has a unimodular element, we can find a monic polynomial $f \in R[X]$ such that P_f contains a unimodular element. But then by Theorem 3.1, P has a unimodular element. $\square$

Remark 3.5. Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$ and P be a projective $R[X_1, \ldots, X_r]$-module $(r \geq 3)$ of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^r(P)$ be an isomorphism. By induction on r, we can define the Euler class group of $R[X_1, \ldots, X_r]$ with respect to the line bundle L, denoted by $E(R[X_1, \ldots, X_r], L)$, as the product of $E(R(X_r)[X_1, \ldots, X_{r-1}], \hat{L})$ and $E(R[X_1, \ldots, X_{r-1}], \overline{L})$.

To the pair (P, χ), we can associate an invariant $e(P, \chi)$ in $E(R[X_1, \ldots, X_r], L)$ as follows:

$$e(P, \chi) = (e(\hat{P}, \hat{\chi}), e(\overline{P}, \overline{\chi}))$$

where $e(\hat{P}, \hat{\chi}) \in E(R(X_r)[X_1, \ldots, X_{r-1}], \hat{L})$ is the Euler class of $(\hat{P}, \hat{\chi})$ and $e(\overline{P}, \overline{\chi}) \in E(R[X_1, \ldots, X_{r-1}], \overline{L})$ is the Euler class of $(\overline{P}, \overline{\chi})$. Finally we have the following result.

Theorem 3.6. *Let R be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$ and P be a projective $R[X_1, \ldots, X_r]$-module of rank n with determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Then $e(P, \chi) = 0$ in $E(R[X_1, \ldots, X_r], L)$ if and only if P has a unimodular element.*

4. Analogue of Roy and Mandal

In this section we will prove (1.4). We begin with the following result from [Ry, Lemma 2.1].

Lemma 4.1. *Let R be a ring and P, Q be two projective R-modules. Suppose that $\phi : Q \longrightarrow P$ is an R-linear map. For an ideal I of R, if ϕ is a split monomorphism modulo I, then $\phi_{1+I} : Q_{1+I} \longrightarrow P_{1+I}$ is also a split monomorphism.*

Lemma 4.2. *Let $(R, \mathcal{M})$ be a local ring and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$. Let P and Q be two projective A-modules and $\phi : Q \longrightarrow P$ be an R-linear map. If ϕ is a split monomorphism modulo $\mathcal{M}$ and if ϕ_f is a split monomorphism for some special monic polynomial $f \in R[X]$, then ϕ is also a split monomorphism.*

Proof. By Lemma 4.1 $\phi_{1+\mathcal{M}A}$ is a split monomorphism. So, there is an element h in $1 + \mathcal{M}A$ such that ϕ_h is a split monomorphism. Since f is a special monic polynomial, $R \hookrightarrow A/f$ is an integral extension and hence, h and f are comaximal. As ϕ_f is also a split monomorphism, it follows that ϕ is a split monomorphism. $\square$

Lemma 4.3. *Let R be a local ring and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$. Let P and Q be two projective A-modules and $\phi, \psi : Q \longrightarrow P$ be R-linear maps. Further assume that $\gamma : P \longrightarrow Q$ is a A-linear map such that $\gamma\psi = f1_Q$ for some special monic polynomial $f \in R[X]$. For large m, there exists a special monic polynomial $g_m \in A$ such that $X\phi + (1 + X^m)\psi$ becomes a split monomorphism after inverting g_m.*

Proof. As in [Ry, M], first we assume that Q is free. We have $\gamma(X\phi + (1 + X^m)\psi) = X\gamma\phi + (1 + X^m)f1_Q$. Since Q is free, $X\gamma\phi + (1 + X^m)f1_Q$ is a matrix. Clearly for large integer m, $det(X\gamma\phi + (1 + X^m)f1_Q)$ is a special monic polynomial which can be taken for g_m.

In the general case, find projective A-module Q' such that $Q \oplus Q'$ is free. Define maps $\phi', \psi' : Q \oplus Q' \longrightarrow P \oplus Q'$ and $\gamma' : P \oplus Q' \longrightarrow Q \oplus Q'$ as $\phi' = \phi \oplus 0$, $\psi' = \psi \oplus f1_{Q'}$

and $\gamma' = \gamma \oplus 1_{Q'}$. By the previous case, we can find a special monic polynomial g_m for some large m such that $(X\phi' + (1+X^m)\psi')_{g_m}$ becomes a split monomorphism. Hence $X\phi + (1+X^m)\psi$ becomes a split monomorphism after inverting g_m. $\square$

The following result generalizes Mandal's [M].

Theorem 4.4. *Let $(R, \mathcal{M})$ be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P and Q be two projective A-modules with rank$(Q) <$ rank(P). If Q_f is a direct summand of P_f for some special monic polynomial $f \in R[X]$, then Q is also a direct summand of P.*

Proof. The method of proof is similar to [Ry, Theorem 1.1], hence we give an outline of the proof.

Since Q_f is a direct summand of P_f, we can find A-linear maps $\psi : Q \longrightarrow P$ and $\gamma : P \longrightarrow Q$ such that $\gamma\psi = f1_Q$ (possibly after replacing f by a power of f).

Let 'bar' denote reduction modulo $\mathcal{M}$. Then we have $\bar{\gamma}\bar{\psi} = \bar{f}1_{\bar{Q}}$. As f is special monic, $\bar{\psi}$ is a monomorphism.

We may assume that $A = R[X, f_1/X^t, \ldots, f_n/X^t]$ with $f_i \in R[X]$. If $f_i \in \mathcal{M}R[X]$, then $\overline{R[X, f_i/X^t]} = \overline{R}[X, Y]/(X^tY)$. If $f_i \in R[X] - \mathcal{M}R[X]$, then $\overline{R[X, f_i/X^t]}$ is either $\overline{R}[X]$ or $\overline{R}[X, X^{-1}]$ depending on whether $\overline{f}_i/X^t$ is a polynomial in $\overline{R}[X]$ or $\overline{F}_i/X^s$ with $\overline{F}_i(0) \neq 0$ and $s > 0$.

In general, $\overline{A}$ is one of $\overline{R}[X]$, $\overline{R}[X, X^{-1}]$ or $\overline{R}[X, Y_1, \ldots, Y_m]/(X^t(Y_1, \ldots, Y_m))$ for some $m > 0$. By [V, Theorem 3.2], any projective $\overline{R}[X, Y_1, \ldots, Y_m]/(X^t(Y_1, \ldots, Y_m))$-module is free. Therefore, in all cases, projective $\overline{A}$-modules are free and hence extended from $\overline{R}[X]$. In particular, $\bar{P}$ and $\bar{Q}$ are extended from $\overline{R}[X]$, which is a PID.

Let rank$(P) = r$ and rank$(Q) = s$. Therefore, using elementary divisors theorem, we can find bases $\{\bar{p}_1, \cdots, \bar{p}_r\}$ and $\{\bar{q}_1, \cdots, \bar{q}_s\}$ for $\bar{P}$ and $\bar{Q}$, respectively, such that $\bar{\psi}(\bar{q}_i) = \bar{f}_i\bar{p}_i$ for some $f_i \in R[X]$ and $1 \leq i \leq s$.

For the rest of the proof, we can follow the proof of [Ry, Theorem 1.1]. $\square$

Now we have the following consequence of (4.4).

Corollary 4.5. *Let R be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P, Q be two projective A-modules such that P_f is isomorphic to Q_f for some special monic polynomial $f \in R[X]$. Then,*

1. *Q is a direct summand of $P \oplus L$ for any projective A-module L.*
2. *P is isomorphic to Q if P or Q has a direct summand of rank one.*
3. *$P \oplus L$ is isomorphic to $Q \oplus L$ for all rank one projective A-modules L.*
4. *P and Q have same number of generators.*

Proof. (1) trivially follows from Theorem 4.4 and (3) follows from (2).

The proof of (4) is same as [Ry, Proposition 3.1 (4)].

For (2), we can follow the proof of [M, Theorem 2.2 (ii)] by replacing doubly monic polynomial by special monic polynomial in his arguments. □

Corollary 4.6. *Let R be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let P be a projective A-module such that P_f is free for some special monic polynomial $f \in R[X]$. Then P is free.*

Proof. Follows from second part of (4.5). □

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