Another definition of Euler class group of a Noetherian ring

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1 Introduction

All the rings are assumed to be commutative Noetherian and all the modules are assumed to be finitely generated.

Let A be a ring of dimension $n \geq 2$ and let L be a projective A-module of rank 1. In [2], Bhatwadekar and Raja Sridharan defined the Euler class group of A with respect to L, denoted by $E(A, L)$. To the pair (P, χ), where P is a projective A-module of rank n with determinant L and $\chi : L \xrightarrow{\sim} \wedge^n P$ is a L-orientation of P, they attached an element of the Euler class group, denoted by $e(P, \chi)$. One of the main result in [2] is that P has a unimodular element if and only if $e(P, \chi)$ is zero in $E(A, L)$.

We will define the Euler class group of A with respect to a projective A-module $F = Q \oplus A$ of rank n, denoted by $E(A, F)$. To the pair (P, χ), where P is a projective A-module of rank n and $\chi : \wedge^n F \xrightarrow{\sim} \wedge^n P$ is a F-orientation of P, we associate an element of the Euler class group, denoted by $e(P, \chi)$ and prove the following result: P has a unimodular element if and only if $e(P, \chi)$ is zero in $E(A, F)$. Note that when $F = L \oplus A^{n-1}$, $E(A, F)$ is same as the Euler class group $E(A, L)$ defined in [2].

2 Preliminaries

Let A be a ring and let M be an A-module. For $m \in M$, we define $O_M(m) = \{\varphi(m) | \varphi \in \mathrm{Hom}_A(M, A)\}$. We say that m is *unimodular* if $O_M(m) = A$. The set of all unimodular elements of M will be denoted by $\mathrm{Um}(M)$. Note that if P is a projective A-module and P has a unimodular element, then $P \xrightarrow{\sim} P_1 \oplus A$.

Let P be a projective A-module. Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p as the composite $P \xrightarrow{\varphi} A \xrightarrow{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and hence $1 + \varphi_p$ is a unipotent automorphism of P.

By a "transvection", we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. We denote by $EL(P)$ the subgroup of $\mathrm{Aut}(P)$ generated by all the transvections of P. Note that $EL(P)$ is a normal subgroup of $\mathrm{Aut}(P)$.

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Recall that if A is a ring of dimension n and if P is a projective A-module of rank n, then any surjection $\alpha : P \twoheadrightarrow J$ is called a *generic* surjection if J is an ideal of A of height n.

The following result is due to Bhatwadekar and Roy ([3], Proposition 4.1)

Proposition 2.1 *Let B be a ring and let I be an ideal of B. Let P be a projective B-module. Then any element of $EL(P/IP)$ can be lifted to an automorphism of P.*

We state some results from [2] for later use.

Lemma 2.2 *([2], Lemma 3.0) Let A be a ring of dimension n and let P be a projective A-module of rank n. Let $\lambda : P \twoheadrightarrow J_0$ and $\mu : P \twoheadrightarrow J_1$ be two surjections, where J_0 and J_1 are ideals of A of height n. Then there exists an ideal I of $A[T]$ of height n and a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J_0$, $I(1) = J_1$, $\alpha(0) = \lambda$ and $\alpha(1) = \mu$.*

For a rank 1 projective A-module L and $P' = L \oplus A^{n-1}$, the following result is proved in ([2], Proposition 3.1). Since the same proof works in our case, we omit the proof.

Proposition 2.3 *Let A be a ring of dimension $n \geq 2$ such that $(n-1)!$ is a unit in A. Let P and $P' = Q \oplus A$ be projective A-modules of rank n and let $\chi : \wedge^n P \xrightarrow{\sim} \wedge^n P'$ be an isomorphism. Suppose that $\alpha(T) : P[T] \twoheadrightarrow I$ be a surjection, where I is an ideal of $A[T]$ of height n. Then there exists a homomorphism $\phi : P' \to P$, an ideal K of A of height $\geq n$ which is comaximal with $(I \cap A)$ and a surjection $\rho(T) : P'[T] \twoheadrightarrow I \cap KA[T]$ such that the following holds:*
(i) $\wedge^n(\phi) = u\chi$, where $u = 1$ modulo $I \cap A$.
(ii) $(\alpha(0) \circ \phi)(P') = I(0) \cap K$.
(iii) $(\alpha(T) \circ \phi(T)) \otimes A[T]/I = \rho(T) \otimes A[T]/I$.
(iv) $\rho(0) \otimes A/K = \rho(1) \otimes A/K$.

The following result is labeled as addition principle ([2], Theorem 3.2).

Theorem 2.4 *Let A be a ring of dimension $n \geq 2$ and let J_1, J_2 be two comaximal ideals of A of height n. Let $P = P_1 \oplus A$ be a projective A-module of rank n and let $\Phi : P \twoheadrightarrow J_1$ and $\Psi : P \twoheadrightarrow J_2$ be two surjections. Then, there exists a surjection $\Theta : P \twoheadrightarrow J_1 \cap J_2$ such that $\Phi \otimes A/J_1 = \Theta \otimes A/J_1$ and $\Psi \otimes A/J_2 = \Theta \otimes A/J_2$.*

The following theorem is labeled as subtraction principle ([2], Theorem 3.3).

Theorem 2.5 *Let A be a ring of dimension $n \geq 2$ and let J and J' be two comaximal ideals of A of height $\geq n$ and n respectively. Let P and $P' = Q \oplus A$ be projective A-modules of rank n and let $\chi : \wedge^n P \xrightarrow{\sim} \wedge^n P'$ be an isomorphism. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : P' \twoheadrightarrow J'$ be surjections. Let "bar" denote reduction modulo J' and let $\overline{\alpha} : \overline{P} \twoheadrightarrow J'/J'^2$ and $\overline{\beta} : \overline{P'} \twoheadrightarrow J'/J'^2$ be surjections induced from α and β respectively. Suppose there exists an isomorphism $\delta : \overline{P} \xrightarrow{\sim} \overline{P'}$ such that $\overline{\beta}\delta = \overline{\alpha}$ and $\wedge^n(\delta) = \overline{\chi}$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

Lemma 2.6 *([2], Proposition 6.7) Let A be a ring of dimension n and let P, P' be stably isomorphic projective A-modules of rank n. Then there exists an ideal J of A of height $\geq n$ such that J is a surjective image of both P and P'. Further, given any ideal K of height ≥ 1, J can be chosen to be comaximal with K.*

We state the following result from ([1], Proposition 2.11) for later use.

Proposition 2.7 *Let A be a ring and let I be an ideal of A of height n. Let $f \in A$ be a non-zerodivisor modulo I and let $P = P_1 \oplus A$ be a projective A-module of rank n. Let $\alpha : P \to I$ be a linear map such that the induced map $\alpha_f : P_f \twoheadrightarrow I_f$ is a surjection. Then, there exists $\Psi \in EL(P_f^*)$ such that*

(i) $\beta = \Psi(\alpha) \in P^*$ and

(ii) $\beta(P)$ is an ideal of A of height n contained in I.

3 Euler class group $E(A, F)$

Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A$ be a projective A-module of rank n. We define the Euler class group of A with respect to F as follows:

Let J be an ideal of A of height n such that J/J^2 is generated by n elements. Let α and β be two surjections from F/JF to J/J^2. We say that α and β are *related* if there exists an automorphism σ of F/JF of determinant 1 such that $\alpha\sigma = \beta$. Clearly, this is an equivalence relation on the set of all surjections from F/JF to J/J^2. Let $[\alpha]$ denote the equivalence class of α. We call $[\alpha]$ a *local F-orientation* of J.

Since $\dim A/J = 0$, $\mathrm{SL}_{A/J}(F/JF) = EL(F/JF)$ and therefore, by (2.1), the canonical map from $\mathrm{SL}_A(F)$ to $\mathrm{SL}_{A/J}(F/JF)$ is surjective. Hence, if a surjection $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\Delta : F \twoheadrightarrow J$, then so can any other surjection β equivalent to α.

A local F-orientation $[\alpha]$ is called a *global F-orientation* of J if the surjection α can be lifted to a surjection from F to J. From now on, we shall identify a surjection α with the equivalence class $[\alpha]$ to which α belongs.

Let $\mathcal{M}$ be a maximal ideal of A of height n and let $\mathcal{N}$ be a $\mathcal{M}$-primary ideal such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let $w_{\mathcal{N}}$ be a local F-orientation of $\mathcal{N}$. Let G be the free abelian group on the set of pairs $(\mathcal{N}, w_{\mathcal{N}})$, where $\mathcal{N}$ is a $\mathcal{M}$-primary ideal and $w_{\mathcal{N}}$ is a local F-orientation of $\mathcal{N}$.

Let $J = \cap \mathcal{N}_i$ be the intersection of finitely many $\mathcal{M}_i$-primary ideals, where $\mathcal{M}_i$ are distinct maximal ideals of A of height n. Assume that J/J^2 is generated by n elements and let w_J be a local F-orientation of J. Then w_J gives rise, in a natural way, to local F-orientations $w_{\mathcal{N}_i}$ of $\mathcal{N}_i$. We associate to the pair (J, w_J), the element $\sum(\mathcal{N}_i, w_{\mathcal{N}_i})$ of G.

Let H be the subgroup of G generated by the set of pairs (J, w_J), where J is an ideal of A of height n and w_J is a global F-orientation of J.

We define the *Euler class group* of A with respect to F, denoted by $E(A, F)$, as the quotient group G/H.

Let A be a ring of dimension n. Let P and $F = Q \oplus A$ be projective A-modules of rank n and let $\chi : \wedge^n F \stackrel{\sim}{\to} \wedge^n P$ be an isomorphism. We call χ a *F-orientation* of P. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(A, F)$ as follows:

Let $\lambda : P \twoheadrightarrow J_0$ be a generic surjection and let "bar" denote reduction modulo the ideal J_0. Then, we obtain an induced surjection $\overline{\lambda} : \overline{P} \twoheadrightarrow J_0/J_0^2$. Since $\dim A/J_0 = 0$, every projective A/J_0-module of constant rank is free. Hence, we choose an isomorphism $\overline{\gamma} : F/J_0F \stackrel{\sim}{\to} P/J_0P$ such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let w_{J_0} be the local F-orientation of J_0 given by $\overline{\lambda} \circ \overline{\gamma} : F/J_0F \twoheadrightarrow J_0/J_0^2$. Let $e(P, \chi)$ be the image in $E(A, F)$ of the element (J_0, w_{J_0}) of G. We say that (J_0, w_{J_0}) is obtained from the pair (λ, χ). We will show that the assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A, F)$ is well defined.

Let $\mu : P \twoheadrightarrow J_1$ be another generic surjection. By (2.2), there exists a surjection $\alpha(T) : P[T] \twoheadrightarrow I$, where I is an ideal of $A[T]$ of height n with $\alpha(0) = \lambda$, $I(0) = J_0$, $\alpha(1) = \mu$ and $I(1) = J_1$. Using (2.3), we get an ideal K of A of height n and a local F-orientation w_K of K such that $(I(0), w_{I(0)}) + (K, w_K) = 0 = (I(1), w_{I(1)}) + (K, w_K)$ in $E(A, F)$. Therefore $(J_0, w_{J_0}) = (J_1, w_{J_1})$ in $E(A, F)$. Hence $e(P, \chi)$ is well defined in $E(A, F)$.

We define the *Euler class* of (P, χ) to be $e(P, \chi)$.

For a projective A-module L of rank 1 and $F = L \oplus A^{n-1}$, the following result is proved in ([2], Proposition 4.1). Since the same proof works in our case, we omit the proof.

Proposition 3.1 *Let A be a ring of dimension $n \geq 2$ and let J, J_1, J_2 be ideals of A of height n such that J is comaximal with J_1 and J_2. Let $F = Q \oplus A$ be a projective A-module of rank n. Assume that $\alpha : F \twoheadrightarrow J \cap J_1$ and $\beta : F \twoheadrightarrow J \cap J_2$ be surjections with $\alpha \otimes A/J = \beta \otimes A/J$. Suppose there exists an ideal J_3 of height n such that*

(i) J_3 is comaximal with J, J_1 and J_2 and

(ii) there exists a surjection $\gamma : F \twoheadrightarrow J_3 \cap J_1$ with $\alpha \otimes A/J_1 = \gamma \otimes A/J_1$.

Then there exists a surjection $\lambda : F \twoheadrightarrow J_3 \cap J_2$ with $\lambda \otimes A/J_3 = \gamma \otimes A/J_3$ and $\lambda \otimes A/J_2 = \beta \otimes A/J_2$.

Using (3.1, 2.4 and 2.5) and following the proof of ([2], Theorem 4.2), the next result follows.

Theorem 3.2 *Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A$ be a projective A-module of rank n. Let J be an ideal of A of height n such that J/J^2 is generated by n elements. Let $w_J : F/JF \twoheadrightarrow J/J^2$ be a local F-orientation of J. Suppose that the image of (J, w_J) is zero in $E(A, F)$. Then w_J is a global F-orientation of J.*

Using (3.2 and 2.5) and following the proof of ([2], Corollary 4.3), the next result follows.

Corollary 3.3 *Let A be a ring of dimension $n \geq 2$. Let P and $F = Q \oplus A$ be projective A-modules of rank n and let $\chi : \wedge^n F \stackrel{\sim}{\to} \wedge^n P$ be a F-orientation of P. Let J be an ideal of A of height n such that J/J^2 is generated by n elements and let w_J be a local F-orientation of J. Suppose $e(P, \chi) = (J, w_J)$ in $E(A, F)$. Then there exists a surjection $\alpha : P \twoheadrightarrow J$ such that (J, w_J) is obtained from (α, χ).*

Using (3.2, 3.3) and following the proof of ([2], Theorem 4.4), the next result follows.

Corollary 3.4 *Let A be a ring of dimension $n \geq 2$. Let P and $F = Q \oplus A$ be projective A-modules of rank n and let $\chi : \wedge^n F \xrightarrow{\sim} \wedge^n P$ be a F-orientation of P. Then $e(P, \chi) = 0$ in $E(A, F)$ if and only if P has a unimodular element.*

Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A$ be a projective A-module of rank n. Let "bar" denote reduction modulo the nil radical N of A and let $\overline{A} = A/N$ and $\overline{F} = F/NF$. Let J be an ideal of A of height n with primary decomposition $J = \cap \mathcal{N}_i$. Then $\overline{J} = (J+N)/N$ is an ideal of $\overline{A}$ of height n with primary decomposition $\overline{J} = \cap \overline{\mathcal{N}}_i$. Moreover, any surjection $w_J : F/JF \twoheadrightarrow J/J^2$ induces a surjection $\overline{w_{\overline{J}}} : \overline{F}/\overline{JF} \twoheadrightarrow \overline{J}/\overline{J^2} = (J+N)/(J^2+N)$. Hence, the assignment sending (J, w_J) to $(\overline{J}, \overline{w_{\overline{J}}})$ gives rise to a group homomorphism $\Phi : E(A, F) \twoheadrightarrow E(\overline{A}, \overline{F})$.

As a consequence of (3.2), we get the following result, the proof of which is same as of ([2], Corollary 4.6).

Corollary 3.5 *The homomorphism $\Phi : E(A, F) \to E(\overline{A}, \overline{F})$ is an isomorphism.*

4 Some results on $E(A, F)$

Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A$ be a projective A-module of rank n. Let J be an ideal of A of height n and let $w_J : F/JF \twoheadrightarrow J/J^2$ be a surjection. Let $\overline{b} \in A/J$ be a unit. Then composing w_J with an automorphism of F/JF with determinant $\overline{b}$, we get another local F-orientation of J, which we denote by $\overline{b} w_J$. Further, if w_J and $\widetilde{w}_J$ are two local F-orientations of J, then it is easy to see that $\widetilde{w}_J = \overline{b} w_J$ for some unit $\overline{b} \in A/J$.

We recall the following two results from ([2], Lemma 2.7 and 2.8) respectively.

Lemma 4.1 *Let A be a ring and let P be a projective A-module of rank n. Assume $0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\to} A \to 0$ is an exact sequence. Let $(a_0, p_0) \in A \oplus P$ be such that $a_0 b - \alpha(p_0) = 1$. Let $q_i = (a_i, p_i) \in P_1$ for $i = 1, \ldots, n$. Then*

(i) the map $\delta : \wedge^n P_1 \to \wedge^n P$ given by $\delta(q_1 \wedge \ldots \wedge q_n) = a_0(p_1 \wedge \ldots \wedge p_n) + \sum_1^n (-1)^i a_i (p_0 \wedge \ldots p_{i-1} \wedge p_{i+1} \ldots \wedge p_n)$ is an isomorphism.

(ii) $\delta(bq_1 \wedge \ldots \wedge q_n) = p_1 \wedge \ldots \wedge p_n$.

Lemma 4.2 *Let A be a ring and let P be a projective A-module of rank n. Assume $0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\to} A \to 0$ is an exact sequence. Then*

(i) The map $\beta : P_1 \to A$ given by $\beta(q) = c$, where $q = (c, p)$, has the property that $\beta(P_1) = \alpha(P)$.

(ii) The map $\Phi : P \to P_1$ given by $\Phi(p) = (\alpha(p), bp)$ has the property that $\beta \circ \Phi = \alpha$ and $\delta \circ \wedge^n \Phi$ is a scalar multiplication by b^{n-1}, where δ is as in (4.1).

The following result can be deduced from (4.1, 4.2). Briefly it says that if there exists a projective A-module P of rank n with a F-orientation $\chi : \wedge^n F \xrightarrow{\sim} \wedge^n P$ such that $e(P, \chi) = (J, w_J)$

and if $\overline{a} \in A/J$ is a unit, then there exists another projective A-module P_1 with $[P_1] = [P]$ in $K_0(A)$ and a F-orientation $\chi_1 : \wedge^n F \xrightarrow{\sim} \wedge^n P_1$ of P_1 such that $e(P_1, \chi_1) = (J, \overline{a^{n-1}w_J})$.

Lemma 4.3 *Let A be a ring of dimension $n \geq 2$. Let P and $F = Q \oplus A$ be projective A-modules of rank n and let $\chi : \wedge^n F \xrightarrow{\sim} \wedge^n P$ be a F-orientation of P. Let $\alpha : P \twoheadrightarrow J$ be a generic surjection and let (J, w_J) be obtained from (α, χ). Let $a, b \in A$ with $ab = 1$ modulo J and let P_1 be the kernel of the surjection $(b, -\alpha) : A \oplus P \twoheadrightarrow A$. Let $\beta : P_1 \twoheadrightarrow J$ be as in (4.2) and let χ_1 be the F-orientation of P_1 given by $\delta^{-1}\chi$, where δ is as in (4.1). Then $(J, \overline{a^{n-1}})$ is obtained from (β, χ_1).*

Using the above results and following the proof of ([2], Lemmas 5.3, 5.4 and 5.5) respectively, the next three results follows.

Lemma 4.4 *Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A^2$ be projective A-module of rank n. Let J be an ideal of A of height n and let $w_J : F/JF \twoheadrightarrow J/J^2$ be a surjection. Suppose w_J can be lifted to a surjection $\alpha : F \twoheadrightarrow J$. Let $\overline{a} \in A/J$ be a unit and let θ be an automorphism of F/JF with determinant $\overline{a^2}$. Then the surjection $w_J \circ \theta : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\gamma : F \twoheadrightarrow J$.*

Lemma 4.5 *Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A^2$ be a projective A-module of rank n. Let J be an ideal of A of height n and let w_J be a local F-orientation of J. Let $\overline{a} \in A/J$ be a unit. Then $(J, w_J) = (J, \overline{a^2} w_J)$ in $E(A, F)$.*

Lemma 4.6 *Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A$ be a projective A-module of rank n. Let J be an ideal of A of height n and let w_J be a local F-orientation of J. Suppose $(J, w_J) \neq 0$ in $E(A, F)$. Then there exists an ideal J_1 of height n which is comaximal with J and a local F-orientation w_{J_1} of J_1 such that $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A, F)$. Further, given any ideal K of A of height ≥ 1, J_1 can be chosen to be comaximal with K.*

The following result is similar to ([2], Lemma 5.6).

Lemma 4.7 *Let A be an affine domain of dimension $n \geq 2$ over a field k and let f be a non-zero element of A. Let $F = Q \oplus A^2$ be a projective A-module of rank n and let J be an ideal of A of height n such that J/J^2 is generated by n elements. Suppose that $(J, w_J) \neq 0$ in $E(A, F)$ but the image of (J, w_J) is zero in $E(A_f, F_f)$. Then there exists an ideal J_2 of A of height n such that $(J_2)_f = A_f$ and $(J, w_J) = (J_2, w_{J_2})$ in $E(A, F)$.*

Proof Since $(J, w_J) \neq 0$ in $E(A, F)$, but its image is zero in $E(A_f, F_f)$, we see that f is not a unit in A. By (4.6), we can choose an ideal J_1 of height n which is comaximal with Jf such that $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A, F)$. Since the image of (J, w_J) is zero in $E(A_f, F_f)$, it follows that the image of (J_1, w_{J_1}) is zero in $E(A_f, F_f)$.

By (3.2), there exists a surjection $\alpha : F_f \twoheadrightarrow (J_1)_f$ such that $\alpha \otimes A_f/(J_1)_f = (w_{J_1})_f$. Choose a positive integer k such that $f^{2k}\alpha : F \to J_1$. Since f is a unit modulo J_1, by (4.5), $(J_1, w_{J_1}) =$

$(J_1, \overline{f^{2kn}} w_{J_1})$ in $E(A,F)$. By (2.7), there exists $\Psi \in EL(F_f^*)$ such that $\beta = \Psi(\alpha) \in F^*$ and $\beta(F) \subset J_1$ is an ideal of height n. Thus $\beta(F) = J_1 \cap J_2$, where J_2 is an ideal of A of height n such that $(J_2)_f = A_f$. Hence $J_1 + J_2 = A$. From the surjection β, we get $(J_1, w_{J_1}) + (J_2, w_{J_2}) = 0$ in $E(A,F)$. Since $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A,F)$, it follows that $(J, w_J) = (J_2, w_{J_2})$ in $E(A,F)$. This proves the result. $\square$

Using (3.3, 4.5 and 4.7) and following the proof of ([2], Lemma 5.8), the following result can be proved.

Lemma 4.8 *Let A be an affine domain of dimension $n \geq 2$ over a field k. Let P and $F = Q \oplus A^2$ be projective A-modules of rank n with $\wedge^n P \xrightarrow{\sim} \wedge^n F$. Let f be a non-zero element of A. Assume that every generic surjection ideal of P is surjective image of F. Then every generic surjection ideal of P_f is surjective image of F_f.*

Using above results and following the proof of ([2], Theorem 5.9), the next result follows.

Theorem 4.9 *Let A be an affine domain of dimension $n \geq 2$ over a real closed field k. Let P and $F = Q \oplus A^2$ be projective A-modules of rank n with $\wedge^n P \xrightarrow{\sim} \wedge^n F$. Assume that every generic surjection ideal of P is surjective image of F. Then P has a unimodular element.*

In particular, if $L = \wedge^n P$ and every generic surjection ideal of P is surjective image of $L \oplus A^{n-1}$, then P has a unimodular element.

5 Weak Euler Class Group

Let A be a ring of dimension $n \geq 2$ and let $F = Q \oplus A$ be a projective A-module of rank n. We define the weak Euler class group $E_0(A,F)$ of A with respect to F as follows:

Let $\mathcal{S}$ be the set of ideals $\mathcal{N}$ of A such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements, where $\mathcal{N}$ is $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of A of height n. Let G be the free abelian group on the set $\mathcal{S}$.

Let $J = \cap \mathcal{N}_i$ be the intersection of finitely many ideals $\mathcal{N}_i$, where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary and $\mathcal{M}_i$'s are distinct maximal ideals of A of height n. Assume that J/J^2 is generated by n elements. We associate to J, the element $\sum \mathcal{N}_i$ of G. We denote this element by (J).

Let H be the subgroup of G generated by elements of the type (J), where J is an ideal of A of height n which is surjective image of F.

We set $E_0(A,F) = G/H$.

Let P be a projective A-module of rank n such that $\wedge^n P \xrightarrow{\sim} \wedge^n F$. Let $\lambda : P \twoheadrightarrow J_0$ be a generic surjection. We define $e(P) = (J_0)$ in $E_0(A,F)$. We will show that this assignment is well defined.

Let $\mu : P \twoheadrightarrow J_1$ be another generic surjection. By (2.2), there exists a surjection $\alpha(T) : P[T] \rightarrow\rightarrow I$, where I is an ideal of $A[T]$ of height n with $\alpha(0) = \lambda$, $I(0) = J_0$, $\alpha(1) = \mu$ and $I(1) = J_1$. Now, as before, using (2.3), we see that $(J_0) = (J_1)$ in $E_0(A,F)$. This shows that $e(P)$ is well defined.

Note that there is a canonical surjection from $E(A,F)$ to $E_0(A,F)$ obtained by forgetting the orientations.

We state the following result which follows from (4.3 and 4.5).

Lemma 5.1 *Let A be a ring of even dimension n. Let P and $F = Q \oplus A^2$ be projective A-modules of rank n and let $\chi : \wedge^n F \xrightarrow{\sim} \wedge^n P$ be a F-orientation of P. Let $e(P,\chi) = (J, w_J)$ in $E(A,F)$ and let $\widetilde{w}_J$ be another local F-orientation of J. Then there exists a projective A-module P_1 with $[P_1] = [P]$ in $K_0(A)$ and a F-orientation χ_1 of P_1 such that $e(P_1, \chi_1) = (J, \widetilde{w}_J)$ in $E(A,F)$.*

Proposition 5.2 *Let A be a ring of even dimension n and let $F = Q \oplus A^2$ be a projective A-module of rank n. Let J_1 and J_2 be two comaximal ideals of A of height n and let $J_3 = J_1 \cap J_2$. If any two of J_1, J_2 and J_3 are surjective images of projective A-modules of rank n which are stably isomorphic to F, then so is the third one.*

Proof Let P_1 and P_2 be two projective A-modules of rank n with $[P_1] = [P_2] = [F]$ in $K_0(A)$ and let $\psi_1 : P_1 \twoheadrightarrow J_1$ and $\psi_2 : P_2 \twoheadrightarrow J_2$ be two surjections. Choose F-orientations χ_1 and χ_2 of P_1 and P_2 respectively. Then $e(P_1, \chi_1) = (J_1, w_{J_1})$ and $e(P_2, \chi_2) = (J_2, w_{J_2})$ in $E(A,F)$.

By (2.6), there exists an ideal J_1' of height n which is surjective image of both P_1 and F. Hence $e(P_1, \chi_1) = (J_1, w_{J_1}) = (J_1', w_{J_1'})$ in $E(A,F)$ for some local F-orientation $w_{J_1'}$ of J_1'. Similarly, there exists an ideal J_2' of height n which is surjective image of both P_2 and F. Hence $e(P_2, \chi_2) = (J_2, w_{J_2}) = (J_2', w_{J_2'})$ in $E(A,F)$ for some local F-orientation $w_{J_2'}$ of J_2'. Further, we may assume that $J_1' + J_2' = A$. Let $(J_1, w_{J_1}) + (J_2, w_{J_2}) = (J_3, w_{J_3})$ in $E(A,F)$.

Let $J_3' = J_1' \cap J_2'$. By addition principle (2.4), J_3' is surjective image of F and $(J_1', w_{J_1'}) + (J_2', w_{J_2'}) = (J_3', w_{J_3'})$ in $E(A,F)$. Hence $(J_3', w_{J_3'}) = (J_3, w_{J_3})$. Since J_3' is surjective image of F, by (5.1), there exists a projective A-module P_3 with $[P_3] = [F]$ in $K_0(A)$ and a F-orientation χ_3 of P_3 such that $e(P_3, \chi_3) = (J_3', w_{J_3'}) = (J_3, w_{J_3})$ in $E(A,F)$. By (3.3), there exists a surjection $\psi_3 : P_3 \twoheadrightarrow J_3$ such that (ψ_3, χ_3) induces (J_3, w_{J_3}). This proves the first part.

Now assume that J_1 and J_3 are surjective images of P_3 and P_1' respectively, where P_3 and P_1' are projective A-modules of rank n with $[P_3] = [P_1'] = [F]$ in $K_0(A)$.

Let $e(P_3, \chi_3) = (J_3, w_3)$ for some F-orientation χ_3 of P_3 and let $(J_3, w_3) = (J_1, w_1) + (J_2, w_2)$ in $E(A,F)$. Let $e(P_1', \chi_1') = (J_1, w_1')$ for some F-orientation χ_1' of P_1'. By (5.1), there exists a projective A-module P_1 of rank n with $[P_1] = [P_1']$ in $K_0(A)$ and a F-orientation χ_1 of P_1 such that $e(P_1, \chi_1) = (J_1, w_1)$ in $E(A,F)$.

By (2.6), there exists an ideal J_4 of height n which is surjective image of F and P_1 both and is comaximal with J_2 such that $e(P_1, \chi_1) = (J_1, w_1) = (J_4, w_4)$. Write $J_5 = J_4 \cap J_2$. Then $(J_4, w_4) + (J_2, w_2) = (J_5, w_5)$. This shows that $e(P_3, \chi_3) = (J_3, w_3) = (J_5, w_5)$.

Let $e(F, \chi) = (J_4, \widetilde{w}_4) = 0$. If $(J_4, \widetilde{w}_4) + (J_2, w_2) = (J_5, \widetilde{w}_5)$, then $(J_2, w_2) = (J_5, \widetilde{w}_5)$. Since $e(P_3, \chi_3) = (J_5, w_5)$, by (5.1), there exists a projective A-module $\widetilde{P}_3$ of rank n with $[\widetilde{P}_3] = [P_3]$ in $K_0(A)$ such that $e(\widetilde{P}_3, \widetilde{w}_3) = (J_5, \widetilde{w}_5) = (J_2, w_2)$. Hence, by (3.3), J_2 is a surjective image of $\widetilde{P}_3$ which is stably isomorphic to F. This completes the proof. $\square$

Proposition 5.3 *Let A be a ring of even dimension n and let $F = Q \oplus A^2$ be a projective A-module of rank n. Let J be an ideal of A of height n. Then $(J) = 0$ in $E_0(A, F)$ if and only if J is a surjective image of a projective A-module of rank n which is stably isomorphic to F.*

Proof Let J_1 be an ideal of A of height n. Assume that J_1 is surjective image of a projective A-module of rank n which is stably isomorphic to F. Assume (J_1, w_{J_1}) is a non-zero element of $E(A, F)$. We will show that there exist height n ideals J_2 and J_3 with local F-orientations w_{J_2} and w_{J_3} respectively such that

(*i*) J_2, J_3 are comaximal with any given ideal of height ≥ 1,

(*ii*) $(J_1, w_{J_1}) = -(J_2, w_{J_2}) = (J_3, w_{J_3})$ in $E(A, F)$ and

(*iii*) J_2, J_3 are surjective images of projective A-modules of rank n which are stably isomorphic to F.

By (4.6), there exists an ideal J_2 of height n which is comaximal with J_1 and any given ideal of height ≥ 1 such that $(J_1, w_{J_1}) + (J_2, w_{J_2}) = 0$ in $E(A, F)$. By (3.2), $J_1 \cap J_2$ is surjective image of F. By (5.2), J_2 is a surjective image of a projective A-module of rank n which is stably isomorphic to F.

Repeating the above with (J_2, w_{J_2}), we get an ideal J_3 of height n which is comaximal with any given ideal of height ≥ 1 such that $(J_2, w_{J_2}) + (J_3, w_{J_3}) = 0$ in $E(A, F)$. Further, J_3 is a surjective image of a projective A-module of rank n which is stably isomorphic to F. Thus, we have $(J_1, w_{J_1}) = -(J_2, w_{J_2}) = (J_3, w_{J_3})$ in $E(A, F)$. This proves the above claim.

From the above discussion, we see that given any element h in kernel of the canonical map $\Phi : E(A, F) \twoheadrightarrow E_0(A, F)$, there exists an ideal $\widetilde{J}$ of height n such that $\widetilde{J}$ is surjective image of a projective A-module of rank n which is stably isomorphic to F and $h = (\widetilde{J}, w_{\widetilde{J}})$ in $E(A, F)$. Moreover, $\widetilde{J}$ can be chosen to be comaximal with any ideal of height ≥ 1.

Now assume $(J) = 0$ in $E_0(A, F)$. Choose some local F-orientation w_J of J. Then $(J, w_J) \in ker(\Phi)$. From previous paragraph, we get that there exists an ideal K of height n comaximal with J such that $-(J, w_J) = (K, w_K)$ in $E(A, F)$. Further, K is surjective image of a projective A-module which is stably isomorphic to F.

By (3.2), $J \cap K$ is surjective image of F. By (5.2), J is surjective image of a projective A-module of rank n which is stably isomorphic to F.

Conversely, assume that J is surjective image of a projective A-module P of rank n which is stably isomorphic to F. Let χ be a F-orientation of P. Then $e(P, \chi) = (J, w_J)$ in $E(A, F)$. By (2.6), there exists an ideal I of height n which is surjective image of P and F both. Then $e(P, \chi) = (J, w_J) = (I, w_I)$ in $E(A, F)$. Therefore $(J) = (I)$ in $E_0(A, F)$ and hence $(J) = 0$ in $E_0(A, F)$. This completes the proof. $\square$

Proposition 5.4 *Let A be a ring of even dimension n and let $F = Q \oplus A^2$ and P be projective A-modules of rank n with $\wedge^n P \xrightarrow{\sim} \wedge^n F$. Then $e(P) = 0$ in $E_0(A, F)$ if and only if $[P] = [P_1 \oplus A]$ in $K_0(A)$ for some projective A-module P_1 of rank $n - 1$.*

Proof Assume that $[P] = [P_1 \oplus A]$ in $K_0(A)$. By (2.6), there exists an ideal J of A of height n which is surjective image of both P and $P_1 \oplus A$. Hence $e(P_1 \oplus A, \chi) = (J, w_J) = 0$ in $E(A, F)$, by (3.4). Hence J is surjective image of F. By (5.3), $e(P) = (J) = 0$ in $E_0(A, F)$.

Conversely, assume that $e(P) = 0$ in $E_0(A, F)$. Let $\psi : P \twoheadrightarrow J$ be a generic surjection and let $e(P, \chi) = (J, w_J)$ in $E(A, F)$ for some F-orientation χ of P. Since $e(P) = (J) = 0$ in $E_0(A, F)$, by (5.3), J is surjective image of a projective A-module P_1 with $[P_1] = [F]$ in $K_0(A)$. By (2.6), there exists an height n ideal J_1 which is surjective image of P_1 and F both. Let $e(P_1, \chi_1) = (J, \widetilde{w}_J) = (J_1, w_{J_1})$ for some F-orientation χ_1 of P_1.

By (5.1), there exists a rank n projective A-module P_2 with $[P_2] = [P]$ in $K_0(A)$ and a F-orientation χ_2 of P_2 such that $e(P_2, \chi_2) = (J, \widetilde{w}_J) = (J_1, w_{J_1})$ in $E(A, F)$. Since J_1 is surjective image of F, $(J_1, \widetilde{w}_{J_1}) = 0$ in $E(A, F)$ for some local F-orientation $\widetilde{w}_{J_1}$ of J_1. By (5.1), there exists a projective A-module P_3 with $[P_3] = [P_2]$ in $K_0(A)$ and a F-orientation χ_3 of P_3 such that $e(P_3, \chi_3) = (J_1, \widetilde{w}_{J_1}) = 0$ in $E(A, F)$. Hence $P_3 = P_4 \oplus A$, by (3.4). Therefore $[P] = [P_2] = [P_4 \oplus A]$ in $K_0(A)$. This completes the proof. $\square$

Proposition 5.5 *Let A be a ring of even dimension n. Let P and $F = Q \oplus A^2$ be projective A-modules of rank n with $\wedge^n P \xrightarrow{\sim} \wedge^n F$. Suppose that $e(P) = (J)$ in $E_0(A, F)$, where J is an ideal of A of height n. Then there exists a projective A-module P_1 of rank n such that $[P] = [P_1]$ in $K_0(A)$ and J is a surjective image of P_1.*

Proof Since P/JP is free and J/J^2 is generated by n elements, we get a surjection $\overline{\psi} : P/JP \twoheadrightarrow J/J^2$. By ([2], Corollary 2.14), we can lift $\overline{\psi}$ to a surjection $\psi : P \twoheadrightarrow J \cap J_1$, where J_1 is an height n ideal comaximal with J. Let $e(P, \chi) = (J, w_J) + (J_1, w_{J_1})$ in $E(A, F)$.

Since $e(P) = (J) = (J \cap J_1)$ in $E_0(A, F)$, $(J_1) = 0$ in $E_0(A, F)$. By (5.3), J_1 is surjective image of a projective A-module P_2 of rank n which is stably isomorphic to F. By (5.1), there exists rank n projective A-module P_3 with $[P_2] = [P_3]$ in $K_0(A)$ and a F-orientation χ_3 of P_3 such that $e(P_3, \chi_3) = (J_1, w_{J_1})$ in $E(A, F)$.

By (2.6), there exists a height n ideal J_2 of A which is comaximal with J and is surjective image of F such that $e(P_3, \chi_3) = (J_1, w_{J_1}) = (J_2, w_{J_2})$ in $E(A, F)$. Hence $e(P, \chi) = (J, w_J) + (J_2, w_{J_2}) = (J \cap J_2, w_{J \cap J_2})$. By (3.3), there exists a surjection $\phi : P \twoheadrightarrow J \cap J_2$. Since $(J_2, \widetilde{w}_{J_2}) = 0$ for some local F-orientation $\widetilde{w}_{J_2}$ of J_2. Let $(J, w_J) + (J_2, \widetilde{w}_{J_2}) = (J \cap J_2, \widetilde{w}_{J \cap J_2})$. By (4.3), there exists rank n projective A-module P_1 with $[P] = [P_1]$ in $K_0(A)$ and $e(P_1, \chi_1) = (J \cap J_2, \widetilde{w}_{J \cap J_2}) = (J, w_J)$. By (3.3), there exists a surjection $\alpha : P_1 \twoheadrightarrow J$. This proves the result. $\square$

The proof of the following result is similar to ([2], Proposition 6.5), hence we omit it.

Proposition 5.6 *Let A be a ring of even dimension n and let J be an ideal of A of height n such that J/J^2 is generated by n elements. Let $F = Q \oplus A^2$ be a projective A-module of rank n and let $\widetilde{w}_J : F/JF \twoheadrightarrow J/J^2$ be a surjection. Suppose that the element $(J, \widetilde{w}_J)$ of $E(A, F)$ belongs to the kernel of the canonical homomorphism $E(A, F) \twoheadrightarrow E_0(A, F)$. Then there exists a projective*

A-module P_1 of rank n such that $[P_1] = [F]$ in $K_0(A)$ and $e(P_1, \chi_1) = (J, \widetilde{w}_J)$ in $E(A, F)$ for some F-orientation χ_1 of P_1.

6 Application

Let A be a ring of dimension $n \geq 2$ and let L be a projective A-module of rank 1. We will define a map Δ from $E(A, L)$ to $E(A, F)$, where $F = Q \oplus A$ is a projective A-module of rank n with determinant L. Let $w_J : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ be a surjection. Since $\dim A/J = 0$, Q/JQ is isomorphic to $L/JL \oplus (A/J)^{n-2}$. Choose an isomorphism $\theta : Q/JQ \xrightarrow{\sim} L/JL \oplus (A/J)^{n-2}$ of determinant one. Let $\widetilde{w}_J = w_J \circ (\theta, id) : Q/JQ \oplus A/J \twoheadrightarrow J/J^2$ be the surjection.

Assume that w_J can be lifted to a surjection $\Phi : L \oplus A^{n-1} \twoheadrightarrow J$. Write $\Phi = (\Phi_1, a)$. We may assume that $\Phi_1(L \oplus A^{n-2}) = K$ is an ideal of height $n-1$. Further, we may assume that the isomorphism $\theta : Q/JQ \xrightarrow{\sim} L/JL \oplus (A/J)^{n-2}$ is induced from an isomorphism $\theta' : Q/KQ \xrightarrow{\sim} L/KL \oplus (A/K)^{n-2}$ (i.e. $\theta' \otimes A/J = \theta$).

Let $(\Phi_2, a) : Q \oplus A \to J = (K, a)$ be a lift of $\widetilde{w}_J$. Then $\Phi_2 \otimes A/K : Q/KQ \twoheadrightarrow K/K^2$ is a surjection. Let $\phi_2 : Q \to K$ be a lift of $\Phi_2 \otimes A/K$. Then $\phi_2(Q) + K^2 = K$. Hence, there exists $e \in K^2$ with $e(1-e) \in \phi_2(Q)$ such that $\phi_2(Q) + Ae = K$. Now it is easy to check that $\phi_2(Q) + Aa = \phi_2(Q) + (e + (1-e)a) = K + Aa = J$ and $(\phi_2, e + (1-e)a) : Q \oplus A \twoheadrightarrow J$ is a lift of $\widetilde{w}_J$.

Hence, we have shown that if w_J can be lifted to a surjection from $L \oplus A^{n-1} \twoheadrightarrow J$, then $\widetilde{w}_J$ can be lifted to a surjection from $Q \oplus A$ to J. Further, if we choose different isomorphism $\theta_1 : Q/JQ \oplus A/J \xrightarrow{\sim} L/JL \oplus (A/J)^{n-1}$ of determinant one and $w_1 = w_J \circ \theta_1 : Q/JQ \oplus A/J \to\to J/J^2$, then $\widetilde{w}_J$ and w_1 are connected by an element of $EL(Q/JQ \oplus A/J)$. Hence, if we define $\Delta : E(A, L) \to E(A, F)$ by $\Delta(w_J) = \widetilde{w}_J$, then this map is well defined. It is easy to see that Δ is a group homomorphism.

Similarly, we can define a map $\Delta_1 : E(A, F) \to E(A, L)$ and it is easy to show that $\Delta \circ \Delta_1 = id$ and $\Delta_1 \circ \Delta = id$. Hence, we get the following interesting result:

Theorem 6.1 *Let A be a ring of dimension $n \geq 2$. Let L and $F = Q \oplus A$ be projective A-modules of rank 1 and n respectively with $\wedge^n F \xrightarrow{\sim} L$. Then $E(A, L)$ is isomorphic to $E(A, F)$.*

Let J be an ideal of A of height n such that J/J^2 is generated by n elements. Further assume that there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$. We will show that J is also a surjective image of $F = Q \oplus A$. Let w_J be the local L-orientation of J induced from α. Then $(J, w_J) = 0$ in $E(A, L)$. Hence $\Delta(J, w_J) = (J, \widetilde{w}_J) = 0$ in $E(A, F)$. Hence, by (3.2), J is a surjective image of F.

We define the map $\widetilde{\Delta} : E_0(A, L) \to E_0(A, F)$ by $(J) \mapsto (J)$. The above discussion shows that $\widetilde{\Delta}$ is well defined. Similarly, we can define a map $\widetilde{\Delta}_1 : E_0(A, F) \to E_0(A, L)$ such that $\widetilde{\Delta} \circ \widetilde{\Delta}_1 = id$ and $\widetilde{\Delta}_1 \circ \widetilde{\Delta} = id$. Thus we get the following interesting result:

Theorem 6.2 *Let A be a ring of dimension $n \geq 2$. Let L and $F = Q \oplus A$ be projective A-modules of rank 1 and n respectively with $\wedge^n F \xrightarrow{\sim} L$. Then $E_0(A, L)$ is isomorphic to $E_0(A, F)$.*

Since, by ([2], 6.8), $E_0(A, L)$ is canonically isomorphic to $E_0(A, A)$, we get the surprising result that $E_0(A, F)$ is canonically isomorphic to $E_0(A, A^n)$ for any projective A-module $F = Q \oplus A$ of rank n.

We end with the following result which follows from (5.3).

Proposition 6.3 *Let A be a ring of even dimension n and let J be an ideal of A of height n such that J/J^2 is generated by n elements. Let L and P be projective A-modules of rank 1 and n respectively such that P is stably isomorphic to $L \oplus A^{n-1}$. Then J is surjective image of P if and only if given any projective A-module Q of rank $n-2$ with determinant L, there exists a projective A-module P_1 which is stably isomorphic to $Q \oplus A^2$ such that J is surjective image of P_1.*

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