Cancellation problem for projective modules over affine algebras

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Abstract: Let A be an affine algebra of dimension n over an algebraically closed field k with $1/n! \in k$. Let P be a projective A-module of rank $n-1$. Then, it is an open question due to N. Mohan Kumar, whether P is cancellative. We prove the following results: (i) If $A = R[T, T^{-1}]$, then P is cancellative.

(ii) If $A = R[T, 1/f]$ or $A = R[T, f_1/f, \ldots, f_r/f]$, where $f(T)$ is a monic polynomial and $f, f_1, \ldots, f_r$ is $R[T]$-regular sequence, then A^{n-1} is cancellative. Further, if $k = \overline{\mathbb{F}}_p$, then P is cancellative.

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1 Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated.

Let A be a ring of dimension d and let P be a projective A-module of rank n. We say that P is *cancellative* if $P \oplus A^m \xrightarrow{\sim} Q \oplus A^m$ for some projective A-module Q implies $P \xrightarrow{\sim} Q$.

A classical result of Bass (2.2) says that if $n > d$, then P is cancellative. It is well known that Bass' result is best possible in general (since tangent bundle of real 2-sphere is stably trivial but not trivial). However, Bass' result can be improved in some specific cases which we describe below. Recall that a field k is called a C_1-field if every homogeneous polynomial $F(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ of degree d has a non-trivial solution in k^n if $n > d$. Some examples of C_1-fields are finite fields $\mathbb{F}_p$ (Chevalley's theorem) and $\overline{k}(T)$, where $\overline{k}$ is an algebraically closed field (Tsen's theorem). Infact, it is easy to see that any field $k \subset \overline{\mathbb{F}}_p$ is a C_1-field.

Theorem 1.1 (i) *If A is an affine algebra of dimension d over an algebraically closed field, then Suslin [21] proved that every projective A-module of rank $\geq d$ is cancellative.*

(ii) *If A is an affine algebra of dimension d over an infinite perfect C_1-field k such that $1/d! \in k$, then Suslin [20] proved that A^d is cancellative. Subsequently, Bhatwadekar ([3], Theorem 4.1) proved that every projective A-module of rank d is cancellative.*

(iii) *If A is an affine algebra of dimension d over $\mathbb{Z}$, then Vaserstein ([24], Corollary 18.1, Theorem 18.2) proved that A^d is cancellative. Subsequently, Mohan Kumar, Murthy and Roy ([12], Corollary 2.5) proved that every projective A-module of rank d is cancellative.*

We note that in (1.1 (ii)), Bhatwadekar's proof [3] uses Suslin's result [20] that A^d is cancellative. Similarly, in (1.1 (iii)), the proof of Mohan Kumar, Murthy and Roy [12] uses Vaserstein's results [24]. Hence, in view of the above results, we can ask the following:

Question 1.2 *Let A be a ring of dimension d. Assume that A^d is cancellative. Is every projective A-module of rank d cancellative?*

In ([2], Example 3.11), Bhatwadekar has given an example of a smooth real affine surface A such that A^2 is cancellative, but $K_A \oplus A$ is not cancellative, where K_A is the canonical module of A. Thus, the above question has negative answer in general. We will modify the above question and prove the following result (3.6).

Theorem 1.3 *Let A be a ring of dimension d. Assume that for every finite extension R of A, R^d is cancellative. Then every projective A-module of rank d is cancellative.*

For a ring k, a finite extension of an affine k-algebra is an affine k-algebra. Hence, in (1.1 (*ii*)), assuming the result of Suslin [20], our result (1.3) gives an alternative proof of Bhatwadekar's result [3]. Similarly, in (1.1 (*iii*)), assuming the result of Vaserstein [24], (1.3) gives an alternative proof of Mohan Kumar, Murthy and Roy's result [12].

Regarding question (1.2), Bhatwadekar ([2], Proposition 3.7) proved the following interesting result: Let A be a ring of dimension 2 and let P be a projective A-module of rank 2. If $\wedge^2(P) \oplus A$ is cancellative, then P is cancellative. In particular, if A^2 is cancellative, then every projective A-module of rank 2 with trivial determinant is cancellative. In view of this result, Bhatwadekar ([4], Question VII) asked the following question which is open for $d \geq 3$.

Question 1.4 *Let A be a ring of dimension d. Assume that A^d is cancellative. Is every projective A-module of rank d with trivial determinant cancellative?*

In [11], Mohan Kumar has given an example of a smooth affine algebra of dimension $n \geq 4$ over which there exist projective modules of rank $n-2$ that are not cancellative. More precisely, he proved the following: let p be a prime integer and let k be any algebraically closed field. Then there exists an $f \in A = k[X_1, \ldots, X_{p+2}]$ and a projective A_f-module P of rank p such that $P \oplus A_f \xrightarrow{\sim} A_f^{p+1}$ but $P \not\xrightarrow{\sim} A_f^p$, i.e. P is not cancellative.

In view of the above results, the only case remaining regarding cancellation problem is when rank $P = \dim A - 1$. We state the following question of Mohan Kumar.

Question 1.5 *Let A be an affine algebra of dimension $n \geq 3$ over an algebraically closed field k. Let P be a projective A-module of rank $n-1$. Is P cancellative?*

This is not known even when $n = 3$ and $P = A^2$. We prove the following result (3.8) which is analogue of (1.3) for affine algebras over $\overline{\mathbb{F}}_p$.

Theorem 1.6 *Let A be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Assume that if R is a finite extension of A, then R^{d-1} is cancellative. Then every projective A-module of rank $d-1$ is cancellative.*

Using above result, we prove the following (4.6): Let R be an affine $\overline{\mathbb{F}}_p$-algebra of dimension d, where $p > d$. Let $f(T) \in R[T]$ be a monic polynomial. Assume that either $A = R[T, 1/f(T)]$ or $A = R[T, f_1/f, \ldots, f_r/f]$ for some $f_1, \ldots, f_r \in R[T]$. Then every projective A-module of rank d is cancellative.

Let R be an affine algebra of dimension $d-1$ over an algebraically closed field k with $1/(d-1)! \in k$. Then Wiemers (2.12) proved that projective $R[X]$-modules of rank $d-1$ are cancellative, thus answering question (1.5) in affirmative in the polynomial ring case. We prove the following two results (4.1) and (6.3, 6.5) which answers question (1.5) in affirmative in some special cases.

Theorem 1.7 *Let k be an algebraically closed field with $1/d! \in k$ and let R be an affine k-algebra of dimension d. Assume that $f(T) \in R[T]$ is a monic polynomial and either*

(i) $A = R[T, 1/f]$ or

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r \in R[T]$ is a regular sequence.

Then A^d is cancellative.

Theorem 1.8 *Let R be an affine algebra of dimension d over an algebraically closed field k with $1/d! \in k$. Let P be a projective $R[X, X^{-1}]$-module of rank d. Then*

(i) P is cancellative and

(ii) the natural map $\text{Aut}\,(P) \to \text{Aut}\,(P/(X-1)P)$ is surjective.

We also prove the analogue of above results for affine algebras over real closed fields (5.2 and 5.3). Note that *(ii)* extends our earlier result ([8], Theorem 4.7), where it was proved for projective A-modules which are extended from R under the assumption that R is smooth. Recall that a field k is called a *real field* if -1 is not a sum of squares in k. Further, a real field k is called a *real closed field* if it has no nontrivial real algebraic extension $k_1 \supset k$, $k_1 \neq k$. By ([6], Theorem 1.2.2), a real field k is a real closed field iff $k[X]/(X^2+1)$ is an algebraically closed field.

Theorem 1.9 *Let k be a real closed field and let R be an affine k-algebra of dimension $d-1$. Assume that $f(T) \in R[T]$ is a monic polynomial which does not belongs to any real maximal ideal. Assume that either*

(i) $A = R[T, 1/f(T)]$ or

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a $R[T]$-regular sequence.

Then every projective A-module of rank d is cancellative. Further, if $R = B[X]$, then A^{d-1} is also cancellative in (i, ii).

2 Preliminaries

Let B be a ring and let P be a projective B-module. Recall that $p \in P$ is called a *unimodular element* if there exists a $\psi \in P^* = \text{Hom}_B(P, B)$ such that $\psi(p) = 1$. We denote by $\text{Um}(P)$, the set of all unimodular elements of P. We write $O(p)$ for the ideal of B generated by $\psi(p)$ for all $\psi \in P^*$. Note that, if $p \in \text{Um}(P)$, then $O(p) = B$. For an ideal $J \subset B$, we denote by $\text{Um}^1(B \oplus P, J)$, the set of all $(a, p) \in \text{Um}(B \oplus P)$ such that $a \in 1 + J$. By $\text{Um}(B \oplus P, J)$, we denote the set of all $(a, p) \in \text{Um}^1(B \oplus P, J)$ such that $p \in JP$.

Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p of P as the composite $P \overset{\varphi}{\to} B \overset{p}{\to} P$. If $\varphi(p) = 0$, then ${\varphi_p}^2 = 0$ and hence $1 + \varphi_p$ is a unipotent automorphism of P. By a *transvection*, we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either $\varphi \in \text{Um}(P^*)$ or $p \in \text{Um}(P)$. We denote by $E(P)$, the subgroup of $\text{Aut}\,(P)$ generated by all transvections of P. Note that $E(P)$ is a normal subgroup of $\text{Aut}\,(P)$.

An existence of a transvection of P pre-supposes that P has a unimodular element. Let $P = B \oplus Q$, $q \in Q, \alpha \in Q^*$. Then the automorphisms Δ_q and Γ_α of P defined by $\Delta_q(b, q') = (b, q' + bq)$ and

$\Gamma_\alpha(b,q') = (b+\alpha(q'),q')$ are transvections of P. Conversely, any transvection Θ of P gives rise to a decomposition $P = B\oplus Q$ in such a way that $\Theta = \Delta_q$ or $\Theta = \Gamma_\alpha$.

For an ideal $J \subset B$, we denote by $EL^1(B\oplus P, J)$, the subgroup of $E(B\oplus P)$ generated by Δ_q and $\Gamma_{a\phi}$, where $q \in P, a \in J, \phi \in P^*$. By $E(B\oplus P, J)$, we denote the subgroup of $E(B\oplus P)$ generated by Δ_q and $\Gamma_{a\phi}$, where $q \in JP, a \in J, \phi \in P^*$.

We begin by stating two classical results due to Serre [18] and Bass [1] respectively.

Theorem 2.1 *Let A be a ring of dimension d. Then any projective A-module of rank $> d$ has a unimodular element. In particular, if $\dim A = 1$, then any projective A-module with trivial determinant is free.*

Theorem 2.2 *Let A be a ring of dimension d and let P be a projective A-module of rank $> d$. Then $E(A\oplus P)$ acts transitively on $\text{Um}(A\oplus P)$. In particular, P is cancellative.*

The following result is due to Quillen [16] and Suslin [23].

Theorem 2.3 *Let A be a ring and let P be a projective $A[T]$-module. Assume that P_f is free for some monic polynomial $f \in A[T]$. Then P is free.*

The following two results are due to Lindel ([10], Theorem 2.6 and Lemma 1.1).

Theorem 2.4 *Let A be a ring of dimension d and $R = A[X_1,\ldots,X_n,Y_1^{\pm 1},\ldots,Y_m^{\pm 1}]$. Let P be a projective R-module of rank $\geq$ max $(2, d+1)$. Then $E(R\oplus P)$ acts transitively on $\text{Um}(R\oplus P)$. In particular, projective R-modules of rank $> d$ are cancellative.*

Lemma 2.5 *Let A be a ring and let P be a projective A-module of rank r. Then there exists $s \in A$ such that the following holds:*

(i) P_s is free,
(ii) there exists $p_1,\ldots,p_r \in P$ and $\phi_1,\ldots,\phi_r \in \text{Hom}(P,A)$ such that $(\phi_i(p_j)) =$ diagonal $(s,\ldots,s)$,
(iii) $sP \subset p_1A + \ldots + p_rA$,
(iv) the image of s in A_{red} is a non-zero-divisor and
(v) $(0 : sA) = (0 : s^2A)$.

The following two results are due to Bhatwadekar and Roy ([5], Theorem 3.1 and Proposition 4.1).

Theorem 2.6 *Let R be a ring of dimension d and let $A = R[X_1,\ldots,X_n]$. Then every projective A-module of rank $> d$ has a unimodular element. In particular, if $d = 1$, then every projective A-module with trivial determinant is free.*

Proposition 2.7 *Let A be a ring and let I be an ideal of A. Let P be a projective A-module. Then any transvection of P/IP can be lifted to an automorphism of P.*

The following result is due to Mohan Kumar, Murthy and Roy ([12], Theorem 2.4).

Theorem 2.8 *Let A be a finitely generated ring of dimension $d \geq 2$ and let $I \subset A$ be an ideal. Let P be a projective A-module of rank d. Then $E(A\oplus P, I)$ acts transitively on $\text{Um}(A\oplus P, I)$.*

Remark 2.9 Let $K \subset \overline{\mathbb{F}}_p$ be a field and let $A = K[X_1, \ldots, X_n]/J$ be an affine algebra of dimension $d \geq 2$. Let $I \subset A$ be an ideal and let P be a projective A-module of rank d. We can choose a finite field $k \subset K$, ideals $J_0 \subset k[X_1, \ldots, X_n]$ and $I_0 \subset B = k[X_1, \ldots, X_n]/J_0$ such that $I = I_0 \otimes_k K$, $A = B \otimes_k K$ and $P = P_0 \otimes_k K$, where P_0 is a projective B-module. Applying (2.8), $E(B \oplus P_0, I_0)$ acts transitively on $\text{Um}(B \oplus P_0, I_0)$. Hence $E(A \oplus P, I)$ acts transitively on $\text{Um}(A \oplus P, I)$. In particular, projective A-modules of rank d are cancellative.

The following result is due to Wiemers ([26], Theorem 3.2, Corollary 3.4).

Proposition 2.10 *Let R be a ring of dimension d and $A = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Let P be a projective A-module of rank $\geq$ max $(2, d+1)$ and $I = (Y_m - 1)A$. Then*

(i) $EL^1(A \oplus P, I)$ acts transitively on $\text{Um}^1(A \oplus P, I)$.

(ii) the natural map $\text{Aut}_A(P) \to \text{Aut}_{A/I}(P/IP)$ is surjective.

We state two results due to Wiemers ([26], Lemma 4.2 and Theorem 4.3) respectively which are very crucial for our results.

Proposition 2.11 *Let A be a ring and let P be an A-module (need not be projective). Assume that there exists $p = [p_1, \ldots, p_n] \in \text{Hom}_A(A^n, P)$, $\phi = [\phi_1, \ldots, \phi_n]^t \in \text{Hom}_A(P, A^n)$ and $s_1, \ldots, s_n \in A$ such that*

(i) $(0 : s_i) = (0 : s_i^2)$ for $i = 1, \ldots, n$,

(ii) $(\phi_i(p_j))_{n \times n} =$ diagonal $(s_1, \ldots, s_n) = N$.

Let $\mathcal{M}$ be the subgroup of $\text{GL}_n(A)$ consisting of all matrices $1_n + T.N^2$ for some matrix T. Then the map $\Phi : \mathcal{M} \to \text{Aut}_A(P)$, defined by $\Phi(1_n + T.N^2) = Id_P + p.T.N.\phi$ is a group homomorphism.

Theorem 2.12 *Let R be a ring of dimension d with $1/d! \in R$ and $A = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Let P be a projective A-module of rank d. If $P/(X_1, \ldots, X_n)P$ is cancellative, then P is cancellative. In particular, if projective R-modules of rank d are cancellative, then projective $R[X_1, \ldots, X_n]$-modules of rank d are also cancellative.*

The following result is due to Keshari ([8], Theorem 3.5). In case P' is free (i.e. P is stably free), this was proved by Murthy ([13], Theorem 2.10).

Theorem 2.13 *Let R be an affine algebra of dimension n over an algebraically closed field k with $1/(n-1)! \in k$. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Let P' be a projective A-module of rank $n-1$ which is extended from R. Let $(a, p) \in \text{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then P is extended from R, i.e. there exists a projective R-module Q such that $P = Q \otimes_R A$. Further, when P' is free, we may assume that $\wedge^{n-1} Q = R$.*

We end this section by stating a result of Keshari ([7], Theorem 3.10) and ([8], Theorem 4.4). In case P' is free (i.e. P is stably free), (i) was proved by Ojanguren and Parimala ([14], Theorem).

Theorem 2.14 *Let R be an affine k-algebra of dimension n, where k is a real closed field. Let $f \in R$ be an element not belonging to any real maximal ideal of A. Assume that either*

(i) $A = R_f$ or

(ii) $A = R[f_1/f, \ldots, f_r/f]$, *where* $f, f_1, \ldots, f_r$ *is a regular sequence in* R.

Let P' *be a projective* A*-module of rank* $\geq n-1$ *which is extended from* R. *Let* $(a,p) \in \mathrm{Um}(A \oplus P')$ *and* $P = A\oplus P'/(a,p)A$. *Then there exists a projective* R*-module* Q *such that* $P = Q \otimes_R A$.

Further, when P' *is free, we may assume that determinant of* Q *is trivial.*

3 Main Theorem

We begin this section with the following result which is very crucial for later use and seems to be well known to experts. Since we are unable to find an appropriate reference, we give the complete proof.

Proposition 3.1 *Let* A *be a ring of dimension* d *and let* J *be an ideal of* A. *Consider the cartesian square*

$$\begin{array}{ccc} C & \xrightarrow{i_1} & A \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A & \xrightarrow[j_2]{} & A/J \end{array}$$

Then C *is finitely generated algebra over* A *of dimension* d. *In particular, if* A *is an affine algebra over a field* k, *then* C *is also an affine algebra over* k.

Proof Recall that C is the subalgebra of $A \times A$ consisting of all elements (a,b) such that $a - b \in J$. First we will show that $C \xrightarrow{\sim} A\oplus J$, where $A\oplus J$ has the obvious ring structure, i.e. $(a,x) + (a',x') = (a+a', x+x')$ and $(a,x).(a',x') = (aa', ax' + a'x + xx')$ for $(a,x),(a',x') \in A\oplus J$.

We define $i_1 : A\oplus J \to A$ by $i_1(a,x) = a + x$ and $i_2 : A\oplus J \to A$ by $i_2(a,x) = a$. Then $j_1 i_1 = j_2 i_2$. It is enough to show that $A\oplus J$ satisfies the universal property of cartesian square. Let B be a ring and let $f_i : B \to A$ be ring homomorphism, $i = 1,2$ such that $j_1 f_1 = j_2 f_2$. To show that there exists a unique ring homomorphism $F : B \to A\oplus J$ such that $i_1 F = f_1$ and $i_2 F = f_2$.

Define $F(b) = (f_2(b), f_1(b) - f_2(b))$. Since $j_1 f_1 = j_2 f_2$, $F : B \to A\oplus J$. Also it is clear that $i_1 F = f_1$ and $i_2 F = f_2$. It remains to show that F is a ring homomorphism. Clearly, $F(b + b') = F(b) + F(b')$ for $b, b' \in B$. We have

$$\begin{aligned} &F(b).F(b') = (f_2(b), f_1(b) - f_2(b)).(f_2(b'), f_1(b') - f_2(b')) \\ &= (f_2(b)f_2(b'), f_2(b)(f_1(b') - f_2(b')) + f_2(b')(f_1(b) - f_2(b)) + (f_1(b) - f_2(b))(f_1(b') - f_2(b'))) \\ &= (f_2(bb'), f_1(bb') - f_2(bb')) = F(bb'). \end{aligned}$$

Uniqueness of F follows from the fact that $i_1 F = f_1$ and $i_2 F = f_2$. This proves that $C \xrightarrow{\sim} A\oplus J$. If $J = (a_1, \ldots, a_r)$, then $A\oplus J$ is generated by $(0,a_1), \ldots, (0,a_r)$ over $A\oplus 0$, since if $x = a_1 x_1 + \ldots + a_r x_r \in J$, then $(0,x) = (x_1,0).(0,a_1) + \ldots + (x_r,0).(0,a_r)$. Hence $A\oplus J$ is a finitely generated algebra over A.

To show that $\dim A\oplus J = \dim A$, we show that $A\oplus J$ is integral over A. It is enough to show that $(0,a_i)$, $i = 1, \ldots, r$ are integral over A. Clearly $(0,a_i)^2 - (a_i,0)(0,a_i) = (0,0)$. This proves the result. $\square$

Corollary 3.2 *Let A be a ring and let $s \in A$. Then the cartesian square of (A, A) over A/sA is $A[X]/(X^2 - sX)$.*

Proof By (3.1), the cartesian square of (A, A) over A/sA is $C = A \oplus sA$, where $(0, s)$ satisfies the integral relation $(0, s)^2 - (0, s)(s, 0) = 0$. Hence C is isomorphic to $A[X]/(X^2 - sX)$. $\square$

The following result is very crucial for later use.

Lemma 3.3 *Let A be a ring and let P be a projective A-module of rank r. Choose $s \in A$ satisfying the properties of (2.5). Assume that R^r is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then $\text{Aut}\,(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$.*

Proof Without loss of generality, we can assume that A is reduced. By (2.5), there exist $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in \text{Hom}\,(P, A)$ such that P_s is free, $(\phi_i(p_j)) =$ diagonal $(s, \ldots, s)$, $sP \subset p_1A + \ldots + p_rA$ and s is a non-zero-divisor.

Let $(f, q) \in \text{Um}^1(A \oplus P, s^2A)$. Since $f = 1 + s^2a$ for some $a \in A$, $\Delta_{-q}(f, q) = (f, -s^2aq)$ and $\Delta_{s^2aq}(f, -s^2aq) = (f, s^4a^2q)$. Hence, replacing (f, q) by $\Delta_{s^2aq}\Delta_{-q}(f, q)$, we may assume that $q \in s^3P$. Since $sP \subset p_1A + \ldots + p_rA$, we can write $q = f_1p_1 + \ldots + f_rp_r$ for some $f_i \in s^2A$, $i = 1, \ldots, r$. Note that $(f, f_1, \ldots, f_r) \in \text{Um}_{r+1}(A, s^2A)$.

By (3.2), R is the cartesian square of (A, A) over A/s^2A.

$$\begin{array}{ccc} R & \xrightarrow{i_1} & A \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A & \xrightarrow[j_2]{} & A/(s^2) \end{array}$$

Patching unimodular rows $(f, f_1, \ldots, f_r)$ and $(1, 0, \ldots, 0)$ over A/s^2A, we get a unimodular row $(c_0, c_1, \ldots, c_r) \in \text{Um}_{r+1}(R)$. Since R^r is cancellative, there exists $\Theta \in \text{GL}_{r+1}(R)$ such that $(c_0, c_1, \ldots, c_r)\Theta = (1, 0, \ldots, 0)$. The projections of this equation gives

$$(f, f_1, \ldots, f_r)\Psi = (1, 0, \ldots, 0),\ (1, 0, \ldots, 0)\widetilde{\Psi} = (1, 0, \ldots, 0)$$

for certain matrices $\Psi, \widetilde{\Psi} \in \text{GL}_{r+1}(A)$ such that $\Psi = \widetilde{\Psi}$ modulo (s^2). Hence $(f, f_1, \ldots, f_r)\Psi\,\widetilde{\Psi}^{-1} = (1, 0, \ldots, 0)$, where $\Psi\widetilde{\Psi}^{-1} = \Delta \in \text{GL}_{r+1}(A, s^2A)$.

Let $\Delta = 1 + TN^2$, where T is some matrix and $N =$ diagonal $(1, s, \ldots, s)$. Applying (2.11) with $n = r + 1$ and $(s_1, \ldots, s_n) = (1, s, \ldots, s)$, we get $\Phi(\Delta) = Id + pTN\phi \in \text{Aut}\,(A \oplus P, sA)$, where $p = [p_1, \ldots, p_n] \in \text{Hom}\,(A^n, P)$ and $\phi = [\phi_1, \ldots, \phi_n]^t \in \text{Hom}\,(P, A^n)$ with $(\phi_i(p_j)) = N =$ diagonal $(1, s, \ldots, s)$. We have

$$\begin{aligned}
&\Phi(\Delta)(f, f_1p_1 + \ldots + f_rp_r) = (Id + pTN\phi)(f, f_1p_1 + \ldots + f_rp_r) \\
&= (f, f_1p_1 + \ldots + f_rp_r) + pTN(f, f_1s, \ldots, f_rs)^t \\
&= p(f, f_1, \ldots, f_r)^t + pT(f_0, f_1s^2, \ldots, f_rs^2)^t \\
&= p(1 + TN^2)(f, f_1, \ldots, f_r)^t = p(1, 0, \ldots, 0)^t = (1, 0).
\end{aligned}$$

This proves the result. $\square$

Remark 3.4 In (3.3), it is easy to see that $\text{Aut}\,(A\oplus P, sA)$ acts transitively on $\text{Um}(A\oplus P, s^2A)$. For, let $(f,q) \in \text{Um}(A\oplus P, s^2A)$, then replacing (f,q) by $\Delta_{-q}(f,q)$, we may assume that $q \in s^3A$. Note that $\Delta_{-q} \in \text{Aut}\,(A\oplus P, sA)$, since $q \in s^2P$. The rest of the proof is same as in (3.3).

Corollary 3.5 *Let A be a ring of dimension d and let P be a projective A-module of rank d. Choose $s \in A$ satisfying the properties of (2.5). Assume that R^d is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then P is cancellative.*

Proof We may assume that A is reduced. By (2.2), $A\oplus P$ is cancellative, hence, we need to show that $\text{Aut}\,(A\oplus P)$ acts transitively on $\text{Um}(A\oplus P)$. Let $(f,q) \in \text{Um}(A\oplus P)$. Since s is a non-zero-divisor, $\dim A/s^2 < \dim A$. Hence, by (2.2), there exists $\theta \in E(\overline{A}\oplus\overline{P})$ such that $\theta(\overline{f},\overline{q}) = (1,0)$, where "bar" denotes reduction modulo (s^2). By (2.7), θ can be lifted to $\Theta \in \text{Aut}\,(A\oplus P)$ and $\Theta(f,q) \in \text{Um}^1(A\oplus P, s^2A)$. By (3.3), there exists $\Theta_1 \in \text{Aut}\,(A\oplus P)$ such that $\Theta_1\Theta(f,q) = (1,0)$. This proves the result. $\square$

As a consequence of above result, we prove our first main result.

Theorem 3.6 *Let A be a ring of dimension d. Assume that for every finite extension R of A, R^d is cancellative. Then every projective A-module of rank d is cancellative.*

Proof Let P be a projective A-module of rank d. Choose $s \in A$ satisfying the properties of (2.5). If $R = A[X]/(X^2 - s^2X)$, then R is finite extension of A and hence R^d is cancellative. By (3.5), P is cancellative. $\square$

Lemma 3.7 *Let A be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let P be a projective A-module of rank $d-1$. Choose $s \in A$ satisfying the properties of* (2.5)*. Assume that R^{d-1} is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then P is cancellative.*

Proof We can assume that A is reduced and hence s is a non-zero-divisor. Since, by Suslin's result (1.1), every projective A-module of rank d is cancellative, it is enough to show that $\text{Aut}\,(A\oplus P)$ acts transitively on $\text{Um}(A\oplus P)$.

Let $(a,p) \in \text{Um}(A\oplus P)$. Let "bar" denotes reduction modulo s^2A. Then $\dim \overline{A} = d-1 \geq 2$. By (2.9), there exists $\overline{\sigma} \in E(\overline{A}\oplus\overline{P})$ such that $\overline{\sigma}(\overline{a},\overline{p}) = (1,0)$. By (2.7), lifting $\overline{\sigma}$ to $\sigma \in \text{Aut}\,(A\oplus P)$ and replacing (a,p) by $\sigma(a,p)$, we may assume that $(a,p) \in \text{Um}^1(A\oplus P, s^2A)$. By (3.3), there exists $\Delta \in \text{Aut}\,(A\oplus P)$ such that $\Delta(a,p) = (1,0)$. This proves the result. $\square$

Theorem 3.8 *Let A be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Assume that if R is a finite extension of A, then R^{d-1} is cancellative. Then every projective A-module of rank $d-1$ is cancellative.*

Proof Let P be a projective A-module of rank $d-1$. Choose $s \in A$ satisfying the properties of (2.5). Since $R = A[X]/(X^2 - s^2X)$ is a finite extension of A, R^{d-1} is cancellative, by hypothesis. By (3.7), P is cancellative. $\square$

Proposition 3.9 *Let A be a ring and let P be a projective A-module of rank r. Choose $s \in A$ satisfying the properties of (2.5). If* $\mathrm{GL}_{r+1}(A, s^2A)$ *acts transitively on* $\mathrm{Um}_{r+1}(A, s^2A)$, *then* $\mathrm{Aut}\,(A \oplus P, sA)$ *acts transitively on* $\mathrm{Um}(A \oplus P, s^2A)$.

Proof Let $(a,p) \in \mathrm{Um}(A \oplus P, s^2A)$. Replacing (a,p) by $\Delta_{-p}(a,p)$, we may assume that $p \in s^3P$. Note that $\Delta_{-p} \in E(A \oplus P, sA)$. Since, by (2.5), $sP \subset p_1A + \ldots + p_rA$, we get $p = a_1p_1 + \ldots + a_rp_r$ for some $a_i \in s^2A$, $i = 1, \ldots, r$. Note that $(a, a_1, \ldots, a_r) \in \mathrm{Um}_{r+1}(A, s^2A)$. By assumption, there exists $\Delta \in \mathrm{GL}_{r+1}(A, s^2A)$ such that $\Delta(a, a_1, \ldots, a_r) = (1, 0, \ldots, 0)$.

Let $\Delta = 1 + TN^2$, where T is some matrix and $N =$ diagonal $(1, s, \ldots, s)$. Applying (2.11) with $n = r+1$ and $(s_1, \ldots, s_n) = (1, s, \ldots, s)$, we get $\Phi(\Delta) = Id + pTN\phi \in \mathrm{Aut}\,(A \oplus P)$. It is easy to see that $\Phi(\Delta) \in \mathrm{Aut}\,(A \oplus P, sA)$. Further, as in the proof of (3.3), we can see that $\Phi(\Delta)(a,p) = (1,0)$. This proves the result. $\square$

Remark 3.10 Let k be a field and let A be an affine k-algebra of dimension d. Assume that characteristic of k is either 0 or $p > d$. Further assume that $cd(k) \leq 1$, where "cd" stands for cohomological dimension [19]. Then A^d is cancellative (Suslin's result). The proof of this result is contained in [20] (see [13], 2.1 - 2.4).

In particular, if A is an affine k-algebra of dimension d, where k is a C_1-field of characteristic 0 or $p > d$. Then A^d is cancellative. Note that we do not need k to be perfect in (1.1 (ii)). By (3.5), we get Bhatwadekar's result (1.1(ii)) that every projective A-module of rank d is cancellative.

4 Over algebraically closed fields

In this section, $\overline{k}$ will denote an algebraically closed field.

Proposition 4.1 *Let R be an affine $\overline{k}$-algebra of dimension d with $1/d! \in \overline{k}$. Let $f(T) \in R[T]$ be a monic polynomial. Assume that either*

(i) $A = R[T, 1/f(T)]$ or

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a regular sequence in $R[T]$.

Then A^d is cancellative.

Proof (i) Assume that $A = R[T, 1/f(T)]$ and let P be a stably free A-module of rank d. Since $A_{1+f\overline{k}[f]}$ is an affine domain of dimension d over a C_1-field $\overline{k}(f)$, by Suslin's result (3.10), $P \otimes A_{1+f\overline{k}[f]}$ is free. Hence, there exists $h \in 1 + f\overline{k}[f]$ such that P_h is free. By ([13], Lemma 2.9), patching P and $(R[T]_h)^d$, we get a projective $R[T]$-module Q of rank d such that $Q_f \xrightarrow{\sim} P$ and Q_h is free. Since $h \in R[T]$ is a monic polynomial, by (2.3), Q is free and hence P is free. This proves that A^d is cancellative.

(ii) Assume that $A = R[T, f_1/f, \ldots, f_r/f]$ and let P be a stably free A-module of rank d. By (2.13), there exists a projective $R[T]$-module Q of rank d such that $P \xrightarrow{\sim} Q \otimes A$. Since $P \oplus A \xrightarrow{\sim} (Q \otimes A) \oplus A$ is free, hence $(Q \oplus R[T]) \otimes R[T, 1/f]$ is free. Since f is a monic polynomial, by (2.3), $Q \oplus R[T]$ is free. By (2.12), $R[T]^d$ is cancellative. Hence Q is free and therefore P is free. This proves that A^d is cancellative. $\square$

Remark 4.2 Let $A = K[T, f(T)/g(T)]$, where K is a field. We can assume that f and g have no common factors. Hence $(f, g) = K[T]$. Let $af + bg = 1$ for some $a, b \in K[T]$. Then $b + (af/g) = 1/g$. Therefore, $1/g \in A$ and hence $A = K[T, 1/g]$ is a PID. Similarly, if $A = K[T, f_1/f, f_2/f, \ldots, f_r/f]$, then assuming $(f_1, f) = K[T]$ by canceling the common factors, we get $K[T, f_1/f] = K[T, 1/f]$ and hence $A = K[T, 1/f]$ is a PID. Hence every projective A-module is free.

Lemma 4.3 *Let R be a reduced ring of dimension d. Assume that either*

(i) $A = R[T, 1/f(T)]$ *for some* $f(T) \in R[T]$ *or*

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$ *for some* $f, f_1, \ldots, f_r \in R[T]$.

Let P be a projective A-module. Then there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.5).

Proof Let S be the set of non-zero-divisors of R. Then $S^{-1}R$ is a direct product of fields.

(i) Assume $A = R[T, 1/f(T)]$. Since $K[T, 1/g(T)]$ is a PID for any field K and $g(T) \in K[T]$, every projective $K[T, 1/g(T)]$-module is free. Hence every projective module of constant rank over $S^{-1}R[T, 1/f(T)]$ is free. Now, it is easy to see that we can choose $s \in S$ satisfying the properties of (2.5).

(ii) Assume $A = R[T, f_1/f, \ldots, f_r/f]$. By (4.2), $S^{-1}P$ is free. Now, we can choose $s \in S$ satisfying the properties of (2.5). $\square$

Theorem 4.4 *Let R be a reduced affine $\overline{k}$-algebra of dimension d with $1/d! \in \overline{k}$. Let $f(T) \in R[T]$ be a monic polynomial and assume that either*

(i) $A = R[T, 1/f(T)]$ *or*

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$, *where $f, f_1, \ldots, f_r$ is $R[T]$-regular sequence.*

Let P be a projective A-module of rank d. By (4.3), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.5). Then $\text{Aut}\,(A \oplus P)$ *acts transitively on* $\text{Um}^1(A \oplus P, s^2 A)$.

Proof (i) Assume $A = R[T, 1/f(T)]$. Let $C = A[X]/(X^2 - s^2X) = B[T, 1/f(T)]$, where $B = R[X]/(X^2 - s^2X)$ is an affine $\overline{k}$-algebra of dimension d. By (4.1), C^d is cancellative. Applying (3.3), we get that $\text{Aut}\,(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$.

(ii) Assume $A = R[T, f_1/f, \ldots, f_r/f]$. Let $C = A[X]/(X^2 - s^2X) = B[T, f_1/f, \ldots, f_r/f]$, where $B = R[X]/(X^2 - s^2X)$ is an affine $\overline{k}$-algebra of dimension d. Since B is a free R-module, $f, f_1, \ldots, f_r$ is a $B[T]$-regular sequence. By (4.1), C^d is cancellative. Applying (3.3), we get that $\text{Aut}\,(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$. $\square$

Remark 4.5 In (4.4), if every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2 A)$ by an automorphism of $A \oplus P$, then P will be cancellative.

Theorem 4.6 *Let R be an affine $\overline{\mathbb{F}}_p$-algebra of dimension d, where $p > d$. Let $f(T) \in R[T]$ be a monic polynomial and either*

(i) $A = R[T, 1/f(T)]$ *or*

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$ *for some* $f_1, \ldots, f_r \in R[T]$.

Then every projective A-module of rank d is cancellative.

Proof Without loss of generality, we may assume that R is reduced. Let P be a projective A-module of rank d. By (4.3), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.5).

(i) Assume $A = R[T, 1/f(T)]$. Let $C = A[Y]/(Y^2 - s^2Y) = B[T, 1/f(T)]$, where $B = R[Y]/(Y^2 - s^2Y)$. By (4.1), C^d is cancellative. Applying (3.7), we get that P is cancellative.

(ii) Assume $A = R[T, f_1/f, \ldots, f_r/f]$. First we prove that A^d is cancellative. Note that if $f, f_1, \ldots, f_r$ was a $R[T]$-regular sequence, then we could have applied (4.1). Let $\widetilde{P}$ be a stably free A-module of rank d. By Suslin's result (1.1(i)), we may assume that $\widetilde{P} \oplus A$ is free. By ([8], Theorem 3.6), $\widetilde{P}$ is extended from $R[T]$. Let Q be a projective $R[T]$-module of rank d such that $\widetilde{P} = Q \otimes A$. Since $\widetilde{P} \oplus A$ is free, $(\widetilde{P} \oplus A) \otimes A_f = (Q \oplus R[T]) \otimes R[T]_f$ is free. Hence, by (2.3), $Q \oplus R[T]$ is free. By (2.12), $R[T]^d$ is cancellative. Hence Q is free and so $\widetilde{P}$ is free.

Now, we will show that P is cancellative. Let $C = A[Y]/(Y^2 - s^2Y) = B[T, f_1/f, \ldots, f_r/f]$, where $B = R[Y]/(Y^2 - s^2Y)$ is an affine $\overline{\mathbb{F}}_p$-algebra of dimension d. Then, as in the previous paragraph, C^d is cancellative. Applying (3.7), we get that P is cancellative. $\square$

Remark 4.7 Note that ([8], Theorem 3.6) is proved only for $d \geq 4$. But, using (2.9) and following the proof of ([8], Theorem 3.6), it can be proved for $d \leq 3$. Hence, we can use it in (4.6).

As a consequence of (4.6), we get the following result which extends a result of Murthy ([13], Corollary 2.13), where it was proved that A^d is cancellative.

Corollary 4.8 *Let $R = \overline{\mathbb{F}}_p[X_1, \ldots, X_{d+1}]$ and let A be a subring of the fraction field of R with $R \subset A$. Suppose $p > d$. Then every projective A-module of rank d is cancellative.*

The following result gives examples of affine algebras of dimension d over which projective modules of rank $\geq d - 1$ are cancellative.

Proposition 4.9 *Let $R = \overline{k}[X_1, \ldots, X_d]$ with $1/(d-1)! \in \overline{k}$ and let $A = R[f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a R-regular sequence. Further, assume that $0 \neq f \in k[X_1]$. Then every projective A-module of rank $d - 1$ is cancellative. In particular, every projective $R[g/f]$-module of rank $d - 1$ is cancellative, where $g \in R$.*

Proof Let P be a projective A-module of rank $d - 1$. Since $A_f = A'[X_2, \ldots, X_d]$, where $A' = \overline{k}[X_1, 1/f]$, every projective $A'[X_d]$-module is stably free and hence free. Thus, P_f is free.

We can choose some positive integer N such that $s = f^N$ satisfies the properties of (2.5). Let $C = A[Y]/(Y^2 - s^2Y) = B[X_2, \ldots, X_d][f_1/f, \ldots, f_r/f]$, where $B = \overline{k}[X_1, Y]/(Y^2 - s^2Y)$ is an affine $\overline{k}$-algebra of dimension 1. Since B is a free $\overline{k}[X_1]$-module, $f, f_1, \ldots, f_r$ is a $B[X_2, \ldots, X_d]$-regular sequence. Let Q be a stably free C-module of rank $d - 1$. By (2.13), there exists a projective $B[X_2, \ldots, X_d]$-module Q' of rank $d - 1$ with trivial determinant such that $Q = Q' \otimes_{B[X_2, \ldots, X_d]} C$. By (2.6), Q' is free, hence Q is free. This proves that C^{d-1} is cancellative. Applying (3.3), we get that $\text{Aut}\,(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. Now, it is enough to show that every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2A)$ by an automorphism of $A \oplus P$.

Let "tilde" denotes reduction modulo fA and "bar" denotes reduction modulo $s^2A = f^{2N}A$. Since $(\widetilde{s^2})$ is a nilpotent ideal in $\widetilde{A}$, it is easy to see that $E(\widetilde{A} \oplus \widetilde{P}) \to E(\overline{A} \oplus \overline{P})$ is a surjective map. Since, by (2.7), every element of $E(\overline{A}\oplus\overline{P})$ can be lifted to an element of Aut $(A\oplus P)$. Hence it is enough to show that $E(\widetilde{A}\oplus\widetilde{P})$ acts transitively on $\mathrm{Um}(\widetilde{A}\oplus\widetilde{P})$.

Since $A = R[f_1/f, \ldots, f_r/f] = R[T_1, \ldots, T_r]/I$, where $I = (fT_1 - f_1, \ldots, fT_r - f_r)$, we get $\widetilde{A} = A/fA = A_0[T_1, \ldots, T_r]$, where $A_0 = \overline{k}[X_1, \ldots, X_d]/(f, f_1, \ldots, f_r)$ is an affine $\overline{k}$-algebra of dimension $d - r - 1 \leq d - 2$. Hence, by (2.4), $E(\widetilde{A}\oplus\widetilde{P})$ acts transitively on $\mathrm{Um}(\widetilde{A}\oplus\widetilde{P})$. This proves the result. □

The following two results generalizes two results of Murthy ([13], Proposition 3.1) and ([13], Theorem 3.6) respectively.

Corollary 4.10 *Let $A = \overline{k}[x_0, x_1, \ldots, x_d]$, where $x_0^2+x_1^2+f(x_2, \ldots, x_d) = 0$ for some $f \in \overline{k}[x_2, \ldots, x_d]$ and $1/(d-1)! \in \overline{k}$. Then every projective A-module of rank $d-1$ is cancellative*

Proof Write $u = x_0+ax_1$ and $v = x_0-ax_1$, where $a \in \overline{k}$ with $a^2 = -1$. Then $A = \overline{k}[u, x_2, \ldots, x_d, f/u]$. Now, the result follows from (4.9). □

Corollary 4.11 *Let $A = \overline{k}[X, Y, T_1, \ldots, T_{d-1}]/(F)$, where $F = X + X^s g(X, T_1, \ldots, T_{d-1}) + X^r Y + f(T_1, \ldots, T_{d-1}) = 0$, with $d \geq 3$, $s \geq 2$, $r \geq 2$ and $1/(d-1)! \in \overline{k}$. Then A is a smooth d dimensional affine $\overline{k}$-algebra. Further, every projective A-module of rank $d-1$ is cancellative.*

Proof Since $\partial F/\partial x$ and $\partial F/\partial y$ have no common zeroes, A is a smooth affine algebra. Further $A = \overline{k}[X, T_1, \ldots, T_{d-1}, M/X^r]$, where $M = X + X^s g(X, T_1, \ldots, T_{d-1}) + f(T_1, \ldots, T_{d-1})$. Now, the result follows from (4.9). □

5 Over real closed fields

In this section, k will denote a real closed field.

Proposition 5.1 *Let R be an affine k-algebra of dimension $d-1$ and let $f(T) \in R[T]$ be a monic polynomial. Assume that $f(T)$ does not belongs to any real maximal ideal of $R[T]$ and either*
(i) $A = R[T, 1/f(T)]$ or
(ii) $A = R[T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a regular sequence in $R[T]$.
Then A^d is cancellative.

Proof (*i*) Let $A = R[T, 1/f(T)]$ and let P be a stably free A-module of rank d. By (2.2), $P\oplus A$ is free. Applying (2.14), P is extended from $R[T]$. Let Q be a projective $R[T]$-module such that $P = Q \otimes_{R[T]} A$. Since $P\oplus A$ is free, $(Q\oplus R[T])_f$ is free. Since f is a monic polynomial, by (2.3), $Q\oplus R[T]$ is free. By Plumstead's result [15], every projective $R[T]$-module of rank $> \dim R$ is cancellative. Hence Q is free and therefore P is free.

(*ii*) Let $A = R[T, f_1/f, \ldots, f_r/f]$ and let P be a stably free A-module of rank d. By (2.2), $P\oplus A$ is free. Applying (2.14), P is extended from $R[T]$. Let Q be a projective $R[T]$-module of rank d such that $P = Q \otimes_{R[T]} A$. Since $P\oplus A$ is free, $(Q\oplus R[T]) \otimes R[T]_f$ is free. As f is a monic polynomial, by (2.3), $Q\oplus R[T]$ is free. By Plumstead's result [15], projective $R[T]$-modules of rank $> \dim R$ are cancellative. Hence Q is free and so P is free. □

Theorem 5.2 *Let R be an affine k-algebra of dimension $d-1$ and let $f(T) \in R[T]$ be a monic polynomial. Assume that $f(T)$ does not belong to any real maximal ideal of $R[T]$ and either*

(i) $A = R[T, 1/f(T)]$ or

(ii) $A = R[T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a regular sequence in $R[T]$.

Then every projective A-module of rank d is cancellative.

Proof Without loss of generality, we may assume that R is reduced. Let P be a projective A-module of rank d. By (4.3), we can choose a non-zero-divisor $s \in R$ satisfying the properties of (2.5).

(i) Assume $A = R[T, 1/f(T)]$. Let $C = A[X]/(X^2 - s^2X) = B[T, 1/f(T)]$, where $B = R[X]/(X^2 - s^2X)$. Since $B[T]$ is a finite extension of $R[T]$, any maximal ideal of $B[T]$ will contract to a maximal ideal of $R[T]$. Therefore, $f(T)$ does not belongs to any real maximal ideal of $B[T]$. By (5.1), C^d is cancellative. Hence, by (3.5), P is cancellative.

(ii) Assume $A = R[T, f_1/f, \ldots, f_r/f]$. Let $C = A[X]/(X^2 - s^2X) = B[T, f_1/f, \ldots, f_r/f]$, where $B = R[X]/(X^2 - s^2X)$. Since $B[T]$ is a finite extension of $R[T]$, $f(T)$ does not belongs to any real maximal ideal of $B[T]$. Also, since $B[T]$ is a free $R[T]$-module, $f, f_1, \ldots, f_r$ is a regular sequence in $B[T]$. By (5.1), C^d is cancellative. Hence, by (3.5), P is cancellative. $\square$

Proposition 5.3 *Let R be an affine k-algebra of dimension $d-2$ and let $f \in R[X, T]$ be a monic polynomial in T. Assume that f does not belong to any real maximal ideal of $R[X, T]$ and either*

(i) $A = R[X, T, 1/f]$ or

(ii) $A = R[X, T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a $R[X, T]$-regular sequence.

Then A^{d-1} is cancellative.

Proof Let P be a stably free A-module of rank $d-1$. By (5.1), we may assume that $P \oplus A$ is free. By (2.14), P is extended from $R[X, T]$. Let Q be a projective $R[X, T]$-module such that $P \xrightarrow{\sim} Q \otimes A$. Since $(P \oplus A) \otimes A_f = (Q \oplus R[X, T]) \otimes R[X, T, 1/f]$ is free and f is a monic polynomial, hence, by (2.3), $Q \oplus R[X, T]$ is free. By Ravi Rao's result ([17], Theorem 2.5), every projective $R[X_1, \ldots, X_n]$-module of rank $> \dim R$ is cancellative. Hence Q is free and therefore P is free. $\square$

Remark 5.4 Let $A = k[X_1, \ldots, X_d, 1/f]$. Assume that f does not belongs to any real maximal ideal of $k[X_1, \ldots, X_d]$. Since every projective A-module is stably free, by (5.3), every projective A-module of rank $\geq d-1$ is free.

Theorem 5.5 *Let R be an affine k-algebra of dimension $d-2$. Let $f \in R[X, T]$ be a monic polynomial in T which does not belongs to any real maximal ideal of $R[X, T]$. Assume that either*

(i) $A = R[X, T, 1/f(T)]$ or

(ii) $A = R[X, T, f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a $R[X, T]$-regular sequence.

Let P be a projective A-module of rank $d-1$. Assume that there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.5). Then $\text{Aut}\,(A \oplus P)$ *acts transitively on* $\text{Um}^1(A \oplus P, s^2A)$.

Proof Let $B = R[Y]/(Y^2 - s^2Y)$ and $C = A[Y]/(Y^2 - s^2Y)$. If $A = R[X, T, 1/f]$, then $C = B[X, T, 1/f]$ and if $A = R[X, T, f_1/f, \ldots, f_r/f]$, then $C = B[X, T, f_1/f, \ldots, f_r/f]$. Since $B[X, T]$ is a finite extension of $R[X, T]$, f does not belongs to any real maximal ideal of $B[X, T]$. Also, as $B[X, T]$ is a free module over $R[X, T]$, $f, f_1, \ldots, f_r$ is a regular sequence in $B[X, T]$. Hence, by (5.3), C^{d-1} is cancellative. Applying (3.3), we get that $\text{Aut}\,(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. $\square$

Remark 5.6 In (5.5), if every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2A)$ by an automorphism of $A \oplus P$, then P will be cancellative.

Proposition 5.7 *Let $R = k[X_1, \ldots, X_d]$ and $A = R[f_1/f, \ldots, f_r/f]$, where $f, f_1, \ldots, f_r$ is a R-regular sequence. Further, assume that $f \in k[X_1]$ does not belongs to any real maximal ideal. Then every projective A-module of rank $d - 1$ is cancellative.*

Proof Let P be a projective A-module of rank $d - 1$. Since $A_f = A'[X_2, \ldots, X_d]$, where $A' = k[X_1, 1/f]$, every projective A_f-module is stably free and hence free (since $\dim A' = 1$). Hence P_f is free.

We can choose some positive integer N such that $s = f^N$ satisfies the properties of (2.5). Let $C = A[Y]/(Y^2 - s^2Y) = B[X_2, \ldots, X_d][f_1/f, \ldots, f_r/f]$, where $B = k[X_1, Y]/(Y^2 - s^2Y)$ is an affine k-algebra of dimension 1. Since B is free $k[X_1]$-module, $f, f_1, \ldots, f_r$ is a $B[X_d]$-regular sequence. Also, since C is an integral extension of A, f does not belong to any real maximal ideal of C. Let Q be a stably free C-module of rank $d - 1$. By (2.14), there exists projective $B[X_2, \ldots, X_d]$-module Q' of rank $d - 1$ with trivial determinant such that $Q = Q' \otimes_{B[X_d]} C$. By (2.6), Q' is free and hence Q is free. Therefore C^{d-1} is cancellative. Applying (3.3), we get that $\text{Aut}\,(A \oplus P, sA)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. Now, it is enough to show that every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2A)$ by an automorphism of $A \oplus P$. The rest of the argument is same as in (4.9). $\square$

Corollary 5.8 *Let $R = k[X_1, \ldots, X_d]$ and $A = R[f/g]$, where $f \in R$ and $g \neq 0 \in k[X_1]$. Assume that g does not belongs to any real maximal ideal of R. Then every projective A-module of rank $\geq d - 1$ is cancellative.*

6 Over Laurent polynomial rings

Lemma 6.1 *Let R be a reduced ring of dimension d and let $A = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Let P be a projective A-module. Then there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.5).*

Proof Let S be the set of non-zero-divisors of R. Then $S^{-1}R$ is a direct product of fields. Suslin ([22], Corollary 7.4) and Swan ([25], Theorem 1.1) independently proved that if K is a field or a PID, then every projective $K[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$-modules are free. Hence $S^{-1}P$ is free. Now, we can choose $s \in S$ satisfying the properties of (2.5). $\square$

Theorem 6.2 *Let R be a ring of dimension d and let $A = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Let P be a projective A-module of rank $\geq d$. By (6.1), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.5). Assume that B^d is cancellative, where $B = A[T]/(T^2 - s^2T)$. Then P is cancellative.*

Proof By (3.3), $\text{Aut}\,(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. Let "bar" denotes reduction modulo s^2A. Since $\dim R/s^2R < d$, by (2.4), $E(\overline{A} \oplus \overline{P})$ acts transitively on $\text{Um}(\overline{A} \oplus \overline{P})$. Further, by (2.7), every element of $E(\overline{A} \oplus \overline{P})$ can be lifted to an element of $\text{Aut}\,(A \oplus P)$. Hence, every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2A)$ by an automorphism of $A \oplus P$. This proves that P is cancellative. $\square$

Theorem 6.3 *Let R be an affine algebra of dimension d over an algebraically closed field $\overline{k}$ with $1/d! \in \overline{k}$. Let $A = R[X, X^{-1}]$. Then every projective A-module of rank d is cancellative.*

Proof We can assume that A is reduced. Let P be a projective A-module of rank d. By (6.1), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.5). Let $B = A[T]/(T^2 - s^2T) = B_1[X, X^{-1}]$, where $B_1 = R[T]/(T^2 - s^2T)$ is an affine algebra over $\overline{k}$ of dimension d. By (4.1), B^d is cancellative. Hence, applying (6.2), we get that P is cancellative. This proves the result. $\square$

Theorem 6.4 *Let R be a ring of dimension d. Let $A = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$ and let P be a projective A-module of rank $\geq d$. By (6.1), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.5). Assume that B^d is cancellative, where $B = B_1[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$ and $B_1 = R[T]/(T^2 - s^2T)$. Then the natural map $\text{Aut}\,(P) \to \text{Aut}_{A/I}(P/IP)$ is surjective, where $I = (Y_m - 1)A$.*

Proof When rank $P > d$, the result follows from (2.10). Hence, we assume that rank $P = d$. Let "bar" denotes reduction modulo I. It is easy to see that we can assume that R is reduced.

Let $\overline{\Lambda} \in \text{Aut}_{\overline{A}}(\overline{P})$, then by (2.10), we can lift $Id_{\overline{A}} \oplus \overline{\Lambda} \in \text{Aut}_{\overline{A}}(\overline{A} \oplus \overline{P})$ to an automorphism Θ of $A \oplus P$. Let $\Theta(1, 0) = (h, p) \in \text{Um}(A \oplus P, I)$. Assume that there exists $\Theta_1 \in \text{Aut}\,(A \oplus P, I)$ such that $\Theta_1(h, p) = (1, 0)$. Then, we have the following commutative diagram

$$\begin{array}{ccccc} A \oplus P & \xrightarrow{\Theta} & A \oplus P & \xrightarrow{\Theta_1} & A \oplus P \\ \downarrow & & \downarrow & & \downarrow \\ \overline{A} \oplus \overline{P} & \xrightarrow[Id_{\overline{A}} \oplus \overline{\Lambda}]{} & \overline{A} \oplus \overline{P} & \xrightarrow{Id} & \overline{A} \oplus \overline{P} \end{array}$$

Note that $\Psi = \Theta_1 \Theta \in \text{Aut}\,(A \oplus P)$ is a lift of $Id_{\overline{A}} \oplus \overline{\Lambda}$. Further $\Psi(1, 0) = (1, 0)$. Hence Ψ induces an automorphism $\Lambda \in \text{Aut}\,(P)$ which is a lift of $\overline{\Lambda}$. Hence, it is enough to show that $\text{Aut}\,(A \oplus P, I)$ acts transitively on $\text{Um}(A \oplus P, I)$.

Let $(f, q) \in \text{Um}(A \oplus P, I)$. Let "tilde" denote reduction modulo s^3A. Since $\dim R/s^3R < \dim R$, by (2.10), $EL^1(\widetilde{A} \oplus \widetilde{P}, I)$ acts transitively on $\text{Um}^1(\widetilde{A} \oplus \widetilde{P}, I)$. By (2.7), after lifting the $EL^1(\widetilde{A} \oplus \widetilde{P}, I)$ transformations, we may assume that $(f, q) = (1, 0)$ modulo s^3I.

Since $q \in s^3IP$, with the notation in (2.5), we can write $q = f_1p_1 + \ldots + f_dp_d$ for some $f_i \in s^2I$, $i = 1, \ldots, d$. Note that $(f, f_1, \ldots, f_d) \in \text{Um}_{d+1}(A, s^2I)$.

Since B^d is cancellative, where B is the cartesian square of (A, A) over $A/(s^2)$, as in the proof of (3.3), there exists $\Delta_1 \in \mathrm{GL}_{d+1}(A, s^2A)$ such that $(f, f_1, \ldots, f_d)\Delta_1 = (1, 0, \ldots, 0)$. Since $(f, f_1, \ldots, f_d) \in \mathrm{Um}_{d+1}(A, s^2I)$, if $\Delta_0 = \Delta_1$ (modulo I), then $\Delta_0 \in \mathrm{GL}_{d+1}(A/IA) \subset \mathrm{GL}_{d+1}(A)$ (since $A/I = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_{m-1}^{\pm 1}]$). Further, going modulo I, we get $(1, 0, \ldots, 0)\Delta_0 = (1, 0, \ldots, 0)$. Hence, if $\Delta_2 = \Delta_1\Delta_0^{-1}$, then $(f, f_1, \ldots, f_d)\Delta_2 = (1, 0, \ldots, 0)$ and $\Delta_2 \in \mathrm{GL}_{d+1}(A, s^2I)$.

Let $\Delta_2 = 1 + TN^2$, where T is some matrix and $N =$ diagonal $(1, s, \ldots, s)$. Applying (2.11) with $n = d + 1$ and $(s_1, \ldots, s_n) = (1, s, \ldots, s)$, we get $\Psi_1 = \Phi(\Delta_2) \in \mathrm{Aut}\,(A \oplus P, I)$ and $\Psi_1(f, f_1p_1 + \ldots + f_dp_d) = (1, 0)$. This proves the result. $\square$

As an application of (6.3) and (6.4), we get the following result.

Corollary 6.5 *Let R be an affine algebra of dimension d over an algebraically closed field $\overline{k}$ with $1/d! \in \overline{k}$. Let $A = R[X, X^{-1}]$ and let P be a projective A-module of rank d. Then the natural map* $\mathrm{Aut}\,(P) \to \mathrm{Aut}_{A/(Y-1)A}(P/(Y-1)P)$ *is surjective.*

We state three results which follow directly from (2.12) by applying (1.1(iii)), (3.10) and (6.3, 6.4) respectively. Note that (6.8(i)) generalizes a result of Keshari ([9], Proposition A.9), where it was proved when A is a smooth affine algebra of dimension $d = 2$ and the determinant of P is trivial.

Theorem 6.6 *Let A be a finitely generated algebra over $\mathbb{Z}$ of dimension d with $1/d! \in A$. Then all projective $A[X_1, \ldots, X_n]$-modules of rank d are cancellative.*

Theorem 6.7 *Let A be an affine algebra of dimension d over a C_1-field k with $1/d! \in k$ and $R = A[X_1, \ldots, X_n]$. Then every projective R-module of rank d is cancellative. Note that any field $k \subset \overline{\mathbb{F}}_p$ is a C_1-field.*

Theorem 6.8 *Let A be an affine algebra of dimension d over an algebraically closed field $\overline{k}$ with $1/d! \in \overline{k}$ and $R = A[X_1, \ldots, X_n, Y^{\pm 1}]$. Let P be a projective R-module of rank d. Then*

(i) P is cancellative and

(ii) the natural map $\mathrm{Aut}\,(P) \to \mathrm{Aut}_{R/(Y-1)R}(P/(Y-1)P)$ *is surjective.*

We end this section with the following question.

Question 6.9 *Let A be an affine algebra of dimension d over an algebraically closed field $\overline{k}$ with $1/d! \in \overline{k}$ and $R = A[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Let P be a projective R-module of rank d. Is P cancellative?*

We remark that if (6.9) has an affirmative answer, then by (6.4), the natural map $\mathrm{Aut}\,(P) \to \mathrm{Aut}_{R/(Y_m-1)R}(P/(Y_m - 1)P)$ will be surjective. Further, we note that to answer (6.9) in affirmative, it is enough to show, by (6.2), that R^d is cancellative in (6.9).

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References

[1] H. Bass, *K-theory and stable algebra*, IHES **22** (1964), 5-60.

[2] S.M. Bhatwadekar, *Cancellation theorem for projective modules over a two-dimensional ring and its polynomial extension*, Compositio Math. **128** (2001), 339-359.

[3] S.M. Bhatwadekar, *A cancellation theorem for projective modules over affine algebras over C_1-fields*, JPAA **183** (2003), 17-26.

[4] S.M. Bhatwadekar, *Projective modules over affine algebras*, Survey article.

[5] S.M. Bhatwadekar and A. Roy, *Some theorems about projective modules over polynomial rings*, J. Algebra **86** (1984), 150-158.

[6] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **36** Springer-Verlag, Berlin, 1998.

[7] M.K. Keshari, *A note on projective modules over real affine algebras*, J. Algebra **278** (2004), 628-637.

[8] M.K. Keshari, *Stability results for projective modules over blowup rings*, J. Algebra **294** (2005), 226-238.

[9] M.K. Keshari, *Euler class group of a Laurent polynomial ring: local case*, J. Algebra **308** vol 2, (2007), 666-685.

[10] H. Lindel, *Unimodular elements in projective modules*, J. Algebra **172** (1995), 301-319.

[11] N. Mohan Kumar, *Stably free modules*, Amer. J. Math. **107** (1985), 1439-1444.

[12] N. Mohan Kumar, M.P. Murthy and A. Roy, *A cancellation theorem for projective modules over finitely generated rings*, in: Hijikata H., et al. (Eds.), Algebraic geometry and commutative algebra in honor of Masayoshi Nagata, vol. 1, (1987), 281-287.

[13] M.P. Murthy, *Cancellation problem for projective modules over certain affine algebras*, Proceedings of the international colloquium on Algebra, Arithmetic and Geometry, Mumbai, Narosa Publishing House, (2000), 493-507.

[14] M. Ojanguren and R. Parimala, *Projective modules over real affine algebras*, Math. Ann. **287** (1990), 181-184.

[15] B. Plumstead, *The conjectures of Eisenbud and Evans*, Amer. J. Math. **105** (1983), 1417-1433.

[16] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167-171.

[17] R.A. Rao, *A question of H. Bass on the cancellative nature of large projective modules over polynomial rings*, Amer. J. Math. **110** (1988), 641-657.

[18] J.P. Serre, *Sur les modules projectifs*, Sem. Dubreil-Pisot **14** (1960-61), 1-16.

[19] J.P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology **3** (1968), 264-277.

[20] A.A. Suslin, *Cancellation over affine varieties*, J.Sov. Math. **27** (1984), 2974-2980.

[21] A.A. Suslin, *A cancellation theorem for projective modules over affine algebras*, Sov. Math. Dokl. **18** (1977), 1281-1284.

[22] A.A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR-Izv. **11** (1977), 221-238.

[23] A.A. Suslin, *Projective modules over a polynomial ring are free*, Sov. Math. Dokl. **17** (1976), 1160-1164.

[24] A.A. Suslin and L.N. Vaserstein, *Serre's problem on projective modules over polynomial rings and algebraic K-theory*, Math. USSR, Izvestija **10** (5) (1976), 937-1001.

[25] R.G. Swan, *Projective modules over Laurent polynomial rings*, Trans. Amer. Math. Soc. **237** (1978), 111-120.

[26] A. Wiemers, *Cancellation properties of projective modules over Laurent polynomial rings*, J. Algebra **156** (1993), 108-124.