Cancellation of projective modules over non-Noetherian rings

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Abstract

(i) Let R be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over R and $f_i \in R[Y_i]$. Then all projective A-modules are free and $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$.

(ii) Let R be a ring of dimension d and A be one of $R[Y]$ of $R[Y, Y^{-1}]$, where Y is a variable over R. Let P be a projective A-module of rank $\geq d+1$ satisfying property $\Omega(R)$ (see 4.8 for definition of property $\Omega(R)$). Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. When P is free, this result is due to Yengui: $A = R[Y]$ and Abedelfatah: $A = R[Y, Y^{-1}]$.

1 Introduction

Rings are assumed to be commutative with unity and modules are finitely generated. The dimension of a ring means its Krull dimension and projective modules are of constant rank.

Let R be a *Noetherian* ring of dimension d and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over R and $f_i \in R[Y_i]$. If P is a projective A-module of rank $\geq max\{2, d+1\}$, then author-Dhorajia ([6], Theorem 3.12) proved that $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. In particular P is cancellative, i.e. $P \oplus A^t \overset{\sim}{\to} Q \oplus A^t$ for some projective A-module $Q \implies P \overset{\sim}{\to} Q$. The case $n = m = 0$ of this result is due to Bass [4], $n = 1, m = 0$ is due to Plumstead [15], $n = m = 1$ and $f_1 = Y_1$ is due to Mandal [14] (he proved that P is cancellative), $m = 0$ is due to Rao [17] (he proved that P is cancellative) and Laurent polynomial case $f_i = Y_i$ is due to Lindel [13].

Heitmann ([9], Corollary 2.7) generalized Bass' result to all commutative non-Noetherian rings. It is natural to ask if analog of above results hold for non-Noetherian rings.

Let R be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n]$ be a polynomial ring in n variables $Y_1, \ldots, Y_n$ over R. Then Brewer-Costa [5] proved that all projective A-modules are free, generalizing the well known Quillen-Suslin theorem [16, 20] (see Ellouz-Lombardi-Yengui [8] for a constructive proof). Abedelfatah [2] generalized Brewer-Costa's result by proving that $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$. We generalize these results as follows (see 3.2, 3.3). This is non-Noetherian analog of author-Dhorajia's result in case $d = 0$.

Theorem 1.1 *Let R be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over R and $f_i \in R[Y_i]$. Then all projective A-modules are free and $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$.*

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Let R be a ring of dimension d and $n \geq d+2$. Then Yengui [23] proved that $E_n(R[Y])$ acts transitively on $\mathrm{Um}_n(R[Y])$ which is non-Noetherian analog of Plumstead's result in free case. Abedelfatah [1] proved that $E_n(R[Y,Y^{-1}])$ acts transitively on $\mathrm{Um}_n(R[Y,Y^{-1}])$ which is non-Noetherian analog of Mandal's result in free case. We generalize both results as follows (4.9). See (4.8) for definition of property $\Omega(R)$.

Theorem 1.2 *Let R be a ring of dimension d and A be one of $R[Y]$ or $R[Y,Y^{-1}]$, where Y is a variable over R. If P is a projective A-module of rank $\geq d+1$ satisfying property $\Omega(R)$, then $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$. In particular P is cancellative.*

We generalize (1.2) for Prüfer domain as follows (see 5.3): *Let R be a Prüfer domain of dimension d and $A = R[Y, f^{-1}]$, where Y is a variable over R and $f \in R[Y]$. If P is a projective A-module of rank $\geq d+1$, then $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$.*

2 Preliminaries

Let A be a ring, J an ideal of A and M an A-module. We say that $m \in M$ is *unimodular* if there exist $\phi \in M^* = Hom_A(M,A)$ such that $\phi(m) = 1$. The set of unimodular elements of M is denoted by $\mathrm{Um}(M)$. We write $\mathrm{Um}^1(A \oplus M, J)$ for the set of $(a,m) \in \mathrm{Um}(A \oplus M)$ such that $a \in 1+J$. We write $\mathrm{Um}(A \oplus M, J)$ for the set of $(a,m) \in \mathrm{Um}^1(A \oplus M, J)$ such that $m \in JM$. We write $\mathrm{Um}_r(A,J)$ for $\mathrm{Um}(A \oplus A^{r-1}, J)$.

The group of A-automorphism of M is denoted by $\mathrm{Aut}_A(M)$. We write $E^1(A \oplus M, J)$ for the subgroup of $\mathrm{Aut}_A(A \oplus M)$ generated by automorphisms $\Delta_{a\varphi}$ and Γ_m, where

$$\Delta_{a\varphi} = \begin{pmatrix} 1 & a\varphi \\ 0 & id_M \end{pmatrix} \quad \text{and} \quad \Gamma_m = \begin{pmatrix} 1 & 0 \\ m & id_M \end{pmatrix} \quad \text{with} \quad a \in J,\ \varphi \in M^*,\ m \in M.$$

We write $E^1(A \oplus M)$ for $E^1(A \oplus M, A)$. Let $E_{r+1}(A)$ denote the subgroup of $\mathrm{SL}_{r+1}(A)$ generated by elementary matrices $I + ae_{ij}$, where $a \in A$, $i \neq j$ and e_{ij} is the matrix with only non-zero entry 1 at (i,j)-th place. We write $E^1_{r+1}(A,J)$ for the subgroup of $E_{r+1}(A)$ generated by $\Delta_{\mathbf{a}}$ and $\Gamma_{\mathbf{b}}$, where

$$\Delta_{\mathbf{a}} = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & id_F \end{pmatrix} \quad \text{and} \quad \Gamma_{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ \mathbf{b}^t & id_F \end{pmatrix}, \quad \text{where} \quad F = A^r,\ \mathbf{a} \in JF,\ \mathbf{b} \in F.$$

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(m) = 0$. Let $\varphi_p \in End(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is an automorphism of M. The automorphism $1 + \varphi_p$ of M is called a *transvection* of M if either $p \in \mathrm{Um}(M)$ or $\varphi \in \mathrm{Um}(M^*)$. We write $E(M)$ for the subgroup of $\mathrm{Aut}(M)$ generated by transvections of M.

Due to following result of Bak-Basu-Rao ([3], theorem 3.10), we can interchange $E(A \oplus P)$ and $E^1(A \oplus P)$.

Theorem 2.1 *Let A be a ring and P a projective A-module of rank ≥ 2. Then $E^1(A \oplus P) = E(A \oplus P)$.*

The following result of Heitmann ([9], Corollary 2.7) generalizes Bass's cancellation [4] to non-Noetherian rings.

Theorem 2.2 *Let A be a ring of dimension d and P a projective A-module of rank $\geq d+1$. Then $E(A\oplus P)$ acts transitively on $\mathrm{Um}(A\oplus P)$. In particular P is cancellative.*

The following result of Brewer-Costa [5] generalizes Quillen-Suslin theorem [16, 20] to all zero-dimensional rings.

Theorem 2.3 *Let R be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n]$ a polynomial ring in n variables $Y_1, \ldots, Y_n$ over R. Then all projective A-modules are free.*

We will state five results which are proved with assumption that rings are Noetherian. But the same proof works for non-Noetherian rings.

Lemma 2.4 *([7], Remark 2.2) Let A be a ring, I an ideal of A and P a projective A-module. Then the natural map $E(A\oplus P) \to E(\frac{A\bigoplus P}{I(A\bigoplus P)})$ is surjective.*

Lemma 2.5 *([6], Lemma 3.1) Let A be a ring, J an ideal of A and P a projective A-module. Let "bar" denote reduction modulo the nil-radical of A. Assume $E^1(\overline{A}\oplus\overline{P}, \overline{J})$ acts transitively on $\mathrm{Um}^1(\overline{A}\oplus\overline{P}, \overline{J})$. Then $E^1(A\oplus P, J)$ acts transitively on $\mathrm{Um}^1(A\oplus P, J)$.*

Lemma 2.6 *([13], Lemma 1.1) Let A be a reduced ring and P an A-module. Assume $s \in A$ is a non-zerodivisor such that P_s is free of rank $r \geq 1$. Then there exist $p_1, \ldots, p_r \in P$, $\phi_1, \ldots, \phi_r \in P^*$ and $t \in \mathbb{N}$ such that*

(i) $s^t P \subset F$ and $s^t P^* \subset G$ with $F = \sum_1^r Ap_i$ and $G = \sum_1^r A\phi_i$.

(ii) $(\phi_i(p_j))_{1\leq i,j\leq r} =$ diagonal $(s^t, \ldots, s^t)$.

Lemma 2.7 *([6] Lemma 3.10) Let A be a reduced ring and P a projective A-module of rank r. Assume there exist a non-zerodivisor $s \in A$ such that P_s is free. Choose $p_1, \ldots, p_r \in P$, $\varphi_1, \ldots, \varphi_r \in P^*$ satisfying (2.6). Let $(a, p) \in \mathrm{Um}(A\oplus P, sA)$ with $p = c_1p_1 + \ldots + c_rp_r$, where $c_i \in sA$ for all i. Assume there exist $\phi \in E^1_{r+1}(A, sA)$ such that $\phi(a, c_1, \ldots, c_r) = (1, 0, \ldots, 0)$. Then there exist $\Phi \in E(A\oplus P)$ such that $\Phi(a, p) = (1, 0)$.*

Lemma 2.8 *([22], Lemma 4.2) Let A be a reduced ring and P an A-module. Assume there exist non-zerodivisors $s_1, \ldots, s_r \in A$, $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in P^*$ such that $(\phi_i(p_j))_{r\times r} =$ diagonal $(s_1, \ldots, s_r) := N$. Let $\mathcal{M}$ be the subgroup of $\mathrm{GL}_r(A)$ consisting of all matrices of the form $I + TN^2$ for $T \in M_r(A)$. Then the map*

$$\Phi : \mathcal{M} \to \mathrm{Aut}_A(P); \quad \Phi(I + TN^2) = id_P + (p_1, \ldots, p_r)\, T\, N\, (\phi_1, \ldots, \phi_r)^t$$

is a group homomorphism.

The following result is from Lam's book ([11], Proposition VI.1.14).

Proposition 2.9 *Let B be a ring and $a, b \in B$ two comaximal elements. Then for any $\sigma \in E_n(B_{ab})$ with $n \geq 3$, there exist $\alpha \in E_n(B_b)$ and $\beta \in E_n(B_a)$ such that $\sigma = (\alpha)_a(\beta)_b$.*

We state Quillen-Suslin theorem [16, 20]. Note that any commutative ring is a filtered union of Noetherian commutative rings. Hence following result will follow from Noetherian case.

Theorem 2.10 *Let R be a ring and P a projective $R[Y]$-module. Let $f \in R[Y]$ be a monic polynomial such that P_f is free. Then P is free.*

We state a result of Yengui [23] and Abedelfatah [1] respectively.

Theorem 2.11 *Let A be a ring of dimension d, Y a variable over A and $n \geq d+2$. Then*
(i) $E_n(A[Y])$ acts transitively on $\text{Um}_n(A[Y])$.
(ii) $E_n(A[Y, Y^{-1}])$ acts transitively on $\text{Um}_n(A[Y, Y^{-1}])$.

3 Zero dimension case

In this section we prove our first result.

Proposition 3.1 *Let $\Sigma(n)$ be set of rings which is closed w.r.t. following properties:*
(i) If $R \in \Sigma(n)$ and $0 \neq f \in R[Y]$ is non-unit, then $R[Y]_{f(1+fR[Y])} \in \Sigma(n)$.
(ii) If $R \in \Sigma(n)$, then all projective modules over $R[Y_1, \ldots, Y_n]$ are free, where $Y_1, \ldots, Y_n$ are variables over R.
Then, for $R \in \Sigma(n)$, all projective modules over $R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$ are free, where $m \leq n$ and $f_i \in R[Y_i]$.

Proof Let P be a projective $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$-module of rank r. If $m = 0$, then P is free by assumption (ii). Assume $m > 0$ and use induction on m. Write $C = R[Y_1, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$, $S = 1 + f_m R[Y_m]$ and $B = R[Y_m]_{f_m S}$. Then $A = C_{f_m}$, $B \in \Sigma(n)$ by assumption (i) and $S^{-1}A = B[Y_1, \ldots, Y_{m-1}, Y_{m+1}, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$. By induction on m, $S^{-1}P$ is free. Since P is finitely generated, we can find $g \in S$ such that P_g is free. Note that f_m and g are comaximal elements of $R[Y_m]$. Consider the fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & C_{f_m} = A \\ \downarrow & & \downarrow \\ C_g & \longrightarrow & C_{f_m g} = A_g \end{array} .$$

Patching projective modules P over C_{f_m} and $(C_g)^r$ over C_g, we get $P \xrightarrow{\sim} Q_{f_m}$, where Q is a projective C-module of rank r. By induction on m, projective modules over C are free. Hence Q is free and therefore P is free. ■

Let $\Sigma(n)$ be the set of rings of dimension 0. If $R \in \Sigma(n)$ and $0 \neq f \in R[Y]$ is a non-unit, then $\dim R[Y] = 1$ and $\dim R[Y]_{f(1+fR[Y])} = 0$. Hence $R[Y]_{f(1+fR[Y])} \in \Sigma(n)$. Using (2.3), projective modules over polynomial ring $R[Y_1, \ldots, Y_n]$ are free. Hence $\Sigma(n)$ satisfies hypothesis (i, ii) of (3.1). Therefore, we get the following generalization of (2.3).

Proposition 3.2 *Let R be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over R and $f_i \in R[Y_i]$. Then all projective A-modules are free.*

Theorem 3.3 *Let R be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over R and $f_i \in R[Y_i]$. Then $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$.*

Proof The case $m = 0$ is due to Abedelfatah [2]. Assume $m > 0$ and use induction on m. Let $v \in E_r(A)$. Write $C = R[Y_1, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$, $S = 1 + f_m R[Y_m]$ and $B = R[Y_m]_{f_m S}$. Then B is 0 dimensional, $A = C_{f_m}$ and $S^{-1}A = B[Y_1, \ldots, Y_{m-1}, Y_{m+1}, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$. By induction on m, $E_r(S^{-1}A)$ acts transitively on $\text{Um}_r(S^{-1}A)$. Hence there exist $\sigma \in E_r(S^{-1}A)$ such that $\sigma(v) = e_1 = (1, 0, \ldots, 0)$. We can find $g \in S$ and $\widetilde{\sigma} \in E_r(C_{f_m g})$ such that $\widetilde{\sigma}(v) = e_1$. Note that f_m and g are comaximal elements of $R[Y_m]$. Consider the fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & C_{f_m} = A \\ \downarrow & & \downarrow \\ C_g & \longrightarrow & C_{f_m g} = A_g \end{array}$$

By (2.9), $\widetilde{\sigma}$ has a splitting $\widetilde{\sigma} = (\alpha)_{f_m}(\beta)_g$, where $\alpha \in E_r(C_g)$ and $\beta \in E_r(C_{f_m})$. We have unimodular elements $\beta(v) \in \text{Um}_r(C_{f_m})$ and $\alpha^{-1}(e_1) \in \text{Um}_r(C_g)$ whose images in $C_{f_m g}$ are same. Hence patching $\beta(v)$ and $\alpha^{-1}(e_1)$, we get $w \in \text{Um}_r(C)$ such that its image in C_{f_m} is $\beta(v)$. By induction on m, $E_r(C)$ acts transitively on $\text{Um}_r(C)$. Hence there exist $\phi \in E_r(C)$ such that $\phi(w) = e_1$. If $\Phi_1 \in E_r(C_{f_m})$ is the image of ϕ, then $\Phi_1(\alpha(v)) = e_1$. Write $\Phi = \Phi_1 \alpha \in E_r(A)$, we are done. ∎

4 Main Theorem

The following result is proved in ([10], Lemma 3.3) with the assumption that ring is Noetherian. Using (2.8), same proof works for non-Noetherian ring. Hence we omit the proof.

Lemma 4.1 *Let A be a reduced ring and P a projective A-module of rank r. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume R^r is cancellative, where $R = A[X]/(X^2 - s^2 X)$. Then any element of $\text{Um}^1(A \oplus P, s^2 A)$ can be taken to $(1, 0)$ by some element of $\text{Aut}(A \oplus P, sA)$.*

An immediate consequence of (4.1) is the following result. Its proof is same as of ([10], Corollary 3.5) using (2.2).

Corollary 4.2 *Let A be a reduced ring of dimension d and P a projective A-module of rank d. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume R^d is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then P is cancellative.*

Let R be a ring and I an ideal of R. For $n \geq 3$, let $E_n(I)$ be the subgroup of $E_n(R)$ generated by $E_{ij}(a) = I + ae_{ij}$ with $a \in I$ and $1 \leq i \neq j \leq n$. Let $E_n(R, I)$ denote the normal closure of $E_n(I)$ in $E_n(R)$. We have two characterisation of $E_n(R, I)$ due to Suslin-Vaserstein [21] and Stein [19] respectively.

Proposition 4.3 *The kernel of the natural map $E_n(R) \to E_n(R/I)$ is isomorphic to $E_n(R, I)$.*

Proposition 4.4 *Consider the following fiber product diagram*

$$\begin{array}{ccc} R(I) & \xrightarrow{p_1} & R \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R & \xrightarrow[j_2]{} & R/I \end{array}$$

Then $E_n(R, I)$ is kernel of the natural surjection $E_n(p_1) : E_n(R(I)) \twoheadrightarrow E_n(R)$.

Using (4.3, 4.4, 2.7) and following the proof of ([7], Lemma 3.3), we get the following result. In [7], it is proved for Noetherian ring.

Lemma 4.5 *Let A be a reduced ring and P a projective A-module of rank r. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then any element of $\text{Um}(A \oplus P, s^2A)$ can be taken to $(1, 0)$ by some element of $E(A \oplus P)$.*

The proof of the following result is same as of ([7], Theorem 3.4) using (4.5, 2.2).

Proposition 4.6 *Let A be a reduced ring of dimension d and P a projective A-module of rank $r \geq d$. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Remark 4.7 By ([11], Exercise 2.34), any reduced ring R can be embedded in a reduced non-Noetherian ring S such that S equals the total quotient ring $Q(S)$ of S and R is a retract of S. In particular, if P is a non-free projective R-module, then $P \otimes_R S$ is a non-free projective S-module. Hence, if R is a reduced non-Noetherian ring and P a projective R-module, then we can not say that P_s is free, for some non-zerodivisor $s \in R$.

Definition 4.8 *Let $R \subset S$ be rings and P a projective S-module. We say that P satisfies property $\Omega(R)$ if for any ideal I of R and $\overline{P} = P/IP$, there exist a non-zerodivisor $\overline{t} \in R/I$ such that $\overline{P}_{\overline{t}}$ is free. The property $\Omega(R)$ avoids situation (4.7).*

The following result generalises (2.11).

Theorem 4.9 *Let R be a ring of dimension d and A is one of $R[Y]$ or $R[Y,Y^{-1}]$, where Y is a variable over R. Let P be a projective A-module of rank $r \geq d+1$ which satisfies property $\Omega(R)$. Then $E(A\oplus P)$ acts transitively on $\text{Um}(A\oplus P)$.*

Proof By (2.5), we may assume R is reduced. If $d = 0$, then P is free by (3.2) and we can use (2.11). Hence assume $d \geq 1$ and use induction on d. Since P satisfies property $\Omega(R)$, we can find a non-zerodivisor $s \in R$ such that P_s is free and (2.6) holds. If $R' = R[X]/(X^2 - s^2X)$, then $\dim R' = d$. Write $B = A[X]/(X^2 - s^2X)$. Then B is one of $R'[Y]$ or $R'[Y,Y^{-1}]$. By (2.11), $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$. Applying (4.5), we get every element of $\text{Um}(A\oplus P, s^2A)$ can be taken to $(1,0)$ by some element of $E(A\oplus P)$. Therefore it is enough to show that every element of $\text{Um}(A\oplus P)$ can be taken to an element of $\text{Um}(A\oplus P, s^2A)$ by some element of $E(A\oplus P)$.

Let "bar" denote reduction modulo s^2A. Then $\dim R/s^2R < d$. By assumption, P/s^2P satisfies property $\Omega(R/s^2R)$. Hence by induction on d, $E(\overline{A}\oplus\overline{P})$ acts transitively on $\text{Um}(\overline{A}\oplus\overline{P})$. Using (2.4), any element of $\text{Um}(A\oplus P)$ can be taken to an element of $\text{Um}(A\oplus P, s^2A)$ by $E(A\oplus P)$. This completes the proof. ∎

5 Some Auxiliary results

Lemma 5.1 *Let R be a ring of dimension d such that dimension of the polynomial ring $A = R[Y_1,\ldots,Y_n]$ is $d+n$. Then every stably free A-module P of rank $\geq d+1$ is free.*

Proof The case $n = 0$ is due to Heitmann (2.2). Assume $n > 0$ and use induction on n. Let S be the set of all monic polynomials in $R[Y_n]$. Then $\dim R[Y_n]_S = d$ and $\dim R[Y_n]_S[Y_1,\ldots,Y_{n-1}] = d+n-1$. Hence by induction on n, $S^{-1}P$ is free. By (2.10), P is free. ∎

Proposition 5.2 *Let R be a ring of dimension d such that dimension of the polynomial ring $R[Y_1,\ldots,Y_n]$ is $d+n$. Let $A = R[Y_1,\ldots,Y_n,(f_1\ldots f_m)^{-1}]$ with $m \leq n$ and $f_i \in R[Y_i]$ a monic polynomial for all i. Then every stably free A-module P of rank $r \geq d+1$ is free.*

Proof The case $m = 0$ follows from (5.1). Assume $m > 0$ and use induction on m. Let $C = R[Y_1,\ldots,Y_n,(f_1\ldots f_{m-1})^{-1}]$. If $S = 1+f_mR[Y_m]$, then $\dim R[Y_m]_{f_mS} = d$ (since $\dim R[Y_m] = d+1$) and $S^{-1}A = R[Y_m]_{f_mS}[Y_1,\ldots,Y_{m-1},Y_{m+1},\ldots,Y_n,(f_1\ldots f_{m-1})^{-1}]$. By induction on m, $S^{-1}P$ is free. Choose $g \in S$ such that P_g is free. Patching projective modules P and C_g^r over C_{f_mg}, we get a projective C-module Q such that $Q_{f_m} = P$. Since P is stably free, $(Q\oplus C^t)_{f_m}$ is free for some t. By (2.10), $Q\oplus C^t$ is free, i.e. Q is stably free. By induction on m, Q and hence P is free. ∎

It is natural to ask if all projective A-modules of rank $\geq d+1$ in (5.2) are cancellative. We give a partial answer.

Theorem 5.3 *Let R be an integral domain of dimension d such that* $\dim R[Y] = d+1$. *Let* $A = R[Y, f^{-1}]$ *with* $f \in R[Y]$ *and P a projective A-module of rank* $r \geq max\{2, d+1\}$. *Then* $E(A\oplus P)$ *acts transitively on* $\mathrm{Um}(A\oplus P)$.

Proof If $d = 0$, then P is free and we are done by (3.3). Assume $d \geq 1$. Choose $0 \neq s \in R$ such that (2.6) holds. Write $R' = R[X]/(X^2 - s^2X)$ and $B = R'[Y, f^{-1}]$. Assume $E_{r+1}(B)$ acts transitively on $\mathrm{Um}_{r+1}(B)$. By (4.5), any $(a,p) \in \mathrm{Um}(A\oplus P, s^2A)$ can be taken to $(1,0)$ by some element in $E(A\oplus P)$. Let "bar" denote reduction modulo s^2A. Then $\dim \overline{A} = d$ and rank $\overline{P} \geq d+1$. Applying (2.2), we get $E(\overline{A}\oplus\overline{P})$ acts transitively on $\mathrm{Um}(\overline{A}\oplus\overline{P})$. Using (2.4), every $V \in \mathrm{Um}(A\oplus P)$ can be taken to $W \in \mathrm{Um}(A\oplus P, s^2A)$ by some element of $E(A\oplus P)$. Therefore, it is enough to show that $E_{r+1}(B)$ acts transitively on $\mathrm{Um}_{r+1}(B)$.

Let $v \in \mathrm{Um}_{r+1}(B)$. If $C = R'[Y]$, then $B = C_f$. Since R' is an integral extension of R, $\dim R[Y] = d+1 = \dim R'[Y]$. Hence $\dim C_{f(1+fR'[Y])} = d$. Applying (2.2), we get $\sigma \in E_{r+1}(C_{f(1+fR'[Y])})$ such that $\sigma(v) = (1,\ldots,0)$. We can find $g \in 1 + fR'[Y]$ and $\widetilde{\sigma} \in E_{r+1}(C_{fg})$ such that $\widetilde{\sigma}(v) = (1,0\ldots,0)$.

By (2.9), $\widetilde{\sigma}$ has a splitting $\widetilde{\sigma} = (\alpha)_f(\beta)_g$, where $\alpha \in E_{r+1}(C_g)$ and $\beta \in E_{r+1}(C_f)$. We have two unimodular elements $\beta(v) \in \mathrm{Um}_{r+1}(C_f)$ and $\alpha^{-1}((1,0,\ldots,0)) \in \mathrm{Um}_{r+1}(C_g)$ whose images in C_{fg} are same. Hence, patching $\beta(v)$ and $\alpha^{-1}((1,0,\ldots,0))$, we get $w \in \mathrm{Um}_{r+1}(C)$ whose image in C_f is $\beta(v)$. By Yengui (2.11), $E_{r+1}(C)$ acts transitively on $\mathrm{Um}_{r+1}(C)$. Hence, we can find $\phi \in E_{r+1}(C)$ such that $\phi(w) = (1,0,\ldots,0)$. If Φ_1 is the image of ϕ in C_f, then $\Phi_1(\alpha(v)) = (1,0,\ldots,0)$ and $\Phi_1\alpha \in E_{r+1}(B)$. This completes the proof. ∎

Remark 5.4 (1) By a result of Seidenberg ([18], Theorem 4), if R is a Prüfer domain, then $\dim R[Y_1,\ldots,Y_n] = \dim R + n$. Hence (5.2, 5.3) holds for a Prüfer domain R and generalizes (4.9).

(2) Lequain-Simis have shown [12] that if R is a Prüfer domain, then projective modules over $R[Y_1,\ldots,Y_n]$ are extended from R. In particular, if R is a valuation domain (local Prüfer domain), then projective $R[Y_1,\ldots,Y_n]$-modules are free. It is natural to ask if projective modules over $R[Y_1,\ldots,Y_n,(f_1\ldots f_m)^{-1}]$ are free, where R is a valuation domain, $m \leq n$ and $f_i \in R[Y_i]$. If each f_i is a monic polynomial, then (5.2) gives a partial answer.

Proposition 5.5 *Let R be a valuation domain of dimension d and* $A = R[X, Y_1,\ldots,Y_n, f^{-1}]$ *with* $f \in R[X]$. *Then every stably free A-module P of rank* $\geq d+1$ *is free.*

Proof If $d = 0$, then P is free, by (3.2). Assume $d \geq 1$. Let $C = R[X, Y_1,\ldots,Y_n]$ and $S = 1+fR[X]$. Since $\dim R[X] = d+1$ by Seidenberg [18], $\dim R[X]_{fS} = d$ and $\dim R[X]_{fS}[Y_1,\ldots,Y_n] = d+n$. By (5.1), $S^{-1}P$ being stably free, is free. Choose $g \in S$ such that P_g is free. Patching projective modules P and $(C_g)^r$ over C_{fg}, we get a projective C-module Q such that $P \stackrel{\sim}{\to} Q_f$. By Lequain-Simis [12], every projective C-module is free. Therefore Q and hence P is free. ∎

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