# Some results on Euler class groups

Manoj K. Keshari

Department of Mathematics, IIT Mumbai, Mumbai - 400076, India; keshari@math.iitb.ac.in

**Abstract:** Let $A$ be a regular domain of dimension $d$ containing an infinite field and let $n$ be an integer with $2n \geq d+3$. For a stably free $A$-module $P$ of rank $n$, we will define the Euler class of $P$ and prove that $(i)$ $P$ has a unimodular element if and only if the euler class of $P$ is zero in $E^n(A)$ and $(ii)$ we define Whitney class homomorphism $w(P) : E^s(A) \to E^{n+s}(A)$, where $E^s(A)$ denotes the $s$th Euler class group of $A$ for $s \geq 1$. Further we prove that if $P$ has a unimodular element, then $w(P)$ is the zero map.

**Mathematics Subject Classification (2000):** Primary 13C10.

**Key words:** Euler class group, Whitney class homomorphism.

## 1 Introduction

Let $A$ be a commutative Noetherian ring of dimension $d$ and let $P$ be a projective $A$-module of rank $r > d$. Then a classical result of Serre [7] says that $P$ has a unimodular element i.e. $P = Q \oplus A$ for some projective $A$-module $Q$ of rank $r-1$. This result is not true in general when $r \leq d$. To find the obstruction for a projective $A$-module $P$ of rank $d$ to have a unimodular element, Bhatwadekar and Raja Sridharan [2] defined the Euler class group of $A$ w.r.t. a rank 1 projective $A$-module $L$, denoted by $E^d(A, L)$. To every pair $(P, \chi)$, where $P$ is a projective $A$-module of rank $d$ with determinant $L$ and $\chi : L \xrightarrow{\sim} \wedge^d P$ is an isomorphism, they associate an element $e(P, \chi)$ of $E^d(A, L)$. Then they proved that $P$ has a unimodular element if and only if $e(P, \chi)$ is zero in $E^d(A, L)$. In other words, the non-vanishing of $e(P, \chi)$ is the precise obstruction for $P$ to have a unimodular element. We would like to have similar obstruction results for projective $A$-modules $P$ of rank $r < d$.

Let $A$ be a regular ring of dimension $d$ containing an infinite field $k$. For a positive integer $n$ with $2n \geq d+3$, Bhatwadekar and Raja Sridharan [3] defined the $n^{th}$ Euler class group of $A$, denoted by $E^n(A)$. For a projective $A$-module $P$ of rank $n$ such that $P \oplus A = A^{n+1}$, they associate an element $e(P)$ of $E^n(A)$ and prove that $P$ has a unimodular element if and only if $e(P)$ is zero in $E^n(A)$. We will generalize this result for all stably free $A$-modules of rank $n$.

For a ring $A$ of dimension $d$, Mandal and Yang [5] defined the $s^{th}$ Euler class group of $A$ for all $1 \leq s \leq d$, denoted by $E^s(A)$. Their definition is a natural generalisation of the one given by Bhatwadekar and Raja Sridharan. For any projective $A$-module $P$ of rank $n < d$, they define a group homomorphism $w(P) : E^{d-n} \to E^d(A)$, called the Whitney class homomorphism. Further they prove that if $P$ has a unimodular element, then $w(P)$ is the zero map.

We will generalize above results as follows. Let $A$ be a regular domain of dimension $d$ containing an infinite field $k$. For a positive integer $n$ with $2n \geq d+3$, we prove the following results:

$(i)$ For a stably free $A$-module $P$ of rank $n$, we will associate an element $e(P)$ of $E^n(A)$ and prove that $e(P) = 0$ in $E^n(A)$ if and only if $P$ has a unimodular element. When $P \oplus A \xrightarrow{\sim} A^{n+1}$, this result is due to Bhatwadekar and Raja Sridharan [3].

(*ii*) Given a stably free $A$-module $Q$ of rank $n$, we define a Whitney class homomorphism $w(Q) : E^s(A) \to E^{n+s}(A)$. Further, we prove that if $Q$ has a unimodular element, then $w(Q)$ is the zero map.

Note that, when $n+s = d$, (*ii*) is proved in [5] for arbitrary projective module $Q$ over any Noetherian ring $A$. We would like to define the above map $w(Q)$ for all projective $A$-module $Q$ of rank $n$. For this, we need to define the euler class of $Q$ in $E^n(A)$ (the non-vanishing of which should be the precise obstruction for $Q$ to have a unimodular element). This is an open problem at present.

# 2 Euler class groups

All the rings considered are commutative Noetherian and all the modules are finitely generated. For a ring $A$ of dimension $d \geq 2$ and $1 \leq n \leq d$, the $n$th Euler class group of $A$, denoted by $E^n(A)$ is defined in [5] as follows:

Let $E_n(A)$ denote the group generated by $n \times n$ elementary matrices over $A$ and let $F = A^n$. A local orientation is a pair $(I, w)$, where $I$ is an ideal of $A$ of height $n$ and $w$ is an equivalence class of surjective homomorphisms from $F/IF$ to $I/I^2$. The equivalence is defined by $E_n(A/I)$-maps.

Let $L^n(A)$ denote the set of all pairs $(I, w)$, where $I$ is an ideal of height $n$ such that $\operatorname{Spec}(A/I)$ is connected and $w : F/IF \twoheadrightarrow I/I^2$ is a local orientation. Similarly, let $L_0^n(A)$ denote the set of all ideals $I$ of height $n$ such that $\operatorname{Spec}(A/I)$ is connected and there is a surjective homomorphism from $F/IF$ to $I/I^2$.

Let $G^n(A)$ denote the free abelian group generated by $L^n(A)$ and let $G_0^n(A)$ denote the free abelian group generated by $L_0^n(A)$.

Suppose $I$ is an ideal of height $n$ and $w : F/IF \twoheadrightarrow I/I^2$ is a local orientation. By ([3], Lemma 4.1), there is a unique decomposition $I = \cap_1^r I_i$, such that $I_i$'s are pairwise comaximal ideals of height $n$ and $\operatorname{Spec}(A/I_i)$ is connected. Then $w$ naturally induces local orientations $w_i : F/I_iF \twoheadrightarrow I_i/I_i^2$. Denote $(I, w) = \sum(I_i, w_i) \in G^n(A)$. Similarly we denote $(I) = \sum(I_i) \in G_0^n(A)$.

We say a local orientation $w : F/IF \twoheadrightarrow I/I^2$ is global if $w$ can be lifted to a surjection $\Omega : F \twoheadrightarrow I$. Let $H^n(A)$ be the subgroup of $G^n(A)$ generated by global orientations. Also let $H_0^n(A)$ be the subgroup of $G_0^n(A)$ generated by $(I)$ such that $I$ is a surjective image of $F$.

The Euler class group of codimension $n$ cycles is defined as $E^n(A) = G^n(A)/H^n(A)$ and the weak Euler class group of codimension $n$ cycles is defined as $E_0^n(A) = G_0^n(A)/H_0^n(A)$.

## 2.1 Euler class of Stably free modules

Let $A$ be a regular ring of dimension $d \geq 3$ containing an infinite field and let $n$ be an integer such that $2n \geq d+3$. In [3], a map from $\mathrm{Um}_{n+1}(A)$ to $E^n(A)$ is defined and it is proved that, if $P$ is a projective $A$-module of rank $n$ defined by the unimodular element $[a_0, \ldots, a_n]$, then $P$ has a unimodular element if and only if the image of $[a_0, \ldots, a_n]$ in $E^n(A)$ is zero ([3], Theorem 5.4) (Note that $P \oplus A \xrightarrow{\sim} A^{n+1}$). We will generalize this result for any stably free $A$-module of rank $n$.

For $r \geq 1$, let $\mathrm{Um}_{r,n+r}(A)$ be the set of all $r \times (n+r)$ matrices $\sigma$ in $M_{r,n+r}(A)$ which has a right inverse, i.e there exists $\tau \in M_{n+r,r}$ such that $\sigma\tau$ is the $r \times r$ identity matrix. For any element $\sigma \in \mathrm{Um}_{r,n+r}(A)$, we have an exact sequence

$$0 \to A^r \xrightarrow{\sigma} A^{n+r} \to P \to 0,$$

where $\sigma(v) = v\sigma$ for $v \in A^r$ and $P$ is a stably free projective $A$-module of rank $n$. Hence, every element of $\mathrm{Um}_{r,n+r}(A)$ corresponds to a stably free projective $A$-module of rank $n$ and conversely, any stably free projective $A$-module $P$ of rank $n$ will give rise to an element of $\mathrm{Um}_{r,n+r}(A)$ for some $r$. We will define a map from $\mathrm{Um}_{r,n+r}(A)$ to $E^n(A)$ which is a natural generalization of the map $\mathrm{Um}_{n+1}(A) \to E^n(A)$ defined in [3].

Let $\sigma$ be an element of $\mathrm{Um}_{r,n+r}(A)$.

$$\sigma = \begin{bmatrix} a_{1,1} & \dots & a_{1,n+r} \\ \vdots & & \vdots \\ a_{r,1} & \dots & a_{r,n+r} \end{bmatrix}$$

Let $e_1, \dots, e_{n+r}$ be the standard basis of $A^{n+r}$ and let

$$P = A^{n+r}/(\sum_{i=1}^{n+r} a_{1,i}\, e_i, \dots, \sum_{i=1}^{n+r} a_{r,i}\, e_i)A.$$

Let $p_1, \dots, p_{n+r}$ be the images of $e_1, \dots, e_{n+r}$ respectively in $P$. Then

$$P = \sum_{i=1}^{n+r} A\, p_i \text{ with relations } \sum_{i=1}^{n+r} a_{1,i}\, p_i = 0, \dots, \sum_{i=1}^{n+r} a_{r,i}\, p_i = 0.$$

To the triple $(P, (p_1, \dots, p_{n+r}), \sigma)$, we associate an element $e(P, (p_1, \dots, p_{n+r}), \sigma)$ of $E^n(A)$ as follows:

Let $\lambda : P \twoheadrightarrow J$ be a generic surjection, i.e. $J \subset A$ is an ideal of height $n$. Since $P \oplus A^r = A^{n+r}$ and $\dim A/J \le d - n \le n - 3$, by [1], $P/JP$ is a free $A/J$-module of rank $n$. Since $J/J^2$ is a surjective image of $P/JP$, $J/J^2$ is generated by $n$ elements.

Let "bar" denote reduction modulo $J$. By Bass result ([1]), there exists $\Theta \in E_{n+r}(A/J)$ such that $[\overline{a_{1,1}}, \dots, \overline{a_{1,n+r}}]\, \Theta = [1, 0, \dots, 0]$. That means the first row of $\Theta^{-1}$ is $[\overline{a_{1,1}}, \dots, \overline{a_{1,n+r}}]$. Let $\overline{\sigma}\, \Theta$ be given by

$$\overline{\sigma}\, \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \overline{b_{2,1}} & \overline{b_{2,2}} & \dots & \overline{b_{2,n+r}} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{b_{r,1}} & \overline{b_{r,2}} & \dots & b_{r,n+r} \end{bmatrix}.$$

Note that $[\overline{b_{2,2}}, \dots, \overline{b_{2,n+r}}] \in \mathrm{Um}_{n+r-1}(A/J)$. Hence, by Bass result, we can find $\Theta_1 \in E_{n+r-1}(A/J)$ such that $[\overline{b_{2,2}}, \dots, \overline{b_{2,n+r}}]\Theta_1 = [1, 0, \dots, 0]$. Further any $\Phi \in E_m(A)$ can be thought of as an element of $E_{m+t}(A)$ as $\begin{bmatrix} Id_t & 0 \\ 0 & \Phi \end{bmatrix}$, where $Id_t$ is $t \times t$ identity matrix. Let

$$\overline{\sigma}\, \Theta\, \Theta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \overline{b_{2,1}} & 1 & \dots & 0 \\ \overline{b_{3,1}} & \overline{c_{3,2}} & \dots & \overline{c_{3,n+r}} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{b_{r,1}} & \overline{c_{r,2}} & \dots & c_{r,n+r} \end{bmatrix}.$$

Continuing this way, we get $\widetilde{\Theta} \in E_{n+r}(A/J)$ such that

$$\overline{\sigma}\,\widetilde{\Theta} = \begin{bmatrix} 1 & 0 & \dots & & & 0 \\ \overline{b_{2,1}} & 1 & 0 & \dots & & 0 \\ \overline{b_{3,1}} & \overline{c_{3,2}} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \overline{b_{r,1}} & \overline{c_{r,2}} & \dots & \overline{d_{r,r-1}} & 1 \quad \dots & 0 \end{bmatrix}.$$

We can find an elementary matrix $\Psi \in E_{n+r}(A/J)$ such that $\overline{\sigma}\widetilde{\Theta}\Psi = [Id_r, \underline{0}]$, where $\underline{0}$ is $r \times n$ zero matrix. Let $\Delta = (\widetilde{\Theta}\Psi)^{-1} \in E_{n+r}(A/J)$, then $\overline{\sigma}$ is the first $r$ rows of $\Delta$, i.e. $\overline{\sigma}$ can be completed to an elementary matrix $\Delta$. Since

$$\sum_{i=1}^{n+r} a_{1,i}\, p_i = 0, \dots, \sum_{i=1}^{n+r} a_{r,i}\, p_i = 0,$$

we get

$$\Delta[\overline{p_1}, \dots, \overline{p_{n+r}}]^t = [0, \dots, 0, \overline{q_1}, \dots, \overline{q_n}]^t,$$

where $t$ stands for transpose.

Thus $(\overline{q_1}, \dots, \overline{q_n})$ is a basis of the free module $P/JP$. Let $w_J : (A/J)^n \twoheadrightarrow J/J^2$ be the surjection given by the set of generators $\overline{\lambda(q_1)}, \dots, \overline{\lambda(q_n)}$ of $J/J^2$.

We define $e(P, (p_1, \dots, p_{n+r}), \sigma) = (J, w_J) \in E^n(A)$. We need to show that $e(P, (p_1, \dots, p_{n+r}), \sigma)$ is independent of the choice of the elementary completion of $\overline{\sigma}$ and the choice of the generic surjection $\lambda$.

We begin with the following result which shows that $e(P, (p_1, \dots, p_{n+r}), \sigma)$ is independent of the choice of the elementary completion of $\overline{\sigma}$.

**Lemma 2.1** *Suppose $\Gamma \in E_{n+r}(A/J)$ is chosen so that its first $r$ rows are $\overline{\sigma}$. Let $\Gamma[\overline{p_1}, \dots, \overline{p_{n+r}}]^t = [0, \dots, 0, \overline{q_1'}, \dots, \overline{q_n'}]^t$. Then there exists $\Psi \in E_n(A/J)$ such that $\Psi[\overline{q_1}, \dots, \overline{q_n}]^t = [\overline{q_1'}, \dots, \overline{q_n'}]$.*

**Proof** The matrix $\Gamma\Delta^{-1} \in E_{n+r}(A/J)$ is such that its first $r$ rows are $[Id_r, \underline{0}]$. Therefore, there exists $\Psi \in \mathrm{SL}_n(A/J) \cap E_{n+r}(A/J)$ such that $\Psi[\overline{q_1}, \dots, \overline{q_n}]^t = [\overline{q_1'}, \dots, \overline{q_n'}]^t$. Since $n > \dim A/J + 1$, by ([8], Theorem 3.2), $\Psi \in E_n(A/J)$. ■

Let $\widetilde{w_J} : (A/J)^n \twoheadrightarrow J/J^2$ be the surjection given by the set of generators $\overline{\lambda(q_1')}, \dots, \overline{\lambda(q_n')}$ of $J/J^2$. Then, by (2.1), $(J, w_J) = (J, \widetilde{w_J})$ in $E^n(A)$. Thus for a given surjection $\lambda : P \twoheadrightarrow J$, the element $e(P, (p_1, \dots, p_{n+r}), \sigma)$ is independent of the choice of the elementary completion of $\overline{\sigma}$.

Now we have to show that $e(P, (p_1, \dots, p_{n+r}), \sigma)$ is independent of the choice of the generic surjection $\lambda$. In other words, we have to show that if $\lambda' : P \twoheadrightarrow J'$ is another generic surjections where $J'$ is an ideal of $A$ of height $n$ and $w_{J'} : (A/J')^n \twoheadrightarrow J'/{J'}^2$ is a surjection obtained as above by completing $\sigma$ modulo $J'$ to an element of $E_{n+r}(A/J')$, then $(J, w_J) = (J', w_{J'})$ in $E^n(A)$.

This independence is proved in ([3], p. 152-153) in case $P \oplus A = A^{n+1}$. The same proof works in the case $P \oplus A^r = A^{n+r}$, hence we omit the proof. Therefore we have a well defined map $e : \mathrm{Um}_{r,n+r}(A) \to E^n(A)$. We denote $e(P, (p_1, \dots, p_{n+r}), \sigma) \in E^n(A)$ by $e(P)$ or $e(\sigma)$.

The following result is proved in ([3], Theorem 5.4) in case $P \oplus A$ is free. Since same proof works in our case, we omit the proof.

**Theorem 2.2** *Let $A$ be a regular ring of dimension $d$ containing an infinite field $k$ and let $n$ be an integer such that $2n \geq d+3$. Let $P$ be a stably free $A$-module of rank $n$ defined by $\sigma \in \mathrm{Um}_{r,n+r}(A)$. Then $P$ has a unimodular element if and only if $e(P) = e(\sigma) = 0$ in $E^n(A)$.*

## 2.2 Whitney class homomorphism

Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field $k$ and let $Q$ be a stably free $A$-module of rank $n$ with $2n \geq d+3$. In (2.2), we proved that $e(Q) = 0$ in $E^n(A)$ if and only if $Q$ has a unimodular element. Using this result we will establish a whitney class homomorphism of stably free modules. When $n+s = d$, then (2.3) is proved in ([5], Theorem 3.1) for any projective $A$-module $Q$. Our proof is similar to [5].

**Theorem 2.3** *Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field $k$. Suppose $Q$ is a stably free $A$-module of rank $n$ defined by $\sigma \in \mathrm{Um}_{r,n+r}(A)$. Then there exists a homomorphism $w(Q) : E^s(A) \to E^{n+s}(A)$ for every integer $s \geq 1$ with $2n+s \geq d+3$.*

**Proof** Write $F = A^n$ and $F' = A^s$. Let $I$ be an ideal of height $s$ and $w : F'/IF' \twoheadrightarrow I/I^2$ be an equivalence class of surjections, where the equivalence is defined by $E_s(A/I) = E(F'/IF')$ maps. To each such pair $(I, w)$, we will associate an element $w(Q) \cap (I, w) \in E^{n+s}(A)$.

First we can find an ideal $\widetilde{I} \subset A$ of height $\geq n+s$ and a surjective homomorphism $\psi : Q/IQ \twoheadrightarrow \widetilde{I}/I$ (this is just the existence of a generic surjection of $Q/IQ$). Let $\psi \otimes A/\widetilde{I} = \widetilde{\psi}$. Then $\widetilde{\psi} : Q/\widetilde{I}Q \twoheadrightarrow \widetilde{I}/(I+\widetilde{I}^2)$ is a surjection.

Since $\dim A/\widetilde{I} \leq d-(n+s) \leq n-3$, $Q/\widetilde{I}Q$ is a free $A/\widetilde{I}$-module, by Bass result [1]. Let "bar" denotes reduction modulo $\widetilde{I}$. Then $\overline{\sigma} \in \mathrm{Um}_{r,n+r}(\overline{A})$ can be completed to an elementary matrix $\Theta \in E_{n+r}(\overline{A})$. This gives a well defined basis $[\overline{q}_1, \ldots, \overline{q}_n]$ for $\overline{Q}$ which does not depends on the elementary completions of $\overline{\sigma}$ (in the sense that any two basis of $\overline{Q}$ obtained this way will be connected by an element of $E_n(\overline{A})$).

Let $\gamma : F/\widetilde{I}F \xrightarrow{\sim} Q/\widetilde{I}Q$ be the isomorphism given by $\gamma(\overline{e_i}) = \overline{q}_i$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the standard basis of the free module $F$. Let $\beta = \widetilde{\psi}\gamma : F/\widetilde{I}F \twoheadrightarrow \widetilde{I}/(I+\widetilde{I}^2)$ be a surjection and let $\beta' : F/\widetilde{I}F \to \widetilde{I}/\widetilde{I}^2$ be a lift of $\beta$.

Further, $w : F'/IF' \twoheadrightarrow I/I^2$ induces a surjection $\widetilde{w} : F'/\widetilde{I}F' \twoheadrightarrow (I+\widetilde{I}^2)/\widetilde{I}^2$. Composing $\widetilde{w}$ with the natural inclusion $(I+\widetilde{I}^2)/\widetilde{I}^2 \subset \widetilde{I}/\widetilde{I}^2$, we get a map $w' : F'/\widetilde{I}F' \to \widetilde{I}/\widetilde{I}^2$.

Combining $w'$ and $\beta'$, it is easy to see that we get a surjective homomorphism

$$\Delta = \beta' \oplus w' : F/\widetilde{I}F \oplus F'/\widetilde{I}F' = (F \oplus F')/\widetilde{I}(F \oplus F') \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$$

(surjectivity follows by considering the exact sequence $0 \to (I+\widetilde{I}^2)/\widetilde{I}^2 \hookrightarrow \widetilde{I}/\widetilde{I}^2 \to \widetilde{I}/(I+\widetilde{I}^2) \to 0$). We have $(\widetilde{I}, \Delta)$ a local orientation of $\widetilde{I}$. We will show that the image of $(\widetilde{I}, \Delta)$ in $E^{n+s}(A)$ is independent of choices of $\psi$, the lift $\beta'$ and the representative of $w$ in the equivalence class.

**Step 1.** First we show that for a fixed $\psi$, $(\widetilde{I}, \Delta)$ in $E^{n+s}$ is independent of the lift $\beta'$ and the representative of $w$.

$(a)$ Suppose $w, w_1 : F'/IF' \twoheadrightarrow I/I^2$ are two equivalent local orientations of $I$. Then $w_1 = w\epsilon$ for some $\epsilon \in E(F'/IF')$. Using the canonical homomorphisms $E(F'/IF') \twoheadrightarrow E(F'/\widetilde{I}F') \to E((F \oplus F')/\widetilde{I}(F \oplus F'))$, we get that $w_1' = w'\epsilon_1$ for some $\epsilon_1 \in E((F \oplus F')/\widetilde{I}(F \oplus F'))$.

Let $\Delta_1$ be the local orientation of $\widetilde{I}$ obtained by using $\beta'$ and $w_1$. Then $\Delta_1 = \Delta\epsilon_1$. Hence $(\widetilde{I}, \Delta) = (\widetilde{I}, \Delta_1)$ in $E^{n+s}(A)$.

$(b)$ Let $\beta'' : F/\widetilde{I}F \to \widetilde{I}/\widetilde{I}^2$ be another lift of $\beta$. Then $\phi = \beta' - \beta'' : F/\widetilde{I}F \to (I + \widetilde{I}^2)/\widetilde{I}^2$. Since $\widetilde{w}_1 : F'/\widetilde{I}F' \twoheadrightarrow (I + \widetilde{I}^2)/\widetilde{I}^2$ is a surjection, there exists $g : F/\widetilde{I}F \to F'/\widetilde{I}F'$ such that $\widetilde{w}_1 g = \phi$.

Let $\epsilon_2 = \left(\begin{smallmatrix} 1 & 0 \\ g & 1 \end{smallmatrix}\right) \in E((F \oplus F')/\widetilde{I}(F \oplus F'))$. Then $(\beta'' \oplus w_1')\epsilon_2 = (\beta' \oplus w_1')$. Therefore, if $\Delta_2 = \beta'' \oplus w_1'$, then $\Delta_2\epsilon_2 = \Delta_1 = \Delta\epsilon_1$.

This completes the proof of the claim in step 1.

**Step 2.** Now we will show that $(\widetilde{I}, \Delta) \in E^{n+s}(A)$ is independent of $\psi$ also (i.e. it depends only on $(I, w)$).

Recall that $w : F'/IF' \twoheadrightarrow I/I^2$ is a surjection. It is easy to see that we can lift $w$ to a surjection $\Omega : F' \twoheadrightarrow I \cap K$, where $K + I = A$ and $K$ is an ideal of height $s$ (or $K = A$).

We can find an ideal $\widetilde{K} \subset A$ of height $\geq n + s$ and a surjective homomorphism $\psi' : Q/KQ \twoheadrightarrow \widetilde{K}/K$. Let $\psi' \otimes A/\widetilde{K} = \widetilde{\psi}'$. Then $\widetilde{\psi}' : Q/\widetilde{K}Q \twoheadrightarrow \widetilde{K}/(K + \widetilde{K}^2)$ is a surjection.

Again, since $\dim A/\widetilde{K} \leq n - 3$, $Q/\widetilde{K}Q$ is a free $A/\widetilde{K}$-module. If "bar" denotes reduction modulo $\widetilde{K}$, then $\overline{\sigma} \in \mathrm{Um}_{r,n+r}(A/\widetilde{K})$ can be completed to an elementary matrix which gives a basis $\overline{p}_1, \ldots, \overline{p}_n$ for $Q/\widetilde{K}Q$. Let $\gamma' : F/\widetilde{K}F \xrightarrow{\sim} Q/\widetilde{K}Q$ be the isomorphism given by $\gamma'(\overline{e_i}) = \overline{p}_i$. Let $\eta = \widetilde{\psi}'\gamma' : F/\widetilde{K}F \twoheadrightarrow \widetilde{K}/(I + \widetilde{K}^2)$ be a surjection and let $\eta' : F/\widetilde{K}F \to \widetilde{K}/\widetilde{K}^2$ be a lift of $\eta$.

The map $\Omega : F' \twoheadrightarrow I \cap K$ induces a surjection $\Omega \otimes A/K = \Omega' : F'/KF' \twoheadrightarrow K/K^2$ which in turn induces a surjection $\Omega' \otimes A/\widetilde{K} = w'' : F'/\widetilde{K}F' \twoheadrightarrow (K + \widetilde{K}^2)/\widetilde{K}^2$. Since $(K + \widetilde{K}^2) \subset \widetilde{K}$, we get a map $w'' : F'/\widetilde{K}F' \to \widetilde{K}/\widetilde{K}^2$.

Combining $w''$ and $\eta'$, we get a surjection $\Delta' = \eta' \oplus w'' : (F \oplus F')/\widetilde{K}(F \oplus F') \twoheadrightarrow \widetilde{K}/\widetilde{K}^2$.

**Claim.** $(\widetilde{I}, \Delta) + (\widetilde{K}, \Delta') = 0$ in $E^{n+s}(A)$.

Since $I + K = A$, we get $\widetilde{I} + \widetilde{K} = A$. Further, we get a surjection

$$\Psi = \psi \oplus \psi' : Q/(I \cap K)Q \simeq Q/IQ \oplus Q/KQ \twoheadrightarrow \widetilde{I}/I \oplus \widetilde{K}/K \simeq (\widetilde{I} \cap \widetilde{K})/(I \cap K).$$

Let $\widetilde{\Psi} : Q \to \widetilde{I} \cap \widetilde{K}$ be a lift of $\Psi$ such that the following holds:

$(i)$ $\widetilde{\Psi} \otimes A/\widetilde{I} = \widetilde{\psi}$, where $\widetilde{\psi} : Q/\widetilde{I}Q \twoheadrightarrow \widetilde{I}/(I + \widetilde{I}^2)$ is a surjection and
$(ii)$ $\widetilde{\Psi} \otimes A/\widetilde{K} = \widetilde{\psi}'$, where $\widetilde{\psi}' : Q/\widetilde{K}Q \twoheadrightarrow \widetilde{K}/(K + \widetilde{K}^2)$ is a surjection.

Let $\widetilde{\Psi}_1 : Q/\widetilde{I}Q \to \widetilde{I}/\widetilde{I}^2$ be a lift of $\widetilde{\Psi} \otimes A/\widetilde{I}$ and let $\widetilde{\Psi}_2 : Q/\widetilde{K}Q \to \widetilde{K}/\widetilde{K}^2$ be a lift of $\widetilde{\Psi} \otimes A/\widetilde{K}$. Then $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ induces a map $\widetilde{\Psi}_3 : Q/(\widetilde{I} \cap \widetilde{K})Q \to (\widetilde{I} \cap \widetilde{K})/(\widetilde{I} \cap \widetilde{K})^2$.

Since $\beta = \widetilde{\psi}\gamma = (\widetilde{\Psi} \otimes A/\widetilde{I})\gamma$ and $\beta' : F/\widetilde{I}F \to \widetilde{I}/\widetilde{I}^2$ is a lift of $\beta$, we get that $\alpha_1 = \beta'\gamma^{-1} - \widetilde{\Psi}_1$ is a map from $Q/\widetilde{I}Q$ to $(I + \widetilde{I}^2)/\widetilde{I}^2 \subset \widetilde{I}/\widetilde{I}^2$. Similarly, $\alpha_2 = \eta'(\gamma')^{-1} - \widetilde{\Psi}_2$ is a map from $Q/\widetilde{K}Q$ to $(K + \widetilde{K}^2)/\widetilde{K}^2 \subset \widetilde{K}/\widetilde{K}^2$.

Since $\widetilde{w} : F'/\widetilde{I}F' \twoheadrightarrow (I + \widetilde{I}^2)/\widetilde{I}^2$ is a surjection, we can find $g_1 : Q/\widetilde{I}Q \to F'/\widetilde{I}F'$ such that $\widetilde{w}g_1 = \alpha_1$. Similarly, we can find $g_2 : Q/\widetilde{K}Q \to F'/\widetilde{K}F'$ such that $w''g_2 = \alpha_2$ (here $w'' = \Omega' \otimes A/\widetilde{K}$).

Let $g$ be given by $g_1, g_2$ and $\widetilde{\gamma}$ be given by $\gamma, \gamma'$. Then

(*a*) $\left(\begin{smallmatrix}\widetilde{\gamma} & 0\\ 0 & 1\end{smallmatrix}\right)$ is an isomorphism from $(F \oplus F')/(\widetilde{I} \cap \widetilde{K})(F \oplus F')$ to $(Q \oplus F')/(\widetilde{I} \cap \widetilde{K})(Q \oplus F')$ and
(*b*) $\left(\begin{smallmatrix}1 & 0\\ g & 1\end{smallmatrix}\right)$ is an automorphism of $(Q \oplus F')/(\widetilde{I} \cap \widetilde{K})(Q \oplus F')$.

Write $\Gamma = \left(\begin{smallmatrix}1 & 0\\ g & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\widetilde{\gamma} & 0\\ 0 & 1\end{smallmatrix}\right)$. Since $\widetilde{\Psi}$ is a lift of $\Psi$, $\Psi$ is a surjection from $Q/(I \cap K)Q$ to $(\widetilde{I} \cap \widetilde{K})/(I \cap K)$ and $\Omega : F' \twoheadrightarrow I \cap K$ is a surjection, we get that $\widetilde{\Psi} \oplus \Omega : Q \oplus F' \twoheadrightarrow \widetilde{I} \cap \widetilde{K}$ is a surjection.

Write $\Theta = (\widetilde{\Psi} \oplus \Omega) \otimes A/(\widetilde{I} \cap \widetilde{K})$. Then $\Theta : (Q \oplus F')/(\widetilde{I} \cap \widetilde{K})(Q \oplus F') \twoheadrightarrow (\widetilde{I} \cap \widetilde{K})/(\widetilde{I} \cap \widetilde{K})^2$. Let $(\Delta, \Delta') : (F \oplus F')/(\widetilde{I} \cap \widetilde{K})(F \oplus F') \twoheadrightarrow (\widetilde{I} \cap \widetilde{K})/(\widetilde{I} \cap \widetilde{K})^2$ be the surjection induced from $\Delta, \Delta'$. We claim that $(\Delta, \Delta') = \Theta\Gamma$. (This follows by checking on $V(\widetilde{I})$ and $V(\widetilde{K})$ separately, but we give a direct proof below.)

Let $\alpha_3 : Q/(\widetilde{I} \cap \widetilde{K})Q \to (\widetilde{I} \cap \widetilde{K})/(\widetilde{I} \cap \widetilde{K})^2$ be the map induced from $\alpha_1, \alpha_2$ and let $\tau : F/(\widetilde{I} \cap \widetilde{K}) \to (\widetilde{I} \cap \widetilde{K})/(\widetilde{I} \cap \widetilde{K})^2$ be the map induced from $\beta', \eta'$. Then we have $\alpha_3 = \tau\widetilde{\gamma}^{-1} - \widetilde{\Psi}_3$. Let $\overline{\Omega} : F'/(\widetilde{I} \cap \widetilde{K})F' \to (\widetilde{I} \cap \widetilde{K})/(\widetilde{I} \cap \widetilde{K})^2$ be the map induced from $\widetilde{w}, w''$. Then we have $\overline{\Omega}g = \alpha_3$.

Now $\Theta\Gamma(0, y) = \Theta(0, y) = \overline{\Omega}(y) = (\Delta, \Delta')(0, y)$ and $\Theta\Gamma(x, 0) = \Theta(\widetilde{\gamma}(x), g\widetilde{\gamma}(x)) = \widetilde{\Psi}_3\widetilde{\gamma}(x) + \overline{\Omega}g\widetilde{\gamma}(x) = \widetilde{\Psi}_3\widetilde{\gamma}(x) + \tau\widetilde{\gamma}^{-1}\widetilde{\gamma}(x) - \widetilde{\Psi}_3\widetilde{\gamma}(x) = \tau(x) = (\Delta, \Delta')(x, 0)$.

This proves that $(\Delta, \Delta') = \Theta\Gamma$. By ([3], Theorem 4.2), we get that $(\widetilde{I}, \Delta) + (\widetilde{K}, \Delta') = 0$ in $E^{n+s}(A)$. Since $(\widetilde{K}, \Delta')$ depends only on $(I, w)$, it follows that $(\widetilde{I}, \Delta)$ is independent of the choice of $\psi$. This establishes the claim in step 2.

If $(I, w)$ is a global orientation, then we can take $K = A$ in the above proof and it will follow that $(\widetilde{I}, \Delta)$ is also a global orientation.

Thus the association $(I, w) \mapsto (\widetilde{I}, \Delta) \in E^{n+s}(A)$ defines a homomorphism $\phi(Q) : G^s(A) \to E^{n+s}(A)$, where $(I, w)$ are the free generators of $G^s(A)$. Further $\phi(Q)$ factors through a homomorphism $w(Q) : E^s(A) \to E^{n+s}(A)$ sending $(I, w) \in E^s(A)$ to $(\widetilde{I}, \Delta) \in E^{n+s}(A)$. This completes the proof of the theorem. ■

**Corollary 2.4** *Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field. Suppose $Q$ is a stably free $A$-module of rank $n$. Then there exists a homomorphism $w_0(Q) : E_0^s(A) \to E_0^{n+s}(A)$ for every integer $s \geq 1$ with $2n + s \geq d + 3$.*

**Proof** The proof is similar to that of (2.3) and we give an outline. Write $F = A^n$ and $F' = A^s$.

Suppose $(I)$ is a generator of $G_0^s(A)$. Here $I$ is an ideal of height $s$, Spec $(A/I)$ is connected and there is a surjection from $F'/IF'$ to $I/I^2$. There is a surjection $\psi : Q/IQ \twoheadrightarrow \widetilde{I}/I$, where $\widetilde{I}$ is an ideal of height $\geq n + s$. For such a generator $(I)$, we associate $(\widetilde{I}) \in E_0^{n+s}(A)$.

For well-definedness, fix a local orientation $w : F'/IF' \twoheadrightarrow I/I^2$ and a surjective lift $\Omega : F' \twoheadrightarrow I \cap K$ of $w$, where $K$ is an ideal of height $\geq s$ and $K + I = A$. Let $\psi' : Q/KQ \twoheadrightarrow \widetilde{K}/K$ be a surjection, where $\widetilde{K}$ is an ideal of height $\geq n + s$. As in (2.3), there exists a surjection from $F \oplus F' \twoheadrightarrow \widetilde{I} \cap \widetilde{K}$. This shows that $(\widetilde{I}) + (\widetilde{K}) = 0$ in $E_0^{n+s}(A)$ and so $(\widetilde{I}) \in E_0^{n+s}(A)$ is independent of the choice of $\psi$.

The association $(I) \mapsto (\widetilde{I}) \in E_0^{n+s}(A)$ extends to a homomorphism $\phi_0 : G_0^s(A) \to E_0^{n+s}(A)$.

If $(I)$ is global (i.e. $I$ is a surjective homomorphism of $F'$), then taking $K = A$ in the above argument, we can prove that $(\widetilde{I})$ is also global. So $\phi_0$ factors through a homomorphism $w_0(Q) : E_0^s(A) \to E_0^{n+s}(A)$. ■

**Definition 2.5** The homomorphism $w(Q)$ in theorem 2.3 will be called the *Whitney class homomorphism.* The image of $(I, w) \in E^s(A)$ under $w(Q)$ will be denoted by $w(Q) \cap (I, w)$.

Similarly, the homomorphism $w_0(Q)$ in (2.4) will be called the *weak Whitney class homomorphism.* The image of $(I) \in E_0^s(A)$ under $w_0(Q)$ will be denoted by $w_0(Q) \cap (I)$.

The proof of the following result is same as ([5], Corollary 3.4), hence we omit it.

**Corollary 2.6** *Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field. Suppose $Q$ is a stably free $A$-module of rank $n$. For every integer $s \geq 1$ with $2n + s \geq d + 3$, we have*

$$w_0(Q)\zeta^s = \zeta^{n+s}w(Q) \text{ and } C^n(Q^*)\eta^s = \eta^{n+s}w_0(Q),$$

*where (i) $\zeta^r : E^r(A) \twoheadrightarrow E_0^r(A)$ is a natural surjection obtained by forgetting the orientation,*

*(ii) $\eta^r : E_0^r(A) \to CH^r(A)$ is a natural homomorphism, sending $(I)$ to $[A/I]$. Here $CH^r(A)$ denotes the Chow group of cycles of codimension $r$ in* Spec $(A)$ *and*

*(iii) $C^n(Q^*)$ denote the top Chern class homomorphism [4].*

The following result is about vanishing of Whitney class homomorphism. When $n+s = d$, it is proved in ([5], Theorem 3.5) for arbitrary projective module $Q$ and our proof is an adaptation of [5]. We will follow the proof of (2.3) with necessary modifications.

**Theorem 2.7** *Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field. Suppose $Q$ is a stably free $A$-module of rank $n$ defined by $\sigma \in \text{Um}_{r,n+r}(A)$. Let $s \geq 1$ be an integer with $2n + s \geq d+3$. Write $F = A^n$ and $F' = A^s$. Let $I$ be an ideal of height $s$ and let $w : F'/IF' \twoheadrightarrow I/I^2$ be a surjection. If $Q/IQ = P_0 \oplus A/I$, then $w(Q) \cap (I, w) = 0$ in $E^{n+s}(A)$.*

*In particular, if $Q = P \oplus A$, then the homomorphism $w(Q) : E^s(A) \to E^{n+s}(A)$ is identically zero. Similar statements hold for $w_0(Q)$.*

**Proof** **Step 1.** We can find an ideal $\widetilde{I} \subset A$ of height $n+s$ and a surjective homomorphism $\psi : Q/IQ \to\to \widetilde{I}/I$. Let $\widetilde{\psi} = \psi \otimes A/\widetilde{I} : Q/\widetilde{I} \twoheadrightarrow \widetilde{I}/(I + \widetilde{I}^2)$.

Let $\Omega : F' \to I$ be a lift of $w$ and let $\overline{w} = w \otimes A/\widetilde{I} : F'/\widetilde{I}F' \twoheadrightarrow I/I\widetilde{I}$. Composing $\overline{w}$ with the natural map $I/I\widetilde{I} \hookrightarrow \widetilde{I}/I\widetilde{I} \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$, we get a map $w' : F'/\widetilde{I}F' \to \widetilde{I}/\widetilde{I}^2$.

Since $Q/IQ = P_0 \oplus A/I$, we can write $\psi = (\theta, \overline{a})$ for some $a \in \widetilde{I}$ and $\theta \in P_0^*$. We may assume that $\psi(P_0) = \widetilde{J}/I$, for some ideal $\widetilde{J} \subset A$ of height $n + s - 1$. Note that $\widetilde{I} = (\widetilde{J}, a)$.

Since $\dim A/\widetilde{J} = d - (n + s - 1) \leq n - 2$ and $P_0/IP_0$ is stably free $A/I$-module of rank $n - 1$, $P_0/\widetilde{J}P_0$ is free. If "prime" denotes reduction modulo $\widetilde{J}$, then $\sigma'$ can be completed to an elementary matrix in $E_{n+r}(A/\widetilde{J})$. This gives a canonical basis of $P_0/\widetilde{J}P_0$, say $q'_1, \ldots, q'_{n-1}$. Let $\gamma' : (A/\widetilde{J})^{n-1} \xrightarrow{\sim} P_0/\widetilde{J}P_0$ be the isomorphism given by $[q'_1, \ldots, q'_{n-1}]$.

Let $\gamma : F/\widetilde{I}F = (A/\widetilde{I})^n \xrightarrow{\sim} Q/\widetilde{I}Q = P_0/\widetilde{I}P_0 \oplus A/\widetilde{I}$ be the isomorphism given by $(\gamma', 1)$, i.e. $\gamma = [\overline{q_1}, \ldots, \overline{q_{n-1}}, 1]$. Let $\beta = \widetilde{\psi}\gamma : F/\widetilde{I}F \twoheadrightarrow \widetilde{I}/(I + \widetilde{I}^2)$ and let $\beta' : F/\widetilde{I}F \to \widetilde{I}/\widetilde{I}^2$ be a lift of $\beta$.

As in the proof of (2.3), combining $w'$ and $\beta'$, we get a surjection $\Delta = \beta' \oplus w' : (F \oplus F')/\widetilde{I}(F \oplus F') \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$ and $(\widetilde{I}, \Delta) = w(Q) \cap (I, w)$. We claim that $(\widetilde{I}, \Delta) = 0$ in $E^{n+s}(A)$.

**Step 2.** In this step, we will prove the claim. The surjection $\theta : P_0 \twoheadrightarrow \widetilde{J}/I$ induces a surjection $\overline{\theta} = \theta \otimes A/\widetilde{J} : P_0/\widetilde{J}P_0 \twoheadrightarrow \widetilde{J}/(I + \widetilde{J}^2)$. Let $\zeta = \overline{\theta}\gamma' : (A/\widetilde{J})^{n-1} \twoheadrightarrow \widetilde{J}/(I + \widetilde{J}^2)$ and let $\zeta' : (A/\widetilde{J})^{n-1} \to \widetilde{J}/\widetilde{J}^2$ be a lift of $\zeta$.

If $\overline{\zeta'}$ denotes the composition of $\zeta' \otimes A/\widetilde{I} : (A/\widetilde{I})^{n-1} \to \widetilde{J}/\widetilde{J}\widetilde{I}$ with natural maps $\widetilde{J}/\widetilde{J}\widetilde{I} \hookrightarrow \widetilde{I}/\widetilde{J}\widetilde{I} \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$, we get that $(\overline{\zeta'}, \overline{a})$ is a lift of $\beta : F/\widetilde{I}F \twoheadrightarrow \widetilde{I}/(I + \widetilde{I}^2)$. Since $w(Q) \cap (I, w)$ is independent of the lift $\beta'$ of $\beta$, we may assume that $\beta' = (\overline{\zeta'}, \overline{a})$.

If $\delta : A^{n-1} \to \widetilde{J}$ is a lift of $\zeta'$, then $(\delta, a, \Omega) : F \oplus F' \to \widetilde{I}$ is a lift of $(\beta', w')$. If $\widetilde{J'}$ is the image of $(\delta, \Omega)$, then $\widetilde{J} = \widetilde{J'} + \widetilde{J}^2$. (To see this, let $y \in \widetilde{J}$, then there exists $x \in A^{n-1}$ such that $\delta(x) - y = y_1 + z$ for some $y_1 \in I$ and $z \in \widetilde{J}^2$. Choose $x_1 \in F'$ such that $y_1 - \Omega(x_1) = z_1 \in I^2 \subset \widetilde{J}^2$. Therefore $\delta(x) - \Omega(x_1) = y$ modulo $\widetilde{J}^2$.)

Since $\widetilde{J} = \widetilde{J'} + \widetilde{J}^2$, we can find $e \in \widetilde{J}^2$ such that $(1-e)\widetilde{J} \subset \widetilde{J'}$ and $\widetilde{J} = (\widetilde{J'}, e)$. Therefore by ([6], Lemma 1), $\widetilde{I} = (\widetilde{J}, a) = (\widetilde{J'}, b)$, where $b = e + (1-e)a$. Thus $(\delta, b, \Omega) : F \oplus F' \twoheadrightarrow \widetilde{I}$ is a surjection which is a lift of $\beta' \oplus w'$. This proves that $(\widetilde{I}, \Delta) = 0$ in $E^{n+s}(A)$. This completes the proof. ■

# References


[1] H. Bass, *K-Theory and stable algebra*, I.H.E.S. **22** (1964), 5-60.

[2] S.M. Bhatwadekar and Raja Sridharan, *The Euler class group of a Noetherian ring*, Compositio Math. **122** (2000), 183-222.

[3] S.M. Bhatwadekar and Raja Sridharan, *On Euler classes and stably free projective modules*, Proceedings of the international colloquium on Algebra, Arithmetic and Geometry, Mumbai 2000, Narosa Publishing House, 139-158.

[4] W. Fulton, *Intersection theory*, Second Ed., Springer Verlag, 1998.

[5] S. Mandal and Y. Yang, *Intersection theory of algebraic obstructions*, J. Pure and Applied Algebra **214** (2010), 2279-2293.

[6] N. Mohan Kumar, *Complete intersections*, J. MAth. Kyoto Univ. **17** (1977), 533-538.

[7] J.P. Serre, *Sur les modules projectifs*, Sem. Dubreil-Pisot **14** (1960-61).

[8] L.N. Vaserstein, *On the stabilisation of the general linear group over a ring*, Math. USSR. Sbornik **8** (1969), (English Translation) 383-400.