A note on cancellation of projective modules

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Dedicated to Professor S.M. Bhatwadekar on his 65th birthday

1 Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated. Let A be a ring and let P be a projective A-module. We say that P is *cancellative* if $P \oplus A^r \xrightarrow{\sim} P' \oplus A^r$ for some positive integer r and some projective A-module P' implies that $P \xrightarrow{\sim} P'$. A classical result of Bass [2] says that *if rank* $P > \dim A$, *then* $E(A \oplus P)$ *acts transitively on* $\text{Um}(A \oplus P)$. *In particular* P *is cancellative.*

Let A be a ring of dimension d and let P be a projective A-module of rank d. It is interesting to know under what conditions A^d is cancellative implies that every projective A-module P of rank d is cancellative. Bhatwadekar ([3], Example 2.11) gave an example of a smooth affine surface A over $\mathbb{R}$ such that A^2 is cancellative but $A \oplus K_A$ is not cancellative, where K_A is the canonical module of A. The second author obtained a sufficient condition ([9], Theorem 3.6) by proving the following result:

Let A be a ring of dimension d. Assume that if R is a finite extension of A then R^d is cancellative. Then every projective A-module P of rank d is also cancellative. In other words, if $\text{GL}_{d+1}(R)$ acts transitively on $\text{Um}_{d+1}(R)$ for every finite extension R of A, then $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

Our first result generalizes above result as follows (3.4).

Theorem 1.1 *Let A be a ring of dimension d and let P be a projective A-module of rank d. Assume that if R is a finite extension of A then $E_{d+1}(R)$ acts transitively on $\text{Um}_{d+1}(R)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

If A is an affine algebra of dimension d over $\mathbb{Z}$ then Vaserstein [14] proved that $E_{d+1}(A)$ acts transitively on $\text{Um}_{d+1}(A)$. As a consequence of (1.1), we get another proof of the following result of Mohan Kumar, Murthy and Roy ([10], Theorem 2.4) that if P is a projective A-module of rank d, then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

Let A be a smooth affine algebra of dimension d over an algebraically closed field $\overline{k}$. Assume that gcd $((d-1)!, char(\overline{k})) = 1$. Then Fasel, Rao and Swan ([5], Theorem 7.3) proved that *stably free A-modules of rank $d-1$ are free*, thus answering an old question of Suslin. Infact if $d \geq 4$, then they proved that A being normal suffices. In view of their result, a natural question arises: Let P be a projective A-module of rank $d-1$. Is P cancellative? We answer this question in affirmative when $\overline{k} = \overline{\mathbb{F}}_p$. More precisely, we prove the following result (3.5).

Theorem 1.2 *Let A be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$, where $p \geq d$. Then every projective A-module of rank $d-1$ is cancellative.*

Finally, as a consequence of the techniques developed for (1.1), we will prove the following result (4.5). Gubeladze proved this result ([7], [8]) in case P is free.

Theorem 1.3 *Let $M \subset \mathbb{Q}_+^r$ be a seminormal monoid such that $M \subset \mathbb{Q}_+^r$ is an integral extension. Let R be a ring of dimension d and let P be a projective $R[M]$-module of rank n. Then $E(R[M]{\oplus}P)$ acts transitively on $\text{Um}(R[M]{\oplus}P)$ whenever $n \geq max\ (2, d+1)$.*

2 Preliminaries

Let A be a ring and let M be an A-module. We say that $m \in M$ is *unimodular* if there exists $\phi \in M^* = \text{Hom}_A(M, A)$ such that $\phi(m) = 1$. The set of all unimodular elements of M will be denoted by $\text{Um}(M)$. We denote by $\text{Aut}_A(M)$, the group of all A-automorphism of M.

For an ideal J of A, we denote by $EL^1(A \oplus M, J)$, the subgroup of $\text{Aut}_A(A \oplus M)$ generated by all the automorphisms $\Delta_{a\varphi} = \left(\begin{smallmatrix} 1 & a\varphi \\ 0 & id_M \end{smallmatrix}\right)$ and $\Gamma_m = \left(\begin{smallmatrix} 1 & 0 \\ m & id_M \end{smallmatrix}\right)$ with $a \in J, \varphi \in M^*$ and $m \in M$. In particular, we denote by $E^1_{r+1}(A, J)$, the subgroup of $E_{r+1}(A)$ generated by $\Delta_{\mathbf{a}} = \left(\begin{smallmatrix} 1 & \mathbf{a} \\ 0 & id_F \end{smallmatrix}\right)$ and $\Gamma_{\mathbf{b}} = \left(\begin{smallmatrix} 1 & 0 \\ \mathbf{b}^t & id_F \end{smallmatrix}\right)$, where $F = A^r$, $\mathbf{a} \in JF$ and $\mathbf{b} \in F$. Further, we will write $EL^1(A{\oplus}M)$ for $EL^1(A{\oplus}M, A)$.

We denote by $\text{Um}(A{\oplus}M, J)$ the set of all $(a, m) \in \text{Um}(A{\oplus}M)$ with $a \in 1 + J$ and $m \in JM$. We will write $\text{Um}_r(A, J)$ for $\text{Um}(A{\oplus}A^{r-1}, J)$.

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(m) = 0$. Let $\varphi_p \in \text{End}(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is a (unipotent) automorphism of M. An automorphism of M of the form $1 + \varphi_p$ is called a *transvection* of M if either $p \in \text{Um}(M)$ or $\varphi \in \text{Um}(M^*)$. We denote by $E(M)$, the subgroup of $\text{Aut}(M)$ generated by all the transvections of M.

The following result is due to Bak, Basu and Rao ([1], Theorem 3.10). In [4], we proved results for $EL^1(A{\oplus}P)$. Due to this result, we can interchange $E(A{\oplus}P)$ and $EL^1(A{\oplus}P)$.

Theorem 2.1 *Let A be a ring and let P be a projective A-module of rank ≥ 2. Then $EL^1(A{\oplus}P) = E(A{\oplus}P)$.*

Remark 2.2 Using (2.1), it is easy to see that if I is any ideal of A, then the natural map $E(A{\oplus}P) \to E((A{\oplus}P)/I(A{\oplus}P))$ is surjective.

The following result is due to Lindel ([11], Lemma 1.1).

Lemma 2.3 *Let A be a ring and let P be a projective A-module of rank r. Then there exists $s \in A$ such that the following holds:*

(i) P_s is free,
(ii) there exists $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in \text{Hom}(P, A)$ such that $(\phi_i(p_j)) =$ diagonal $(s, \ldots, s)$,
(iii) $sP \subset p_1A + \ldots + p_rA$,
(iv) the image of s in A_{red} is a non-zero-divisor and
(v) $(0 : sA) = (0 : s^2A)$.

The following two results are from ([4], Lemma 3.1 and Lemma 3.10).

Lemma 2.4 *Let A be a ring and let P be a projective A-module. Let "bar" denote reduction modulo the nil radical of A. If $E(\overline{A}{\oplus}\overline{P})$ acts transitively on $\text{Um}(\overline{A}{\oplus}\overline{P})$, then $E(A{\oplus}P)$ acts transitively on $\text{Um}(A{\oplus}P)$.*

Lemma 2.5 *Let A be a ring and let P be a projective A-module of rank r. Choose $s \in A$, $p_1, \ldots, p_r \in P$ and $\varphi_1, \ldots, \varphi_r \in P^*$ satisfying the properties of (2.3). Let $(a, p) \in \text{Um}(A{\oplus}P, sA)$ with $p = c_1p_1 + \ldots + c_rp_r$, where $c_i \in sA$ for $i = 1, \ldots, r$. Assume there exists $\phi \in E^1_{r+1}(A, sA)$ such that $\phi(a, c_1, \ldots, c_r) = (1, 0, \ldots, 0)$. Then there exists $\Phi \in E(A{\oplus}P)$ such that $\Phi(a, p) = (1, 0)$. (Infact, we get a map from $E^1_{r+1}(A, sA)$ to $E(A{\oplus}P)$.)*

When A is an affine algebra of dimension d over an algebraically closed field $\overline{k}$, then Suslin [13] proved that Bass cancellation theorem [2] can be strengthened as follows: *If P is a projective A-module of rank d, then* $\mathrm{Aut}(A \oplus P)$ *acts transitively on* $\mathrm{Um}(A \oplus P)$*, i.e. P is cancellative.* Mohan Kumar, Murthy and Roy ([10], Theorem 2.4) generalized Suslin's result in case $\overline{k} = \overline{\mathbb{F}}_p$ as follows.

Theorem 2.6 *Let A be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$. Let P be a projective A-module of rank d. Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$.

The following result ([9], Theorem 3.8) is very crucial for the proof of (3.5).

Theorem 2.7 *Let A be an affine algebra of dimension d over $\overline{\mathbb{F}}_p$. Assume that if R is a finite extension of A then R^{d-1} is cancellative. Then every projective A-module P of rank $d-1$ is cancellative.*

We end this section with a result due to Fasel, Rao and Swan ([5], Corollary 7.4).

Proposition 2.8 *Let R be an affine algebra of dimension $d \geq 4$ over an algebraically closed field $\overline{k}$. Assume that gcd $((d-1)!, char(k)) = 1$. Let J be the ideal defining the singular locus of R. Then for any $v \in \mathrm{Um}_d(R, J)$, there exists $\Theta \in \mathrm{GL}_d(R)$ such that $v\Theta = (1, 0, \ldots, 0)$.*

3 Main Theorem

In this section, we prove our main result.

Let A be a ring and I an ideal of A. For an integer $n \geq 3$, define $E_n(I)$ as the subgroup of $E_n(A)$ generated by $E_{ij}(a) = Id + ae_{ij}$, where $a \in I$, $1 \leq i \neq j \leq n$ and only non-zero entry of the matrix ae_{ij} is a at the (i,j)th place.

Consider the cartesian square

$$\begin{array}{ccc} A(I) & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A & \xrightarrow[j_2]{} & A/I \end{array}$$

The relative group $E_n(A, I)$ is defined in [12] by the exact sequence

$$1 \to E_n(A, I) \to E_n(A(I)) \xrightarrow{E_n(p_1)} E_n(A) \to 1$$

and it is shown ([12], Proposition 2.2) that $E_n(A, I)$ is isomorphic to the kernel of the natural map $E_n(A) \to E_n(A/I)$. Further, $E_n(A, I)$ is the normal closure of $E_n(I)$ in $E_n(A)$ ([13], Section 2).

The following result is proved in ([13], Lemma 2.7).

Lemma 3.1 *Let R be a ring and I an ideal of R. If $n \geq 3$, then $E_n(R, I^2) \subset E_n(I)$.*

Lemma 3.2 *Let R be a ring and I an ideal of R. If $n \geq 3$, then $E_n(I) \subset E_n^1(R, I)$. In particular, $E_n(R, I^2) \subset E_n^1(R, I)$.*

Proof Let $E_{ij}(x) \in E_n(I)$, where $x \in I$. If $i = 1$ or $j = 1$, then $E_{ij}(x) \in E_n^1(R, I)$. Assume $i \neq 1$ and $j \neq 1$. Then $E_{ij}(x) = E_{i1}(1)E_{1j}(x)E_{i1}(-1)E_{1j}(-x) \in E_n^1(R, I)$. $\square$

Lemma 3.3 *Let A be a ring and let P be a projective A-module of rank r. Choose $s \in A$ satisfying the conditions in (2.3). Assume that if $R = A[X]/(X^2 - s^2X)$ then $E_{r+1}(R)$ acts transitively on* $\mathrm{Um}_{r+1}(R)$*. Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P, s^2A)$.

Proof Without loss of generality, we may assume that A is reduced. By (2.3), there exist $p_1, \dots, p_r \in P$ and $\phi_1, \dots, \phi_r \in \text{Hom}(P, A)$ such that P_s is free, $(\phi_i(p_j)) = \text{diagonal } (s, \dots, s)$, $sP \subset p_1 A + \dots + p_r A$ and s is a non-zerodivisor.

Let $(a, p) \in \text{Um}(A \oplus P, s^2 A)$. Replacing p by $p - ap$, we may assume that $p \in s^3 P$. Since $sP \subset \sum_1^r Ap_i$, we get $p = f_1 p_1 + \dots + f_r p_r$ for some $f_i \in s^2 A$. Note that $v = (a, f_1, \dots, f_r) \in \text{Um}_{r+1}(A, s^2 A)$.

Consider the following cartesian square

$$\begin{array}{ccc} R & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A & \xrightarrow[j_2]{} & A/(s^2) \end{array}$$

Patching unimodular rows $(a, f_1, \dots, f_r)$ and $(1, 0, \dots, 0)$ over $A/s^2 A$, we get a unimodular row $(c_0, c_1, \dots, c_r) \in \text{Um}_{r+1}(R)$. Since $E_{r+1}(R)$ acts transitively on $\text{Um}_{r+1}(R)$, there exists $\Theta \in E_{r+1}(R)$ such that $(c_0, c_1, \dots, c_r)\Theta = (1, 0, \dots, 0)$. The projections of this equation gives

$$(a, f_1, \dots, f_r)\Psi = (1, 0, \dots, 0) \text{ and } (1, 0, \dots, 0)\widetilde{\Psi} = (1, 0, \dots, 0)$$

where $\Psi, \widetilde{\Psi} \in E_{r+1}(A)$ such that $\Psi = \widetilde{\Psi}$ modulo (s^2). Hence $(a, f_1, \dots, f_r)\Psi\, \widetilde{\Psi}^{-1} = (1, 0, \dots, 0)$, where $\Psi\widetilde{\Psi}^{-1} = \Delta \in E_{r+1}(A, s^2 A)$.

By (3.2), $\Delta \in E^1_{r+1}(A, sA)$. Hence by (2.5), there exists $\Theta \in E(A \oplus P)$ such that $(a, p)\Theta = (1, 0)$. This completes the proof. $\square$

As a consequence of (3.3), we prove our first result.

Theorem 3.4 *Let A be a ring of dimension d and let P be a projective A-module of rank d. Assume that if R is a finite extension of A then $E_{d+1}(R)$ acts transitively on $\text{Um}_{d+1}(R)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Proof Let $(a, p) \in \text{Um}(A \oplus P)$. Choose $s \in A$ satisfying the conditions in (2.3). Let "bar" denote reduction modulo $s^2 A$. Since $\dim \overline{A} = d - 1$, by Bass cancellation theorem [2], there exists $\overline{\sigma} \in E(\overline{A} \oplus \overline{P})$ such that $(\overline{a}, \overline{p})\overline{\sigma} = (1, 0)$. By (2.2), we can lift $\overline{\sigma}$ to $\theta \in E(A \oplus P)$. If $(a, p)\theta = (b, q)$, then $(b, q) \in \text{Um}(A \oplus P, s^2 A)$. By (3.3), there exists $\theta_1 \in E(A \oplus P)$ such that $(b, q)\theta_1 = (a, p)\theta\theta_1 = (1, 0)$. This proves the result. $\square$

The next result generalize a result of Fasel, Rao and Swan ([5], Theorem 7.3) in the case $\overline{k} = \overline{\mathbb{F}}_p$ as follows.

Theorem 3.5 *Let A be an affine algebra of dimension $d \geq 4$ over the field $\overline{\mathbb{F}}_p$, where $p \geq d$. Let P be a projective A-module of rank $d - 1$. Then P is cancellative.*

Proof By (2.7), it is enough to show that if R is any affine algebra of dimension d over $\overline{\mathbb{F}}_p$, then R^{d-1} is cancellative. Let $v \in \text{Um}_d(R)$ be any unimodular row of length d. It is enough to show that there exists $\Delta \in \text{GL}_d(R)$ such that $v\Delta = e_1 = (1, 0, \dots, 0)$. Without loss of generality, we may assume that R is reduced.

Let J be the ideal of R defining the singular locus of R. Since R is reduced, height of J is ≥ 1. Let "bar" denote reduction modulo J. Then $\dim \overline{R} \leq d - 1$. By (2.6), there exits $\overline{\sigma} \in E_d(\overline{R})$ such that $\overline{v}\,\overline{\sigma} = e_1$. By (2.2), we can lift $\overline{\sigma}$ to $\theta \in E_d(R)$. Then $v\theta = e_1$ modulo J. Applying (2.8), we get $\theta_1 \in \text{GL}_d(R)$ such that $v\theta\theta_1 = e_1$. Hence v is completable to an invertible matrix $(\theta\theta_1)^{-1}$, i.e. R^{d-1} is cancellative. This completes the proof. $\square$

4 Extension of Gubeladze's results

In this section we extend some results of Gubeladze. We begin by recalling three results due to Gubeladze from [6], ([7], Theorem 8.1) and ([8], Theorem 10.1) respectively. See [8] for the definition of a monoid M of Φ-simplicial growth.

Theorem 4.1 *Let M be a commutative, torsion-free, seminormal and cancellative monoid. Then for any principal ideal domain R, projective modules over $R[M]$ are free.*

Theorem 4.2 *Let R be a ring of dimension d and let $M \subset \mathbb{Q}_+^r$ be a submonoid such that $M \subset \mathbb{Q}_+^r$ is an integral extension. Then $E_n(R[M])$ acts transitively on $\mathrm{Um}_n(R[M])$ whenever $n \geq$ max $(3, d+2)$.*

Theorem 4.3 *Let R be a ring of dimension d and let M be a monoid of Φ-simplicial growth. Then $E_n(R[M])$ acts transitively on $\mathrm{Um}_n(R[M])$ whenever $n \geq$ max $(3, d+2)$.*

We will generalize above results of Gubeladze as follows. Since we are not assuming that M is seminormal, we need to assume that $S^{-1}P$ is free due to the following result of Gubeladze [6]: *If M is commutative, torsion-free and cancellative monoid such that projective $k[M]$-modules are free for all fields k, then M is seminormal.*

Theorem 4.4 *Let M be as in (4.2) or (4.3). Let R be a ring of dimension d and let P be a projective $R[M]$-module of rank n. Assume that $S^{-1}P$ is free, where S is the set of non-zerodivisors of R. Then $E(R[M]\oplus P)$ acts transitively on $\mathrm{Um}(R[M]\oplus P)$ whenever $n \geq$ max $(2, d+1)$.*

Proof By (2.4), we may assume that the ring $A = R[M]$ is reduced. We will use induction on d. If $d = 0$, then by assumption, projective modules of constant rank over $R[M]$ are free. Hence we are done by (4.2) and (4.3).

Assume $d > 0$. By assumption $S^{-1}P$ is free. Hence we can choose $s \in S$ such that P_s is free and conditions of (2.3) are satisfied.

Let $(a, p) \in \mathrm{Um}(A\oplus P)$ and let "bar" denote reduction modulo s^2A. Since $\dim \overline{R} = d - 1$, by induction hypothesis, there exists $\phi \in E(\overline{A}\oplus\overline{P})$ such that $(\overline{a}, \overline{p})\phi = (1, 0)$. Let $\Phi \in E(A\oplus P)$ be a lift of ϕ, by (2.2). Then $(a, p)\Phi \in \mathrm{Um}(A\oplus P, s^2A)$. By Gubeladze's theorem in the free case, $E_{n+1}(B[M])$ acts transitively on $\mathrm{Um}_{n+1}(B[M])$, where $B = R[X]/(X^2 - s^2X)$ is a ring of dimension d. Applying (3.3) to $(a, p)\Phi \in \mathrm{Um}(A\oplus P, s^2A)$, there exists $\Phi_1 \in E(A\oplus P)$ such that $(a, p)\Phi\Phi_1 = (1, 0)$. This completes the proof. $\square$

Using (4.1, 4.4), we get the following.

Theorem 4.5 *Let M be as in (4.2) or (4.3). Further assume that M is seminormal. Let R be a ring of dimension d and let P be a projective $R[M]$-module of rank n. Then $E(R[M]\oplus P)$ acts transitively on $\mathrm{Um}(R[M]\oplus P)$ whenever $n \geq$ max $(2, d+1)$.*

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References

[1] A. Bak, R. Basu and Ravi A. Rao, *Local-global principle for transvection groups*, Proc. Amer. Math. Soc. **138** (2010), 1191-1204.

[2] H. Bass, *K-theory and stable algebra*, Publ. Math. Inst. Hautes Etudes Sci. **22** (1964), 5-60.

[3] S.M. Bhatwadekar, *Cancellation theorem for projective modules over a two dimensional ring and its polynomial extension*, Compositio Math. **128** (2001), 339-359.

[4] A.M. Dhorajia and M.K. Keshari, *Projective modules over overrings of polynomial rings*, J. Algebra **323** (2010), 551-559.

[5] J. Fasel, Ravi A. Rao and R.G. Swan, *On stably free modules over affine algebras*, Preprint.

[6] J. Gubeladze, *Anderson's conjecture and the maximal class of monoids over which projective modules are free*, Math. USSR-Sb. **63** (1988), 165-188.

[7] J. Gubeladze, *The elementary action on unimodular rows over a monoid ring*, J. Algebra **148** (1992), 135-161.

[8] J. Gubeladze, *The elementary action on unimodular rows over a monoid ring II*, J. Algebra **155** (1993), 171-194.

[9] M.K. Keshari, *Cancellation problem for projective modules over affine algebras*, J. K-Theory **3** (2009), 561-581.

[10] N. Mohan Kumar, M.P. Murthy and A. Roy, *A cancellation theorem for projective modules over finitely generated rings*, in: Hijikata H., et al. (Eds.), Algebraic geometry and commutative algebra in honor of Masayoshi Nagata, vol. 1, (1987), 281-287.

[11] H. Lindel, *Unimodular elements in projective modules*, J. Algebra **172** (1995), 301-319.

[12] M.R. Stein, *Relativizing functors on rings and algebraic K-theory*, J. Algebra **19** (1971), 140-152.

[13] A.A. Suslin, *A cancellation theorem for projective modules over affine algebras*, Sov. Math. Dokl. **18** (1977), 1281-1284.

[14] A.A. Suslin and L.N. Vaserstein, *Serre's problem on projective modules over polynomial rings and algebraic K-theory*, Math. USSR Izvestija **10** (1976), 937-1001.