K_0 of hypersurfaces defined by $x_1^2 + \ldots + x_n^2 = \pm 1$

Manoj K. Keshari[1] and Satya Mandal[2]

[1]Department of Mathematics, IIT Mumbai, Mumbai - 400076, India

[2]Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045

keshari@math.iitb.ac.in, mandal@math.ku.edu

Abstract: Let k be a field of characteristic $\neq 2$ and let $Q_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m) = x_1^2 + \ldots + x_n^2 - (y_1^2 + \ldots + y_m^2)$ be a quadratic form over k. Let $R(Q_{n,m}) = R_{n,m} = k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(Q_{n,m} - 1)$. In this note we will calculate $\widetilde{K}_0(R_{n,m})$ for every $n, m \geq 0$. We will also calculate $CH_0(R_{n,m})$ and the Euler class group of $R_{n,m}$ when $k = \mathbb{R}$.

1 Introduction

In this paper, k **will denote a field of characteristic** $\neq 2$. Let $A_{n,k} = k[x_1, \ldots, x_n]/(\sum_1^n x_i^2 - 1)$. It is well known (see [1]) that $\widetilde{K}_0(A_{n,\mathbb{R}})$ is periodic of period 8. More precisely, $\widetilde{K}_0(A_{n,\mathbb{R}})$ is $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ or 0 depending on whether n is $\{1, 5\}$, $\{2, 3\}$ or $\{0, 4, 6, 7\}$ modulo 8. Similarly, $\widetilde{K}_0(A_{n,\mathbb{C}})$ is periodic of period 2. More precisely, $\widetilde{K}_0(A_{n,\mathbb{C}})$ is $\mathbb{Z}$ or 0 depending on whether n is odd or even.

It will be interesting to know if $\widetilde{K}_0(A_{n,k})$ is also periodic for arbitrary field k. Further, if $\widetilde{A}_{n,k} = k[x_1, \ldots, x_n]/(\sum_1^n x_i^2 + 1)$, then we would like to know if $\widetilde{K}_0(\widetilde{A}_{n,k})$ is periodic. In this paper we answer these questions in affirmative.

Some experts may consider these results as easy computations. However, there is no written reference to these results. These results are derived by application of the celebrated results of Swan [8]. We are confident that this article will serve as valuable resource for the researchers and graduate students in this area.

For $R_{n,m} = k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(\sum_1^n x_i^2 - \sum_1^m y_j^2 - 1)$, we will prove following results.

Theorem 1.1 *Assume that $x^2 + y^2 + z^2 = 0$ has only trivial zero in k^3 (equivalently the quaternion algebra $\frac{(-1,-1)}{k}$ is a division algebra over k). Then $\widetilde{K}_0(R_{n,0})$ and $\widetilde{K}_0(R_{0,m})$ are periodic of period 8. More precisely,*

(1) *$\widetilde{K}_0(R_{n,0})$ is $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ or 0 depending on whether n is $\{1, 5\}$, $\{2, 3\}$ or $\{0, 4, 6, 7\}$ modulo 8.*

(2) *$\widetilde{K}_0(R_{0,m})$ is $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ or 0 depending on whether m is $\{3, 7\}$, $\{5, 6\}$ or $\{0, 1, 2, 4\}$ modulo 8.*

(3) *$\widetilde{K}_0(R_{n,m}) = \widetilde{K}_0(R_{n-m,0})$ if $n \geq m$ and $\widetilde{K}_0(R_{n,m}) = \widetilde{K}_0(R_{0,m-n})$ if $n < m$.*

Theorem 1.2 *Assume $\sqrt{-1} \in k$. Then $\widetilde{K}_0(R_{n,m})$ is $\mathbb{Z}$ or 0 depending on whether $n + m$ is odd or even.*

Theorem 1.3 *Assume that $\sqrt{-1} \notin k$ and -1 is a sum of two squares in k (equivalently, the quaternion algebra $(\frac{-1,-1}{k})$ is not a division algebra over k). Then $\widetilde{K}_0(R_{0,n})$ and $\widetilde{K}_0(R_{n,0})$ are periodic of period 4. More precisely,*

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(i) $\widetilde{K}_0(R_{0,n}) = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ *or* 0 *depending on whether* n *is* $\{3\}$, $\{2\}$ *or* $\{0,1\}$ *modulo* 4.
(ii) $\widetilde{K}_0(R_{n,0}) = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ *or* 0 *depending on whether* n *is* $\{1\}$, $\{2\}$ *or* $\{0,3\}$ *modulo* 4.
(iii) $\widetilde{K}_0(R_{n,m}) = \widetilde{K}_0(R_{n-m,0})$ *if* $n \geq m$ *and* $\widetilde{K}_0(R_{n,m}) = \widetilde{K}_0(R_{0,m-n})$ *if* $n < m$.

2 Preliminaries

We will recall some results from [7] for later use. If $q(x_1, \ldots, x_n)$ is a non-degenerate quadratic form over k, then $R(q)$ will denote the k-algebra $k[x_1, \ldots, x_n]/(q(x_1, \ldots, x_n) - 1)$ and $C(q)$ will denote the Clifford algebra of q.

If $q = a_1x_1^2 + \cdots + a_nx_n^2$ with $a_i \in k$, then $C(q)$ is generated by $e_1, \ldots, e_n$ with relations $e_ie_j + e_je_i = 0$ for $i \neq j$ and $e_i^2 = a_i$. The elements $e_{i_1} \cdots e_{i_r}$ with $1 \leq i_1 < \ldots < i_r \leq n$ form a k-base for $C(q)$. Further, define $det\, q := a_1 \ldots a_n$ and $ds\, q := (-1)^{n(n-1)/2} det\, q$.

A binary quadratic form is called hyperbolic if it has the form $h(x, y) = x^2 - y^2$. By a linear change of variables this is equivalent to $h'(x, y) = xy$.

Lemma 2.1 *([7], 8.1 and 8.2) If* b *is a binary quadratic form, then* $C(b \perp q) \xrightarrow{\sim} C(b) \otimes C((ds\, b)q)$. *In particular, if* h *is hyperbolic, then* $C(q \perp h) \xrightarrow{\sim} C(q) \otimes C(h)$.

Lemma 2.2 *([7], 8.3) (a) If* q *has even rank, then* $C(q)$ *is central simple over* k *and is a tensor product of quaternion algebras.*

(b) If q *has odd rank, then (i) if* $\sqrt{ds\, q} \in k$, *then* $C(q) = A \times A$, *where* A *is central simple over* k *and is a tensor product of quaternion algebras, (ii) otherwise* $C(q)$ *is simple with center* $k(\sqrt{ds\, q})$ *and is a tensor product of its center with quaternion algebras over* k.

It follows from (2.2) that all simple $C(q)$-modules have the same dimension over k. We denote this dimension by $d(q)$.

Lemma 2.3 *([7], Lemma 8.4) (a)* $d(q \perp 1)$ *is either* $d(q)$ *or* $2d(q)$.

(b) If $C(q) = A \times A$, *then* $d(q \perp 1) = 2d(q)$.

See [7] for the definition of $ABS(q)$.

Proposition 2.4 *([7], Proposition 8.5) (a) If* $C(q) = A \times A$, *i.e rank of* q *is odd and* $\sqrt{ds\, q} \in k$, *then* $ABS(q) = \mathbb{Z}$ *generated by either of the simple* $C(q)$*-modules.*

(b) If $C(q)$ *is simple, then (i)* $ABS(q) = 0$ *if* $d(q \perp 1) = d(q)$ *and (ii)* $ABS(q) = \mathbb{Z}/2\mathbb{Z}$ *if* $d(q \perp 1) = 2d(q)$.

We state the following result of Swan ([8], Corollary 10.8)

Theorem 2.5 *Assume that* R *is regular,* $1/2 \in R$ *and* $q \perp <-1>$ *is a non-singular quadratic form. Then* $ABS(q) \xrightarrow{\sim} K_0(R(q))/K_0(R)$.

In particular, if R *is a field, then* $ABS(q) \xrightarrow{\sim} \widetilde{K}_0(R(q))$.

Using (2.4 and 2.5), we get the following result which will be used later.

Theorem 2.6 *If $q(x_1, \ldots, x_n) \perp < -1 >$ is a non-singular quadratic form over k, then*

(i) If $C(q) = A \times A$ (i.e. rank of q is odd and $\sqrt{ds\, q} \in k$), then $\widetilde{K}_0(R(q)) = \mathbb{Z}$.

(ii) If $C(q)$ is simple, then (a) $\widetilde{K}_0(R(q)) = 0$ if $d(q \perp 1) = d(q)$ and (b) $\widetilde{K}_0(R(q)) = \mathbb{Z}/2\mathbb{Z}$ if $d(q \perp 1) = 2d(q)$.

3 Main Theorem

In this section, we fix quadratic forms $q_n = -(x_1^2 + \cdots + x_n^2)$ and $q'_n = x_1^2 + \cdots + x_n^2$ over k. We write C_n and C'_n for the Clifford algebras $C(q_n)$ and $C(q'_n)$. Then we have the following result. In ([1], Proposition 4.2), it is proved for $k = \mathbb{R}$, but the same proof works over any field k.

Proposition 3.1 *There exist isomorphisms $C_n \otimes_k C'_2 \xrightarrow{\sim} C'_{n+2}$ and $C'_n \otimes_k C_2 \xrightarrow{\sim} C_{n+2}$.*

3.1 -1 is not a sum of two squares in k

We begin with the following well known result (see [6], p. 15). For $a, b \in k$, the quaternion algebra $\frac{(a,b)}{k}$, which is a k-algebra defined by i and j with relations $i^2 = a$, $j^2 = b$ and $ij + ji = 0$, is a division algebra if and only if $x^2 = ay^2 + bz^2$ has only trivial zero.

In this section we will assume that $x^2 + y^2 + z^2 = 0$ has only trivial zero in k^3 which is same as the quaternion algebra $\frac{(-1,-1)}{k}$ is a division algebra over k (e.g. any real field). We denote the division algebra $\frac{(-1,-1)}{k}$ by $\mathcal{H}$. Let $\mathcal{C}$ be the subalgebra of $\mathcal{H}$ generated by i over k. Then $\mathcal{C} = k[x]/(x^2 + 1)$ is a field.

The following is a well known result. We will give proof for completeness. Recall that $F(n)$ denote the algebra of $n \times n$ matrices over F.

Lemma 3.2 *If F denote one of k, $\mathcal{C}$ or $\mathcal{H}$, then we have the following identities (i) $F(n) \xrightarrow{\sim} k(n) \otimes_k F$, (ii) $k(n) \otimes_k k(m) \xrightarrow{\sim} k(nm)$, (iii) $\mathcal{C} \otimes_k \mathcal{C} \xrightarrow{\sim} \mathcal{C} \oplus \mathcal{C}$, (iv) $\mathcal{H} \otimes_k \mathcal{C} \xrightarrow{\sim} \mathcal{C}(2)$, (v) $\mathcal{H} \otimes_k \mathcal{H} \xrightarrow{\sim} k(4)$.*

In particular, when $k = \mathbb{R}$ the field of real numbers, then $\mathcal{C} = \mathbb{C}$ and $\mathcal{H} = \mathbb{H}$.

Proof (*i*) and (*ii*) are straightforward.

(*iii*) The map $\mathcal{C} \oplus \mathcal{C} \to \mathcal{C} \otimes_k \mathcal{C}$ defined by $(1, 0) \mapsto 1/2(1 \otimes 1 + i \otimes i)$ and $(0, 1) \mapsto 1/2(1 \otimes 1 - i \otimes i)$ is an isomorphism.

(*iv*) Since $\mathcal{H}$ is a $\mathcal{C}$ vector space under left multiplication, the map $\pi : \mathcal{C} \times \mathcal{H} \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H})$ defined by $\pi_{y,z}(x) = yx\overline{z}$ is k-bilinear, where $y \in \mathcal{C}$, $x, z \in \mathcal{H}$ and $\overline{z} = a1 - bi - cj - dij$ is the conjugate of $z = a1 + bi + cj + dij$ with $a, b, c, d \in k$. Hence, we get a k-linear map $\pi : \mathcal{C} \otimes_k \mathcal{H} \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H})$. Since $\pi_{y,z} \circ \pi_{y',z'} = \pi_{yy',zz'}$, the map π is an k-algebra homomorphism. Further, it is easy to see that π is injective. Since $\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H}) \xrightarrow{\sim} \mathcal{C}(2)$, we get $\dim_k \mathcal{C} \otimes_k \mathcal{H} = 8 = \dim_k \mathcal{C}(2)$ (note that $\dim_{\mathcal{C}} \mathcal{C}(2) = 4$). Hence π is an isomorphism.

(*v*) Define a map $\pi : \mathcal{H} \times \mathcal{H} \to \mathrm{Hom}_k(\mathcal{H}, \mathcal{H})$ by $\pi_{y,z}(x) = yx\overline{z}$, where $y, x, z \in \mathcal{H}$. Then π is k-bilinear. Hence it induces a k-linear map $\pi : \mathcal{H} \otimes_k \mathcal{H} \to \mathrm{Hom}_k(\mathcal{H}, \mathcal{H})$, which is an algebra homomorphism ($\pi_{y,z} \circ \pi_{y',z'} = \pi_{yy',zz'}$). Further, π is injective. Since both sides are vector spaces of dimension 16 over k, π is an isomorphism. Note that $\mathrm{Hom}_k(\mathcal{H}, \mathcal{H}) \tilde{\to} k(4)$. This proves the result. $\square$

Let us begin the proof of our first result. It is easy to see that $C_1 = \mathcal{C}$, $C_2 = \mathcal{H}$, $C_1' = k \oplus k$ and $C_2' = k(2)$. Using (3.1), we get that

n	C_n	C_n'	$d(q_n)$	$d(q_n')$
1	$\mathcal{C}$	$k\oplus k$	2	1
2	$\mathcal{H}$	$k(2)$	4	2
3	$\mathcal{H}\oplus\mathcal{H}$	$\mathcal{C}(2)$	4	4
4	$\mathcal{H}(2)$	$\mathcal{H}(2)$	8	8
5	$\mathcal{C}(4)$	$\mathcal{H}(2)\oplus\mathcal{H}(2)$	8	8
6	$k(8)$	$\mathcal{H}(4)$	8	16
7	$k(8)\oplus k(8)$	$\mathcal{C}(8)$	8	16
8	$k(16)$	$k(16)$	16	16

Note that $C_4 \tilde{\to} C_4'$, $C_{n+4} \tilde{\to} C_n \otimes_k C_4$, $C_{n+8} \tilde{\to} C_n \otimes C_8$. Further $C_8 \tilde{\to} k(16)$. Hence, if $C_n = F(m)$, then $C_{n+8} \tilde{\to} F(16m)$. Similarly, if $C_n' = F(m)$, then $C_{n+8}' = F(16m)$.

If $h = x^2 - y^2$, then $C(h) \tilde{\to} k(2)$. From (2.1), if $h^r = h \perp \ldots \perp h$ (r times), then $C(h^r) = k(2) \otimes \ldots \otimes k(2) \tilde{\to} k(2^r)$. Now, if $C(q) = F(m)$, then $C(q \perp h^r) = F(m) \otimes k(2^r) \tilde{\to} F(2^r m)$.

Since $q_k \perp 1 = q_{k-1} \perp h$, $C(q_k \perp 1) = C(q_{k-1}) \otimes k(2)$. Write $q_k \perp 1$ as $\widetilde{q}_k$. Further $q_k' \perp 1 = q_{k+1}'$. Hence $C(q_k' \perp 1) = C(q_{k+1}')$ and $d(q_k' \perp 1) = d(q_{k+1}')$.

Write $s = 16^r$. Then we have the following table.

n	C_{8r+n}	C_{8r+n}'	$C(\widetilde{q}_{8r+n})$	$d(q_{8r+n})$	$d(q_{8r+n}')$	$d(\widetilde{q}_{8r+n})$
1	$\mathcal{C}(s)$	$k(s)^2$	$k(2s)$	$2s$	s	$2s$
2	$\mathcal{H}(s)$	$k(2s)$	$\mathcal{C}(2s)$	$4s$	$2s$	$4s$
3	$\mathcal{H}(s)^2$	$\mathcal{C}(2s)$	$\mathcal{H}(2s)$	$4s$	$4s$	$8s$
4	$\mathcal{H}(2s)$	$\mathcal{H}(2s)$	$\mathcal{H}(2s)^2$	$8s$	$8s$	$8s$
5	$\mathcal{C}(4s)$	$\mathcal{H}(2s)^2$	$\mathcal{H}(4s)$	$8s$	$8s$	$16s$
6	$k(8s)$	$\mathcal{H}(4s)$	$\mathcal{C}(8s)$	$8s$	$16s$	$16s$
7	$k(8s)^2$	$\mathcal{C}(8s)$	$k(16s)$	$8s$	$16s$	$16s$
8	$k(16s)$	$k(16s)$	$k(16s)^2$	$16s$	$16s$	$16s$

Using (2.6), we get the following result.

Theorem 3.3 *Assume $x^2 + y^2 + z^2 = 0$ has only trivial zero in k^3. Note $R_{0,n} = R(q_n)$ and $R_{n,0} = R(q_n')$. Then*

(1) $\widetilde{K}_0(R_{n,0})$ *is* $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ *or* 0 *depending on whether* n *is* $\{1,5\}$, $\{2,3\}$ *or* $\{0,4,6,7\}$ *modulo* 8.

(2) $\widetilde{K}_0(R_{0,m})$ *is* $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ *or* 0 *depending on whether* m *is* $\{3,7\}$, $\{5,6\}$ *or* $\{0,1,2,4\}$ *modulo* 8.

For n, m be positive integers, consider $Q_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_1^n x_i^2 - \sum_1^m y_i^2$.

Assume $n \geq m$. Then $Q_{n,m} \xrightarrow{\sim} q'_{n-m} \perp h^m$ and $C(Q_{n,m}) \xrightarrow{\sim} C'_{n-m} \otimes k(2^m)$. Hence $d(Q_{n,m}) = d(q'_{n-m})2^m$. Further, $Q_{n,m} \perp 1 \xrightarrow{\sim} q'_{n-m+1} \perp h^m$ and $d(Q_{n,m} \perp 1) = d(q'_{n-m+1})2^m$. Hence $d(Q_{n,m} \perp 1)/d(Q_{n,m}) = d(q'_{n-m})/d(q'_{n-m+1})$.

Assume $n < m$. Then $Q_{n,m} \xrightarrow{\sim} q_{m-n} \perp h^n$ and $Q_{n,m} \perp 1 \xrightarrow{\sim} q_{m-n-1} \perp h^{n+1}$. Further $C(Q_{n,m}) \xrightarrow{\sim} C(q_{m-n}) \otimes k(2^n)$ and $C(Q_{n,m} \perp 1) = C(q_{m-n-1}) \otimes k(2^{n+1})$. Hence, $d(Q_{n,m}) = d(q_{m-n})2^n$ and $d(Q_{n,m} \perp 1) = d(q_{m-n-1})2^{n+1}$. The quotient $d(Q_{n,m} \perp 1)/d(Q_{n,m})$ is equal to $2d(q_{m-n-1})/d(q_{m-n})$. Using (2.6), we get

Theorem 3.4 *Assume $x^2 + y^2 + z^2 = 0$ has only trivial zero in k^3. Then $\widetilde{K}_0(R(Q_{n,m}))$ is same as $\widetilde{K}_0(R(q'_{n-m}))$ when $n \geq m$ and $\widetilde{K}_0(R(q_{m-n}))$ when $n < m$.*

Remark 3.5 We note that the following classical result generalizes (3.4) (see [7], 10.1). Let $f \in k[x_1, \ldots, x_n]$ be non-zero. Let $A = k[x_1, \ldots, x_n]/(f)$ and $B = k[x_1, \ldots, x_n, u, v]/(f + uv)$. Then $\widetilde{G}_0(A) \xrightarrow{\sim} \widetilde{G}_0(B)$. However, for a regular ring R, it is well known that $\widetilde{G}_0(R) \xrightarrow{\sim} \widetilde{K}_0(R)$. In this paper, we have computed $\widetilde{K}_0(R(q_n))$ explicitly.

3.2 $\sqrt{-1} \in k$, i.e. -1 is a square in k

In this case $C_n \xrightarrow{\sim} C'_n$. Further, using (3.1), we get $C_{n+2} \xrightarrow{\sim} C_n \otimes C_2$. Since $C_1 = k \oplus k$ and $C_2 = k(2)$, we get $C_{2n} = k(2^n)$ and $C_{2n+1} = k(2^n) \oplus k(2^n)$. Therefore, by (2.6), we get the following result.

Theorem 3.6 *If $\sqrt{-1} \in k$, then $\widetilde{K}_0(R(q_{2n})) = 0$ and $\widetilde{K}_0(R(q_{2n+1})) = \mathbb{Z}$.*

3.3 -1 is a sum of two squares and $\sqrt{-1} \notin k$

Assume $\sqrt{-1} \notin k$ but $x^2+y^2+z^2 = 0$ has a non-trivial zero in k^3. We denote the field $k[x]/(x^2+1)$ by $\mathcal{C}$. Recall that a quaternion algebra $(\frac{a,b}{k})$ is isomorphic to $M_2(k)$ if and only if it is not a division algebra.

It is easy to see that $C_1 = \mathcal{C}$, $C'_1 = k \oplus k$, $C_2 = k(2) = C'_2$. Further, $C_3 = C'_1 \otimes C_2 = k(2) \oplus k(2)$, $C'_3 = C_1 \otimes C'_2 = \mathcal{C}(2)$ and $C_4 = C'_2 \otimes C_2 = k(4) = C'_4$.

For $n = 4r + i$, where $i \in \{1, 2, 3, 4\}$, we have $C_n = C'_{n-2} \otimes C_2 = C_{n-4} \otimes C'_2 \otimes C_2 = C_{n-4} \otimes C_4 = C_{n-4} \otimes k(4) = \ldots = C_i \otimes k(4^r)$. Similarly, $C'_n = C'_i \otimes k(4^r)$.

Write $s = 4^r$. Then we have the following table.

n	C_{4r+n}	C'_{4r+n}	$C(q_{4r+n} \perp 1)$	$d(q_{4r+n})$	$d(q'_{4r+n})$	$d(q_{4r+n} \perp 1)$
1	$\mathcal{C}(s)$	$k(s)^2$	$k(2s)$	$2s$	s	$2s$
2	$k(2s)$	$k(2s)$	$\mathcal{C}(2s)$	$2s$	$2s$	$4s$
3	$k(2s)^2$	$\mathcal{C}(2s)$	$k(4s)$	$2s$	$4s$	$4s$
4	$k(4s)$	$k(4s)$	$k(4s)^2$	$4s$	$4s$	$4s$

By (2.6), we get the following result.

Theorem 3.7 *Assume $\sqrt{-1} \notin k$ and -1 is a sum of two squares in k. Let $R_{n,m} = k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(\sum_1^n x_i^2 - \sum_1^m y_j^2 - 1)$. Then*

(i) $\widetilde{K}_0(R_{0,n}) = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ or 0 depending on whether n is $\{3\}$, $\{2\}$ or $\{0,1\}$ modulo 4.

(ii) $\widetilde{K}_0(R_{n,0}) = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ or 0 depending on whether n is $\{1\}$, $\{2\}$ or $\{0,3\}$ modulo 4.

(iii) $\widetilde{K}_0(R_{n,m}) = \widetilde{K}_0(R_{n-m,0})$ if $n \geq m$ and $\widetilde{K}_0(R_{n,m}) = \widetilde{K}_0(R_{0,m-n})$ if $n < m$.

4 Some Auxiliary Results

1. Let $A = \mathbb{R}[x_0, \ldots, x_n]/(a_0x_0^2 + \ldots + a_nx_n^2 - b)$ with $a_i, b \in \mathbb{R}$ and let $E(A)$ be the Euler class group of A with respect to A (see [3] for definition). Let $E^{\mathbb{C}}(A)$ be the subgroup of $E(A)$ generated by all the complex maximal ideals of A. By ([4], Lemma 4.2), all the complex maximal ideals of A are generated by n elements, hence $E^{\mathbb{C}}(A) = 0$. Using ([5], Theorem 2.3), we get the following results.

   (*i*) $E(A) \xrightarrow{\sim} E(\mathbb{R}(X))$, where $X = \text{Spec}\,(A)$ and $\mathbb{R}(X)$ is the localization A_S of A with S as the set of all elements of A which do not have any real zero.

   (*ii*) $CH_0(A) \xrightarrow{\sim} CH_0(\mathbb{R}(X))$.

   Further, there is a natural surjection $E(A) \twoheadrightarrow CH_0(A)$.

2. Assume that $A = \mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 + 1)$. Then A has no real maximal ideal and hence $E(A) = E^{\mathbb{C}}(A) = 0$ and hence $CH_0(A) = 0$.

   For $A = \mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 - 1)$, it is known that $E(A) = \mathbb{Z}$ and $CH_0(A) = \mathbb{Z}/2\mathbb{Z}$.

3. Assume $A = \mathbb{R}[x_0, \ldots, x_n]/(\sum_0^m x_i^2 - \sum_{m+1}^n x_i^2 - 1)$ with $m < n$ and $X = \text{Spec}\,(A)$. Then $X(\mathbb{R})$ has no compact connected component. Hence, by ([2], Theorem 4.21), $E(\mathbb{R}(X)) = 0$. From above, we get $E(A) = 0$ and $CH_0(A) = 0$.

4. In general, let $A = \mathbb{R}[x, y, z_1, \ldots, z_n]/(xy + f(z_1, \ldots, z_n))$ and let $X = \text{Spec}\,(A)$. Then $X(\mathbb{R})$ has no compact connected component. All the connected components of $X(\mathbb{R})$ is unbounded. For this, note that if $(a, b, c_1, \ldots, c_n) \in X(\mathbb{R})$, then $f(c_1, \ldots, c_n) = -ab$ and if (x_0, y_0) is any point on the hyperbola $xy = ab$, then $(x_0, y_0, c_1, \ldots, c_n) \in X(\mathbb{R})$.

   By ([2], Theorem 4.21), $E(\mathbb{R}(X)) = 0$ and hence $E(A) = E^{\mathbb{C}}(A)$. Using ([5], Theorem 2.3), we get $E(A) \xrightarrow{\sim} CH_0(A)$. Further, it is known (see [3], Theorem 5.5) that for a smooth affine domain A of dimension ≥ 2 over $\mathbb{R}$, $CH_0(A) \xrightarrow{\sim} E_0(A)$, the weal Euler class group of A. Hence $E(A) \xrightarrow{\sim} E_0(A) \xrightarrow{\sim} CH_0(A)$ and $E(A)$ is generated by complex maximal ideals of A. In particular, if all the complex maximal ideals of A are generated by n elements, then $E(A) = 0$ as is the case in (2) above.

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